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AN EXPLICIT FORMULA FOR GENERALIZED EULER POLYNOMIALS

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Abstract In this paper, we develop two results dealing with these objects. The first one proposes an identity for the generalized Euler polynomials. The second result allows the deduction of similar identities for classical Euler numbers, classical Euler polynomials, as well as for classical Genocchi polynomials and numbers.

1 Introduction

Recognizing the great importance of Euler numbers and Euler polynomials in various branches of mathematics, Roman [5] defined generalized Euler polynomials $E_n^{(\alpha)}(x)$ for every $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$, where $\mathbb{N}$ and $\mathbb{C}$ are the set of positive integers and the set of complex numbers, respectively.

$$\sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right)^{\alpha} e^{tx}. \tag{1.1}$$

We let Ψ_α denote the automorphism of the vector space $\mathbb{C}[x]$ defined for any $\alpha \in \mathbb{C}$ by

$$\Psi_\alpha(x^n) = E_n^{(\alpha)}(x),\ n \in \mathbb{N}. \tag{1.2}$$

Khaldi and Boumehdi [4] obtained some interesting expressions for classical Euler polynomials. This paper briefly reviews some properties of generalized Euler polynomials, as well as two related lemmas that help to prove the second result presented in the third section and the first one is an explicit formula for generalized Euler polynomials and the second result proposes an identity for the generalized Euler polynomials, which leads to further generalizations for several relations involving classical Euler numbers and Euler polynomials and also for classical Genocchi polynomials and numbers as we will see in the last section of this article.

2 Some properties of the generalized Euler polynomials and lemmas

The identity operator I, the finite difference operator Δ, and the operator Λ are the operators that need to be examined. These operators can be defined over any endomorphism of the vector space $\mathbb{C}[x]$. These operators are defined, in turn, by

$$I(x^n) = x^n,\ \Delta(x^n) = (x+1)^n - x^n, \text{ and } \Lambda(x^n) = (x+1)^n + x^n,\ n \in \mathbb{N}. \tag{2.1}$$

Along with

$$\Delta^k(x^n) = \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}(x+i)^n, \tag{2.2}$$

furthermore

$$\text{For } k > n,\ \Delta^k(x^n) = 0. \tag{2.3}$$

Generalized Euler polynomials form a sequence of Appell polynomials [2], satisfying the following well-known properties [5]:

$$E_0^{(\alpha)}(x) = 1, \tag{2.4}$$

$$D(E_n^{(\alpha)}(x)) = nE_{n-1}^{(\alpha)}(x),\ n \geq 1, \tag{2.5}$$

$$E_n^{(\alpha)}(x+y) = \sum_{k=0}^{n}\binom{n}{k} y^{n-k} E_n^{(\alpha)}(x), \tag{2.6}$$

$$\Lambda(E_n^{(\alpha)}(x)) = 2E_n^{(\alpha-1)}(x), \tag{2.7}$$

$$E_n^{(\alpha)}(\alpha - x) = (-1)^n E_n^{(\alpha)}(x), \tag{2.8}$$

$$\Delta(E_n^{(\alpha+1)}(x)) = 2E_{n-1}^{(\alpha)}(x) \text{ for } n \geq 1. \tag{2.9}$$

Lemma 2.1. *For every non-negative n and for all complex numbers α and γ, we have*

$$\Psi_\alpha((x+\gamma)^n) = E_n^{(\alpha)}(x+\gamma). \tag{2.10}$$

Lemma 2.2. *Given any $\alpha \in \mathbb{C}$, we get*

$$D \circ \Psi_\alpha = \Psi_\alpha \circ D, \tag{2.11}$$

$$\Psi_\alpha \circ \Lambda = 2\Psi_{\alpha-1}. \tag{2.12}$$

Proof of Equation (2.11). Given that $(E_n^{(\alpha)}(x))$ is an Appell polynomial in the sense that

$$\begin{aligned}(D \circ \Psi_\alpha)(x^n) &= D(E_n^{(\alpha)}(x)) \\ &= nE_{n-1}^{(\alpha)}(x) \\ &= n\Psi_\alpha(x^{n-1}) \\ &= \Psi_\alpha(nx^{n-1}) \\ &= (\Psi_\alpha \circ D)(x^n).\end{aligned}$$

□

Proof of Equation (2.12). It is simple to demonstrate that

$$E_n^{(\alpha)}(x+1) + E_n^{(\alpha)}(x) = 2E_n^{(\alpha-1)}(x).$$

It follows that

$$\begin{aligned}(\Psi_\alpha \circ \Lambda)(x^n) &= \Psi_\alpha((x+1)^n + x^n) \\ &= E_n^{(\alpha)}(x+1) + E_n^{(\alpha)}(x) \\ &= 2E_n^{(\alpha-1)}(x) \\ &= 2\Psi_{\alpha-1}(x^n).\end{aligned}$$

