Palestine Journal of Mathematics

Vol 14(4)(2025) , 624–633

HYPO-EP OPERATORS ON HILBERT C^*- MODULES

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Communicated by: Harikrishnan Panackal

MSC 2010 Classifications: Primary 47S10, 47A05; Secondary 46S10, 46C99.

Keywords and phrases: Hypo-EP Operators, Hilbert C^*-modules, Moore-Penrose inverse.

The authors would like to thank the reviewers and the editor for their constructive comments and valuable suggestions that improved the quality of our article. The first author thanks the Vellore Institute of Technology (VIT) Vellore, for providing financial support.

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Abstract Closed range operators defined on Hilbert spaces such that their commutant with their own Moore-Penrose inverse is a positive operator are introduced by M.Itoh in the name of hypo-EP operators. This class contains EP and closed range normal operators. We extend this class of operators to Hilbert C^*-modules. Although Hilbert C^*-modules are a natural generalisation of Hilbert spaces, it does not enjoy many properties like Hilbert spaces. In this article, we define hypo-EP operators on Hilbert C^*-modules and discuss some results, including characterizations.

1 Introduction

A bounded linear operator defined on an arbitrary Hilbert space with closed range is said to be an EP operator if the ranges of the operator and its adjoint are equal. The EP operators are defined by Campbell and Mayer [13]. They characterized the EP operators as operators that commute with their own Moore-Penrose inverse. The closed range of the operator guarantees the unique existence of the Moore-Penrose inverse. Equivalently, the commutant of the Moore-Penrose inverse and the operator is zero ($T^\dagger T - TT^\dagger = 0$). M. Itoh in [11] defined a new class of operators by relaxing this condition as commutant of the Moore-Penrose inverse of the operator and the operator itself is positive ($T^\dagger T - TT^\dagger \geq 0$). Such a general class, which contained the EP operators, is named the hypo-EP operators. Various properties and characterizations of hypo-EP operators on Hilbert spaces can be found in [1, 11]. Vinoth and Sam Johnson discussed the sum, restrictions in [4], and products and factorizations of hypo-EP operators in [3]. In [1], the authors discuss hypo-EP operator matrices and also prove that any natural power of a hypo-EP operator is hypo-EP.

A Hilbert space is a complex vector space equipped with a complex valued inner product. A natural generalization of this is the Hilbert C^*- modules in which the underlying complex field is replaced by a C^*-algebra and a module over the C^*-algebra is considered instead of a linear space.[5]. This introduces a new category that falls in between Banach spaces and Hilbert spaces. Hilbert C^*-modules have recently been studied by many for various aspects of applications. The extension of notions of frame theory to Hilbert C^*-modules is a remarkable one among them. Significant and recent developments towards this direction can be found in [12] and [2]

Although Hilbert C^*-modules represent a natural extension of Hilbert spaces, they do not adhere to many of the properties and results of Hilbert spaces. As a consequence of the C^*-algebra valued nature of the inner product, the geometric and analytic structure of a Hilbert C^*-module differs significantly from that of a Hilbert space. The standard properties, such as self duality and the existence of orthogonal decompositions, are not generally true for Hilbert C^*-modules. In addition, bounded linear operators on a Hilbert C^*-module do not necessarily admit adjoints[7]. To develop a consistent theory, we work with the class of bounded linear operators that admit an adjoint, known as the adjointable operators. Any adjointable linear operator on a Hilbert C^*- module has to be bounded [5]. It is also well known that any bounded linear adjointable operator with a closed range admits a unique Moore-Penrose inverse [6]. Also, the Moore-Penrose inverse of an adjointable closed range operator is adjointable.

In this article, we define the class of hypo-EP operators in the setting of Hilbert C^*- modules. Hypo-EP operators are defined in terms of the commutant of operators with their Moore-Penrose inverse. In this framework, we discuss various characterizations and structural properties of hypo-EP operators. In particular, we establish that the existence of a specific decomposition of the Moore-Penrose inverse provides a necessary and sufficient condition for an operator to be hypo-EP. Furthermore, we prove that any operator that is unitarily equivalent to a hypo-EP operator is hypo-EP. Also, it is shown that any natural power of hypo-EP operators remains hypo-EP.

The paper is organized as follows. Section 2 consists of the basic definitions, results, and notations that are used in this article. This includes the formal definition of Hilbert C^*- modules and adjoints of operators on them, to emphasize that the underlying structure of a Hilbert C^*- module is distinct from that of a Hilbert space. In Section 3, we define hypo-EP operators on a Hilbert C^*- module and discuss some of the equivalent characterizations of it. Hypo-EP operators are characterized in terms of range, null space, and Moore-Penrose inverse. It is shown that the unitary equivalence of the operators preserves hypo-EP properties. Section 4 discusses the natural powers of hypo-EP operators along with the necessary results.

2 Notations and prerequisites

Definition 2.1. [5] A Hilbert A- module H is a right A-module equipped with an A valued inner product$\langle\cdot,\cdot\rangle$ as $\langle\cdot,\cdot\rangle : H\times H \to A$ such that H is complete with respect to the induced norm $\|\cdot\| = \|\langle\cdot,\cdot\rangle\|^{\frac{1}{2}}$ and H is said to be a Hilbert C^*- module if A is a C^* algebra. Any complex Hilbert space is a Hilbert C^*- module with the complex field $\mathbb{C}$ as the C^* algebra.

As a convention, we use A and H to denote a unital C^*- algebra and a Hilbert C^*- module, respectively, unless specified otherwise. We say that a closed submodule G of a Hilbert C^*-module H is orthogonally complemented if $H = G \oplus G^{\perp}$. If $G \subseteq H$ is an orthogonally complemented submodule, then any element $x \in H$ can be uniquely written as $x = x_1 + x_2$ where $x_1 \in G$ and $x_2 \in G^{\perp}$. In the framework of a Hilbert space, any closed subspace is orthogonally complemented. But it is not the case with Hilbert C^*- modules. One can refer [5] for further information and counterexamples.

Definition 2.2. [5] An operator T on a Hilbert C^*-module H is said to be adjointable if there exists an operator T^* on H such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in H$.

Any adjointable operator on a Hilbert A-module H must be A-linear and bounded [5]. By $L(H)$ we denote the collection of all adjointable operators on H. Throughout this article, $R(T)$ and $N(T)$ denote the range and kernel of the operator T.

Definition 2.3. [6] An operator $T \in L(H)$ is said to be Moore-Penrose invertible if there exists a unique operator $S \in L(H)$ such that $TST = T$, $STS = S$, $(TS)^* = TS$ and $(ST)^* = ST$. $T^\dagger$.

If the operator T is Moore-Penrose invertible, then the corresponding S as in the definition is referred to as the Moore-Penrose inverse of T and is represented by $T^\dagger$.

Lemma 2.4 (Theorem *2.2* of [15]). *Let $T \in L(H)$. Then T is Moore-Penrose invertible if and only if T has a closed range. That is, $R(T)$ is closed in H.*

The commutant of two operators T, S on H, denoted by $[T, S]$ is defined as $[T, S] = TS - ST$. The commutant of two mutually commuting operators will be zero. There are a plenty of concepts in operator theory defined and characterized in terms of commutants. For instance, normal and hyponormal operators are defined as operators such that the commutant of an operator with its own adjoint is zero or positive, respectively. Subsequently, EP and hypo-EP operators are defined. The theory of hypo-EP operators has been developed in considerable depth within the setting of Hilbert spaces, and has also been generalized to other frameworks, including C^*-algebras[14]. This paper broadens the scope of the definition and certain properties of hypo-EP operators to the framework of Hilbert C^*- modules.

3 Hypo-EP Operators on Hilbert C^*-modules

Definition 3.1. Let A be a C^*- algebra and H be a Hilbert A- module. An adjointable operator T on H is said to be hypo-EP if $R(T)$ is closed and $[T^\dagger, T] \geq 0$, that is $T^\dagger T - TT^\dagger$ is a positive operator on H.

Example 3.2. Consider the standard Hilbert C^*- module H_A over a C^* algebra A. That is,

$$H_A = \{(x_i)_{i=1}^{\infty}, x_i \in A \mid \Sigma x_i^* x_i \; converges \; in \; A\}$$

with the inner product $\langle x, y\rangle = \Sigma x_i^* y_i,$ where $x = (x_i)_{i=1}^{\infty} \in H \; ; \; y = (y_i)_{i=1}^{\infty} \in H$
Let T be the unilateral right shift operator on H_A, given as

$$T(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots).$$

One can observe that $T^\dagger$ is the left shift operator on H_A,

$$T^\dagger(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, \cdots).$$

$$Now, \;\; T^\dagger T(x_1, x_2, x_3, \cdots) = (x_1, x_2, x_3, \cdots) \; and \; TT^\dagger(x_1, x_2, x_3, \cdots) = (0, x_2, x_3, \cdots).$$

Then, $T^\dagger T - TT^\dagger(x_1, x_2, x_3, \cdots) = (x_1, 0, 0, \cdots)$, which is a positive operator.

Example 3.3. Consider the unilateral left shift operator S on the Hilbert C^*- module H_A as above. Clearly $R(S) = H_A$, which is trivially closed. Its Moore-Penrose inverse $S^\dagger$ is the right shift operator on H_A. Then, $S^\dagger S - SS^\dagger(x_1, x_2, x_3, \cdots) = (-x_1, 0, 0, \cdots)$ is not a positive operator. Thus, the left shift operator on H_A is not hypo-EP.