□

3 Main result

3.1 Explicit formula for generalized Euler polynomials

Theorem 3.1. *For every $\alpha \in \mathbb{C}$, we get*

$$E_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{2^k} \binom{k+\alpha-1}{k} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} (x+i)^n. \tag{3.1}$$

Proof. First, we give the following definition of the generalized Euler polynomials:

$$E_n^{(\alpha)}(x) = \left(\frac{2}{e^D+1}\right)^\alpha (x^n),$$

subsequently

$$\Delta = e^D - 1 = \sum_{k=1}^{\infty} \frac{D^k}{k!}.$$

Observe that

$$e^D + 1 = (e^D - 1) + 2 = \Delta + 2$$

So

$$\left(\frac{2}{e^D+1}\right)^\alpha = \left(\frac{2}{\Delta+2}\right)^\alpha = \left(\frac{\Delta+2}{2}\right)^{-\alpha}.$$

Therefore

$$\left(\frac{2}{e^D+1}\right)^\alpha (x^n) = \left(1+\frac{1}{2}\Delta\right)^{-\alpha} (x^n).$$

Conversely, though

$$(1+at)^r = \sum_{k=0}^{\infty} \binom{r}{k} a^k t^k, \quad a \in \mathbb{C}, r \in \mathbb{N}.$$

Consequently, we have for any formal series $(S(t))$

$$(1+S(t))^r = \sum_{k=0}^{r} \binom{r}{k} S^k(t)$$

Substituting $-\alpha$ for r and $\frac{1}{2}\Delta$ for $S(t)$, we obtain

$$\left(\frac{2}{e^D+1}\right)^\alpha (x^n) = \sum_{k=0}^{\infty} \binom{-\alpha}{k} \left(\frac{1}{2}\Delta\right)^k (x^n).$$

Notice that

$$\binom{-\alpha}{k} = \frac{-\alpha(-\alpha-1)\ldots(-\alpha-k+1)}{k!}$$
$$= (-1)^k \frac{\alpha(\alpha+1)\ldots(\alpha+k-1)}{k!}$$
$$= (-1)^k \binom{\alpha+k-1}{k}.$$

Consequently

$$\left(\frac{2}{e^D+1}\right)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} \frac{(-1)^k}{2^k} \Delta^k$$

Thus

$$E_n^{(\alpha)}(x) = \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} \frac{(-1)^k}{2^k} \Delta^k(x^n).$$

Equations (2.2) and (2.3) enable us to swiftly determine (3.1). □

Numerical application

For $n=2$,

$$E_2^{(\alpha)}(x) = \sum_{k=0}^{2} \frac{(-1)^k}{2^k} \binom{\alpha+k-1}{k} \Delta^k(x^2)$$
$$= \binom{\alpha-1}{0}\Delta^0(x^2) + (-1)\binom{\alpha}{1}\Delta^1(x^2) + \frac{1}{4}\binom{\alpha+1}{2}\Delta^2(x^2)$$
$$= x^2 - \alpha\left(-x^2+(x+1)^2\right) + \frac{1+\alpha}{2}\left(x^2-2(x+1)^2+(x+2)^2\right)$$
$$= x^2 - \alpha x - \frac{\alpha}{2} + \frac{\alpha(1+\alpha)}{4}$$
$$= x^2 - \alpha x + \frac{\alpha(\alpha-1)}{4}.$$

When we replace α by 1, we have $E_2(x) = x^2 - x$, where $E_2(x) = E_2^{(1)}(x)$ is the second classical Euler polynomial.

3.2 Another result for generalized Euler polynomials

Theorem 3.2. *For all complex numbers α, λ, and for all non-negative integers ℓ, n, r, and s, we have*

$$\sum_{k=0}^{n+r} \lambda^{n+r-k} \binom{n+r}{k} \binom{\ell+k+r}{r} E_{\ell+k}^{(\alpha)}(x)$$
$$+ (-1)^{\ell+n+r+s+1} \sum_{k=0}^{\ell+r} \lambda^{\ell+r-k} \binom{\ell+r}{k} \binom{n+k+r}{r} E_{n+k}^{(\alpha)}(\alpha-s-\lambda-x)$$
$$= 2\Psi_{\alpha-1}\left(\frac{D^r}{r!} \sum_{k=0}^{s-1} (-1)^k (x+k)^{\ell+r}(x+\lambda+k)^{n+r}\right). \tag{3.2}$$

Proof. Let us consider the polynomial $Q(x)$ defined by

$$Q(x) = \sum_{k=0}^{s-1} Q_k(x),$$

where

$$Q_k(x) = (-1)^k \frac{D^r}{r!}\left((x+k)^{\ell+r}(x+\lambda+k)^{n+r}\right).$$

In this context, it is important to note the equality $Q_k(x+1) = -Q_{k+1}(x)$ and thus

$$Q(x+1) + Q(x) = \sum_{k=0}^{s-1} Q_k(x) - \sum_{k=0}^{s-1} Q_{k+1}(x)$$
$$= \sum_{k=0}^{s-1} Q_k(x) - \sum_{k=1}^{s} Q_k(x)$$
$$= Q_0(x) - Q_s(x).$$