Now, we establish certain equivalent characterizations of hypo-EP operators on Hilbert C^*-modules. Although similar properties have been studied in the context of Hilbert spaces, the inherent structural differences between Hilbert spaces and Hilbert-C^*-modules necessitate a fundamentally separate study. To this end, we begin by establishing a number of foundational results specific to Hilbert C^*-modules, which serve as an essential tool for the subsequent development in the theory. The Lemma 3.4 establishes some standard results concerning the range and null space of a closed range operator T, its Moore-Penrose inverse $T^\dagger$, adjoint T^* and the products $T^\dagger T$ and $TT^\dagger$ on a Hilbert C^*-module H. The proof is based on the characterization of Moore-Penrose inverse given in Definition 1.2, along with Theorem 1.3 of [8]

Lemma 3.4. *Let $T \in L(H)$ be Moore-Penrose invertible and $T^\dagger$ be its Moore-Penrose inverse.Then,*

(i) $R(T) = R(TT^\dagger)$*and* $R(T^\dagger) = R(T^\dagger T) = R(T^*)$.

(ii) $N(T) = N(T^\dagger T)$ *and* $N(T^\dagger) = N(TT^\dagger) = N(T^*)$.

(iii) $T^\dagger T$ *is the orthogonal projection on to* $R(T^\dagger) = R(T^*)$.

(iv) $TT^\dagger$ *is the orthogonal projection on to* $R(T)$.

The following results concerning Hilbert C^* modules, their submodules, and orthogonal projection onto these submodules provide the necessary structural framework for analyzing the operators $T^\dagger T$ and $TT^\dagger$, which play a central role in the study of hypo-EP operators.

Theorem 3.5. *Let H be a Hilbert C* module and M, N be two orthogonally complemented submodules of H. Let P and Q be the orthogonal projections onto M and N, respectively. Then the following are equivalent.*

(i) $P \leq Q$

(ii) $\|Px\| \leq \|Qx\|, \forall x \in H$

(iii) $M \subseteq N$

(iv) $QP = P$

(v) $PQ = P$

Proof. $(i) \Rightarrow (ii)$**:** Assume that $P \leq Q$. Then for any $x \in H$, we have

$$\|Px\|^2 = \|\langle Px, x\rangle\| \leq \|\langle Qx, x\rangle\| = \|Qx\|^2,$$

which implies $\|Px\| \leq \|Qx\|$.

$(ii) \Rightarrow (iii)$**:** Suppose $\|Px\| \leq \|Qx\|$ for all $x \in H$. If $N = H$ or $N^\perp = \{0\}$, the conclusion is immediate. Assume instead that $N \neq H$ and $N^\perp \neq \{0\}$. Let $x \in N^\perp$ with $x \neq 0$. Since Q is the orthogonal projection onto the orthogonally complemented submodule N, we have $N(Q) = N^\perp$, and hence $Qx = 0$. Then, by the assumption,

$$\|Px\| \leq \|Qx\| = 0 \quad \Rightarrow \quad \|Px\| = 0 \quad \Rightarrow \quad Px = 0.$$

Thus, $x \in N(P) = M^\perp$, where the identity follows from the fact that P is the orthogonal projection onto M. Therefore, $N^\perp \subseteq M^\perp$, which implies $M \subseteq N$ by taking orthogonal complements.

$(iii) \Rightarrow (iv)$**:** Assume $M \subseteq N$. For any $x \in H$, we have $Px \in M \subseteq N$, and since Q is the orthogonal projection onto N, it follows that

$$QPx = Px.$$

As this holds for all $x \in H$, we conclude that $QP = P$.

$(iv) \Rightarrow (v)$**:** Assuming $QP = P$, we take adjoints on both sides to obtain

$$P^* = (QP)^* = P^*Q^* = PQ,$$

where we have used the fact that P and Q are self-adjoint projections. Hence, $PQ = P$.

$(v) \Rightarrow (i)$**:** Assume $PQ = P$. Then, for any $x \in H$,

$$\langle Px, x\rangle = \langle P^2x, x\rangle = \langle Px, P^*x\rangle = \langle Px, Px\rangle = \|Px\|^2 = \|PQx\|^2 \leq \|Qx\|^2 = \langle Qx, x\rangle$$

and it follows that $P \leq Q$. □

The equivalent definitions and characterizations of hypo-Ep operators on Hilbert spaces have been extensively studied, with the majority of approaches grounded in algebraic techniques and a few in analytical concepts[3][1]. In what follows, a collection of theorems is provided that offers characterizations of hypo-EP operators on Hilbert C^*-modules, bridging the gap between the two theoretical frameworks and extending the scope of existing theory.

Theorem 3.6. *Let $T \in L(H)$ be such that $R(T)$ is closed. Then the following are equivalent.*

(i) T *is hypo EP;*

(ii) $\|T^\dagger Tx\| \geq \|TT^\dagger x\|$ *; for any* $x \in M$*;*

(iii) $R(T) \subseteq R(T^*)$*;*

(iv) $T^\dagger T^2 T^\dagger = TT^\dagger$*;*

(v) $T(T^\dagger)^2 T = TT^\dagger$.

Proof. Let $P = TT^\dagger$, $Q = T^\dagger T$, $M = R(TT^\dagger)$ and $N = R(T^\dagger T)$. By Lemma 3.4, P and Q are orthogonal projections onto M and N respectively.
Suppose that T is a hypo EP operator on H. Then by definition, $[T^\dagger, T] \geq 0$, or in expansion $T^\dagger T - TT^\dagger \geq 0$. This, in terms of the projections P and Q is $Q \geq P$. According to Theorem 3.5, this inequality is equivalent to the following conditions.

$$\|Px\| \leq \|Qx\| \; \forall x \in H, \;\; M \subseteq N, \;\; QP = P, \;\; PQ = P.$$

Rewriting this in terms of the original operator T we obtain

$$\|T^\dagger Tx\| \geq \|TT^\dagger x\| \forall x \in M, \;\; R(T) \subseteq R(T^*), \;\; T^\dagger T^2 T^\dagger = TT^\dagger \;\; T(T^\dagger)^2 T = TT^\dagger.$$

This proves the theorem. □

Corollary 3.7. *Let H and T be as in the assumptions of the previous theorem. Then T is a hypo-EP operator if and only if any of the following holds.*

(i) $T^n = T^{\dagger}T^{n+1}$ *;* $\forall n \in \mathbb{N}$

(ii) ${T^{\dagger}}^n = {T^{\dagger}}^{n+1}T$*;* $\forall n \in \mathbb{N}$

Proof. Let T be a hypo-EP operator on H. By part *(iv)* of Theorem 3.6, we have

$$T^{\dagger}T^2T^{\dagger} = TT^{\dagger}.$$

Right-multiplying both sides by T^n for any $n \in \mathbb{N}$ yields

$$T^{\dagger}T^{n+1} = T^n, \quad \text{for all } n \in \mathbb{N}.$$

Taking $n = 1$ and further right multiplying both sides by $T^{\dagger}$ gives the converse. Using similar reasoning, the equivalence between the second statement and the part *(v)* of the theorem can be established. □

Theorem 3.6 established a characterization of hypo-EP operators in terms of the ranges of the operator and its adjoint. On the other hand, the next theorem gives another significant characterization for hypo-EP operators on Hilbert C^*- modules in terms of null spaces of these operators.

Theorem 3.8. *Let H be a Hilbert C^*-module. Then a closed range operator $T \in L(H)$ is a hypo-EP operator if and only if $N(T) \subseteq N(T^*)$.*

Proof. Let T be a hypo-EP operator on H. Suppose there exists an $x \in N(T)$ such that $x \notin N(T^*)$. Since $N(T) = N(T^{\dagger}T)$ (by Lemma 3.4), it follows that $\|T^{\dagger}Tx\| = 0$.
On the other hand, Lemma 3.4 also states that $N(T^*) = N(TT^{\dagger})$. Then since $x \notin N(T^*)$, it follows that $|\, TT^{\dagger}x\| > 0$.
Therefore, $\|T^{\dagger}Tx\| < \|TT^{\dagger}x\|$, which contradicts the Theorem 3.6. This contradiction establishes that $N(T) \subseteq N(T^*)$.
Conversely, assume that $N(T) \subseteq N(T^*)$, which implies on taking orthogonal complements that $N(T^*)^{\perp} \subseteq N(T)^{\perp}$. By Theorem 3.2 of [5], we have $N(T)^{\perp} = R(T^*)$ and $N(T^*)^{\perp} = R(T)$.
Hence, $R(T) \subseteq R(T^*)$. which shows that T is a hypo-EP operator on H. □

Corollary 3.9. *If $T \in L(H)$ is such that $R(T)$ is closed, then the following are equivalent.*

(i) *T is hypo-EP.*

(ii) $N(T^{\dagger}T) \subseteq N(TT^{\dagger})$

(iii) $R(TT^{\dagger}) \subseteq R(T^{\dagger}T)$

Proof. By Lemma 3.4, we have $N(T^{\dagger}T) = N(T)$ and $N(TT^{\dagger}) = N(T^*)$. Then the previous theorem proves the equivalence of statements (i) and (ii). Lemma 3.4 also states that $R(TT^{\dagger}) = R(T)$ and $R(T^{\dagger}T) = R(T^*)$. By (iii) of Theorem 3.6 (i) and (iii) are equivalent. □

In [10] K. Sharifi introduces EP modular operators on Hilbert C^* modules, defining them as adjointable linear operators T on H whose range is closed and both T and T^* have the same range. Equivalently, these are the operators that commute with their own Moore-Penrose inverse ($T^{\dagger}T = TT^{\dagger}$). This definition closely parallels the definition of EP operators on Hilbert spaces. It is evident from the definition that every EP operator is hypo-EP. However, example 3.2 demonstrates that the converse is not true.

It is a well-known result that every bounded below operator on Hilbert spaces has a closed range. The following theorem evidently proves a stronger result- namely, that any bounded below operator on a Hilbert C^*- module not only has closed range but is, in fact, a hypo-EP operator.

Theorem 3.10. *Let $T \in L(H)$ be bounded below. Then T is hypo-EP.*

Proof. It suffices to show that $R(T)$ is closed and $N(T) \subseteq N(T^*)$. Let T be an operator that is bounded below. That is, there exist $c > 0$ such that

$$\|Tx\| \geq c\|x\| \,; x \in H.$$

Let $\{y_n\}$ be a sequence in $R(T)$ that converges to some $y \in H$. Then for each n, there exists $x_n \in H$ such that $Tx_n = y_n$ and since $y_n \to y$, we have $Tx_n \to y$.
As H is Cauchy complete, the convergent sequence Tx_n must be Cauchy. That is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|Tx_n - Tx_m\| < \epsilon.$$

Since T is bounded below, it follows that

$$c\|x_n - x_m\| \leq \|Tx_n - Tx_m\| < \epsilon.$$

That is, the sequence x_n is a Cauchy sequence on H and by Cauchy completeness of H it converges to some $x \in H$. By continuity of T, we have $Tx_n \to Tx$. Since $Tx_n \to y$ and by the uniqueness of the limits $y = Tx$. That is, $y \in R(T)$. So $R(T)$ is closed. Now, consider $x_0 \in N(T)$. Since T is bounded below, we have

$$\|Tx_0\| \geq c\|x_0\|.$$

But $Tx_0 = 0$ which gives $\|x\| = 0$ or simply $x_0 = 0$. Therefore, $N(T) = \{0\} \subseteq N(T^*)$, as desired. □

Corollary 3.11. *Any linear adjointable bounded injective map with closed range on a Hilbert C^*−module is hypo-EP.*

Theorem 3.12. *Let H be Hilbert C^*-module and $T \in L(H)$ be of closed range. Then T is a hypo-EP operator if and only if there exists an operator E such that $T^\dagger = ET$. Moreover, if T is EP, then there exists an operator D such that $T = DT^\dagger$.*

Proof. Let T be a hypo EP operator on H. Define a map $\widetilde{E} : R(T) \to R(T^\dagger)$ such that

$$\widetilde{E}(Tz) = T^\dagger z.$$

We first show that $\widetilde{E}$ is well defined and bounded. Let $x_1, x_2 \in R(T)$ be such that $x_1 = x_2$. We need to show that $\widetilde{E}(x_1) = \widetilde{E}(x_2)$. Let $x_1 = Tz_1$ and $x_2 = Tz_2$. By the definition of $\widetilde{E}$, $\widetilde{E}(x_1) = T^\dagger z_1$ and $\widetilde{E}(x_2) = T^\dagger z_2$ Then,

$$\widetilde{E}(x_1) - \widetilde{E}(x_2) = T^\dagger z_1 - T^\dagger z_2.$$

But $0 = x_1 - x_2 = Tz_1 - Tz_2 = T(z_1 - z_2)$ implies $z_1 - z_2 \in N(T)$. Since T is hypo-EP, we have $N(T) \subseteq N(T^*) = N(T^\dagger)$, so $z_1 - z_2 \in N(T^\dagger)$. Thus,

$$T^\dagger(z_1 - z_2) = T^\dagger z_1 - T^\dagger z_2 = 0 \Rightarrow \widetilde{E}(x_1) = \widetilde{E}(x_2)$$

Therefore, $\widetilde{E}$ is well defined. Since $T \in L(H)$ is with closed range, $T^\dagger$ is a bounded operator. So,there exists $M > 0$ such that $\|\widetilde{E}(Tz)\| = \|T^\dagger z\| \leq M\|Z\|$.
As T has closed range, we have the orthogonal decomposition [15],

$$H = R(T) \oplus N(T^*) = R(T) \oplus N(T^\dagger), \ \because N(T^*) = N(T^\dagger)$$

Now define the linear operator $E : H \to H$ as

$$E(y) = \begin{cases} \widetilde{E}(y) & if \ y \in R(T) \\ 0 & if \ y \in N(T^\dagger) \end{cases}$$

Then for any $x \in H$, we have

$$ETx = \widetilde{E}(Tx) = T^\dagger x.$$

Which implies that,

$$ET = T^{\dagger}.$$

Conversely let there exist an operator $E \in L(H)$ such that $T^{\dagger} = ET$. Right multiplying with $T^{\dagger}T$ and left multiplying with T, we get

$$T(T^{\dagger})^2T = TETT^{\dagger}T = TET = TT^{\dagger}$$

Then by part (v) of Theorem 3.6, T is hypo-EP. Now assume further that T is EP. Similarly to the previous part, define a map $\widetilde{D} : R(T^{\dagger}) \to R(T)$ such that

$$\widetilde{D}(T^{\dagger}z) = Tz$$

We show that $\widetilde{D}$ is well defined. Let $x_1, x_2 \in R(T^{\dagger})$ be such that $x_1 = x_2$. We need to show that $\widetilde{D}(x_1) = \widetilde{D}(x_2)$. Let $x_1 = T^{\dagger}z_1$ and $x_2 = T^{\dagger}z_2$. Then by the definition of $\widetilde{D}$,

$$\widetilde{D}(x_1) - \widetilde{D}(x_2) = Tz_1 - Tz_2$$

We have $0 = x_1 - x_2 = T^{\dagger}z_1 - T^{\dagger}z_2 = T^{\dagger}(z_1 - z_2)$ and it follows that $z_1 - z_2 \in N(T^{\dagger})$. For an EP operator T, $N(T) = N(T^*) = N(T^{\dagger})$. So $z_1 - z_2 \in N(T)$ and thus $T(z_1 - z_2) = \widetilde{D}(x_1) - \widetilde{D}(x_2) = 0$.
Also, $\|\widetilde{D}(T^{\dagger}z)\| = \|Tz\| \leq M\|Z\|$, for some $M > 0$, since T is bounded.
Because T is EP, $R(T)$ is closed and we have the orthogonal decomposition

$$H = N(T) \oplus R(T^*) = N(T) \oplus R(T^{\dagger}), \ \because R(T^*) = R(T^{\dagger})$$

Now define the linear operator $D : H \to H$ as

$$D(y) = \begin{cases} \widetilde{D}(y) & if\ y \in R(T^{\dagger}) \\ 0 & if\ y \in N(T) \end{cases}$$

Then for any $x \in H$, $DT^{\dagger}x = \widetilde{D}(T^{\dagger}x) = Tx$. So we have, $DT^{\dagger} = T$.

□

In the case where T is an EP operator, a natural example of an operator D satisfying the conditions of the preceding theorem emerges. Recalling the fundamental identity associated with Moore-Penrose inverse $T = TT^{\dagger}T$ and utilizing the EP property ($T^{\dagger}T = TT^{\dagger}$), one can obtain the representation $T = T^2T^{\dagger}$, suggesting a viable candidate T^2 for D. The following corollary is an immediate consequence of the above theorem.

Corollary 3.13. *Let H be Hilbert C^*-module and $T \in L(H)$ has closed range and $T^{\dagger}$ be its Moore-Penrose inverse. Then T is a hypo-EP operator if and only if $T^{\dagger} = (T^{\dagger})^2T$.*

Proof. Let T be hypo-EP. By *(v)* of Theorem 3.6 we have

$$T(T^{\dagger})^2T = TT^{\dagger}T^{\dagger}T = TT^{\dagger}$$

Left multiplying with $T^{\dagger}$ will give

$$T^{\dagger}TT^{\dagger}T^{\dagger}T = T^{\dagger}TT^{\dagger}$$

Then, by one of the defining properties of Moore-Penrose inverse, that is $T^{\dagger}TT^{\dagger} = T^{\dagger}$, one can conclude that $(T^{\dagger})^2T = T^{\dagger}$.
Conversely let $T^{\dagger} = (T^{\dagger})^2T$.Since $T^{\dagger} \in L(H)$ and $L(H)$ is an algebra, $(T^{\dagger})^2 \in L(H)$. Then by Theorem 3.12, T is hypo-EP. □

Corollary 3.14. *Let H be Hilbert C^*-module and $T \in L(H)$. If T is EP, Then T^n commutes with $T^{\dagger}$ for any $n \in \mathbb{N}$*

Proof. By corollary 3.13, we have $T = T^2T^{\dagger}$. Left multiplying with T^{n-1}, for any $n > 1$ implies $T^n = T^{n+1}T^{\dagger}$. Also, since every EP operator is hypo EP, by corollary 3.7 we have $T^n = T^{\dagger}T^{n+1}$. Equating these two gives the desired result. □

Definition 3.15. Let H be a Hilbert C^*-module. An operator $T \in L(H)$ is said to be unitarily equivalent to an operator $S \in L(H)$ if there exists a unitary operator R such that $S = R^{-1}TR$. An operator $R \in L(H)$ is said to be unitary if it is invertible and $R^{-1} = R^*$.

Theorem 3.16. *Let $T \in L(H)$ be hypo-EP and $S \in L(H)$ be unitarily equivalent to T. Then S is hypo-EP. That is, the property of being hypo-EP is preserved under unitary equivalence on Hilbert C^*- modules.*

Proof. The operator S is unitarily equivalent to T. So, there exists a unitary operator R on H such that

$$S = R^{-1}TR.$$

First we will show that S has a closed range. Since T is hypo-EP, $R(T)$ is closed in H and its Moore-Penrose inverse $T^\dagger$ exists in $L(H)$.
Define the operator

$$P = R^{-1}T^\dagger R.$$

$L(H)$ is a C^*- algebra and being a product of three elements in $L(H)$, $P \in L(H)$. Now, we propose that P is the Moore-Penrose inverse of T.
For we need to verify the four Moore-Penrose conditions.

(i) $SPS = R^{-1}TRR^{-1}T^\dagger RR^{-1}TR = R^{-1}TT^\dagger TR = R^{-1}TR = S.$

(ii) $PSP = R^{-1}T^\dagger RR^{-1}TRR^{-1}T^\dagger R = R^{-1}T^\dagger TT^\dagger R = R^{-1}T^\dagger R = P.$

(iii) $(PS)^* = (R^{-1}T^\dagger RR^{-1}TR)^* = (R^{-1}T^\dagger TR)^*$. By the property of adjoint and unitary nature of R this can be simplified as $R^*(T^\dagger T)^*(R^{-1})^* = R^{-1}(T^\dagger T)^*R$. Since $(T^\dagger T)^* = T^\dagger T$, we obtain $(PS)^* = R^{-1}T^\dagger TR = R^{-1}T^\dagger RR^{-1}TR = PS$

(iv) A similar calculation shows that $(SP)^* = SP$.

This proves that P is the Moore-Penrose inverse of S, say $S^\dagger$, and in turn by Theorem 2.2 of [15] that S has a closed range. By (iv) of Theorem 3.6, to demonstrate that S is hypo-EP, it suffices to show that $S^\dagger S^2 S^\dagger = SS^\dagger$ $S^\dagger S^2 S^\dagger = R^{-1}T^\dagger RR^{-1}TRR^{-1}TRR^{-1}T^\dagger R = R^{-1}T^\dagger T^2 T^\dagger R = R^{-1}TT^\dagger R$, since T is hypo-EP. On the other hand $SS^\dagger = R^{-1}TRR^{-1}T^\dagger R = R^{-1}TT^\dagger R$. That is $S^\dagger S^2 S^\dagger = SS^\dagger$ and this concludes the proof. □

4 Powers of hypo-EP operators

In general, the composition of closed range operators need not have a closed range, nor do the natural powers of such operators necessarily retain this property. For instance, as demonstrated in [4], consider the operators A and B defined on the Hilbert space ℓ_2 of all square summable sequences of real numbers as

$$A(x_1, x_2, x_3, \cdots) = (x_1, 0, x_2, 0, x_3, 0, \cdots)$$

and

$$B(x_1, x_2, x_3, \cdots) = (\frac{x_1}{1} + x_2, \frac{x_3}{3} + x_4, \cdots)$$

Although both A and B have closed ranges, the range of BA is not closed in ℓ_2.
In contrast, we prove that if an operator on a Hilbert C^*- module is hypo-EP, then any natural power of it is of closed range and further hypo-EP. That is, if $T \in L(H)$ is hypo-EP, then T^n is hypo-EP for any natural number n. To demonstrate this, we first present an elementary characterization of hypo-EP operators on Hilbert C^*-modules in terms of the range and null spaces of their powers. This, along with the subsequent lemmas, paves the foundation for the main result stated as Theorem 4.4.

Lemma 4.1. *Let H be a Hilbert C^*-module and $T \in L(H)$ be a hypo-EP operator. Then, for any natural number n $N(T) = N(T^n)$.*

Proof. The inclusion $N(T) \subseteq N(T^n)$ is trivially true for any $n \in \mathbb{N}$. It remains to show that $N(T^n) \subseteq N(T)$ for any $n \in \mathbb{N}$. We proceed by mathematical induction on n.
For $n = 2$, let $x \in N(T^2)$. That is, $T^2x = 0$. Then $Tx \in N(T)$. Since T is hypo EP, it follows that $Tx \in N(T^*)$ and thus $T^*Tx = 0$.
Taking the inner product with x yields

$$\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = 0 \Rightarrow \|\langle Tx, Tx\rangle\| = 0 \Rightarrow \|Tx\| = 0$$

which implies $Tx = 0$ and therefore $x \in N(T)$. Hence, $N(T^2) \subseteq N(T)$. Assume that the result holds for some $n = k$. That is, $N(T^k) \subseteq N(T)$. We show that $N(T^{k+1}) \subseteq N(T)$.
let $x \in N(T^{k+1})$. Then $Tx \in N(T^k)$. By the induction hypothesis $Tx \in N(T) \subseteq N(T^*)$, and thus

$$\langle T^*Tx, x\rangle = 0 \Rightarrow \langle Tx, Tx\rangle = 0 \Rightarrow \|\langle Tx, Tx\rangle\| = 0 \Rightarrow \|Tx\| = 0 \Rightarrow Tx = 0.$$

Therefore, $x \in N(T)$ and hence $N(T^{k+1}) \subseteq N(T)$. By the principle of mathematical induction, $N(T^n) \subseteq N(T)$ for all $n \in \mathbb{N}$. Combining this with the trivial inclusion $N(T) \subseteq N(T^n)$, we obtain the desired equality $N(T) = N(T^n)$ for all $n \in \mathbb{N}$. □

Lemma 4.2. *(Lemma 2.3 of [9]) Let $T \in L(H)$. Then T has closed range if and only if $N(T)$ is orthogonally complemented in H and T is bounded below on $N(T)^\perp$. That is, $\|Tx\| \geq c\|x\|$ for all $x \in N(T)^\perp$*

Lemma 4.3. *Let $T \in L(H)$ be hypo EP, where H is a Hilbert C^* module. Then, $R(T^n)$ is closed for every n.*

Proof. To show that $R(T^n)$ is closed , it suffices to show that $N(T^n)^\perp$ is orthogonally complemented in H and T^n is bounded below on $N(T^n)^\perp$ by Lemma 4.2.
Claim 1:$N(T^n)^\perp$ **is orthogonally complemented**
Since T is a hypo-EP operator, by Lemma 4.1, we have $N(T^n) = N(T)$ for any natural number n and $N(T)$ is orthogonally complemented in H due to the closedness of $R(T)$ [8]. Hence, $N(T^n)$ is also orthogonally complemented.
Claim 2:$N(T)^\perp$ **is invariant under** T^m **for any** m**.**
Fo, let $x \in N(T)$. Since T is hypo–EP, it follows that $x \in N(T^*)$, and thus $T^*x = 0$. Therefore, $T^*(N(T)) \subseteq N(T)$, which implies

$$T(N(T)^\perp) \subseteq N(T)^\perp.$$

By induction, we conclude that

$$T^m(N(T)^\perp) \subseteq N(T)^\perp \quad \text{for all } m \in \mathbb{N}.$$

Now let $x \in N(T^n)^\perp$. By Lemma 4.1, we have $x \in N(T)^\perp$. The invariance established in claim 2 ensures that $T^m x \in N(T)^\perp$ for all m, and in particular $T^{n-1}x \in N(T)^\perp$.
Since $R(T)$ is closed, Lemma 3.4 guarantees the existence of a constant $c > 0$ such that

$$\|Ty\| \geq c\|y\| \quad \text{for all } y \in N(T)^\perp.$$

Applying this recursively, we obtain:

$$\begin{aligned}
\|T^n x\| &= \|T(T^{n-1}x)\| \geq c\|T^{n-1}x\|, \\
\|T^{n-1} x\| &= \|T(T^{n-2}x)\| \geq c\|T^{n-2}x\|, \\
&\vdots \\
\|T^2 x\| &\geq c\|Tx\| \geq c^2\|x\|, \\
\Rightarrow \|T^n x\| &\geq c^n\|x\|.
\end{aligned}$$

Thus, T^n is bounded below on $N(T^n)^\perp$. This completes the proof. □

Theorem 4.4. *If T is a hypo-EP operator on H, Then T^n is hypo EP for all n*

Proof. By Lemma 4.3, the operator T^n has closed range. To establish that T^n is hypo-EP, it therefore suffices to show that $N(T^n) \subseteq N((T^n)^*)$.
Let $x \in N(T^n)$. By Lemma 4.1, we have $x \in N(T)$. Since T is hypo-EP, it follows that $x \in N(T^*)$. Moreover, it is clear that $N(T^*) \subseteq N((T^*)^n)$ and hence $x \in N((T^*)^n)$. Noting that$N((T^n)^*) = N((T^*)^n)$, we conclude that $x \in N((T^n)^*)$. Therefore, $N(T^n) \subseteq N((T^n)^*)$ and since T^n has closed range, it follows that T^n is hypo-EP.

□

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Received: 2025-02-10
Accepted: 2025-09-09