We have

$$\begin{aligned}Q_0(x) &= \frac{D^r}{r!}\left(x^{\ell+r}(x+\lambda)^{n+r}\right)\\ &= \frac{D^r}{r!}\left(\sum_{k=0}^{n+r}\lambda^{n+r-k}\binom{n+r}{k}x^{\ell+k+r}\right)\\ &= \sum_{k=0}^{n+r}\lambda^{n+r-k}\binom{n+r}{k}\binom{\ell+k+r}{r}x^{\ell+k}\end{aligned}$$

and

$$\begin{aligned}Q_s(x) &= (-1)^s\frac{D^r}{r!}\left(((x+\lambda+s)-\lambda)^{\ell+r}(x+\lambda+s)^{n+r}\right)\\ &= (-1)^s\frac{D^r}{r!}\left(\sum_{k=0}^{n+r}(-\lambda)^{\ell+r-k}\binom{\ell+r}{k}(x+\lambda+s)^{n+k+r}\right)\\ &= (-1)^{\ell+n+r+s}\sum_{k=0}^{n+r}\lambda^{\ell+r-k}\binom{\ell+r}{k}\binom{n+k+r}{r}(-1)^{n+k}(x+\lambda+s)^{n+k}\end{aligned}$$

Equations (2.8) and (2.10) yield

$$\Psi_\alpha(Q_0(x)) = \sum_{k=0}^{n+r}\lambda^{n+r-k}\binom{n+r}{k}\binom{\ell+k+r}{r}E^{(\alpha)}_{\ell+k}(x) \tag{3.3}$$

and

$$\begin{aligned}\Psi_\alpha(Q_s(x)) &= (-1)^{\ell+n+r+s}\sum_{k=0}^{\ell+r}\lambda^{\ell+r-k}\binom{\ell+r}{k}\binom{n+k+r}{r}(-1)^{n+k}E^{(\alpha)}_{n+k}(x+\lambda+s)\\ &= (-1)^{\ell+n+r+s}\sum_{k=0}^{\ell+r}\lambda^{\ell+r-k}\binom{\ell+r}{k}\binom{n+k+r}{r}E^{(\alpha)}_{n+k}(\alpha-\lambda-s-x).\end{aligned} \tag{3.4}$$

So

$$\begin{aligned}\Psi_\alpha(Q(x+1)+Q(x)) &= \Psi_\alpha(\Lambda(Q(x)))\\ &= \sum_{k=0}^{n+r}\lambda^{n+r-k}\binom{n+r}{k}\binom{\ell+k+r}{r}E^{(\alpha)}_{\ell+k}(x)\\ &\quad - (-1)^{\ell+n+r+s}\sum_{k=0}^{\ell+r}\lambda^{\ell+r-k}\binom{\ell+r}{k}\binom{n+k+r}{r}E^{(\alpha)}_{n+k}(\alpha-\lambda-s-x).\end{aligned}$$

Finally, by employing Equation (2.12) in Lemma 2.2, we have

$$\begin{aligned}\Psi_\alpha(\Lambda(Q(x))) &= 2\Psi_{\alpha-1}(Q(x))\\ &= 2\Psi_{\alpha-1}\Big(\sum_{k=0}^{s-1}(-1)^k\frac{D^r}{r!}((x+k)^{\ell+r}(x+\lambda+k)^{n+r}\Big)\\ &= 2\Psi_{\alpha-1}\Big(\frac{D^r}{r!}\sum_{k=0}^{s-1}(-1)^k(x+k)^{\ell+r}(x+\lambda+k)^{n+r}\Big).\end{aligned}$$

□

4 Applications

The following corollary arises when $s = 0$, according to Theorem 3.2:

Corollary 4.1. *For all complex number α, we have*

$$\begin{aligned}&\sum_{k=0}^{n+r}x^{n+r-k}\binom{n+r}{k}\binom{\ell+k+r}{r}E^{(\alpha)}_{\ell+k}(y)\\ &= (-1)^{\ell+n+r}\sum_{k=0}^{\ell+r}x^{\ell+r-k}\binom{\ell+r}{k}\binom{n+k+r}{r}E^{(\alpha)}_{n+k}(\alpha-x-y).\end{aligned} \tag{4.1}$$

Then, in particular, for $\alpha = 1$, $r = 0$, and $x = 1$, we get

$$(-1)^n\sum_{k=0}^{n+r}\binom{n}{k}E_{\ell+k}(y) = (-1)^\ell\sum_{k=0}^{\ell+r}\binom{\ell}{k}E_{n+k}(-y), \tag{4.2}$$

and for $\alpha = 1$, $r = 1$, and $x = 1$, we obtain

$$(-1)^n \sum_{k=0}^{n+1} \binom{n+1}{k} (\ell+k+1) E_{\ell+k}(y) + (-1)^\ell \sum_{k=0}^{\ell+1} \binom{\ell+1}{k} (n+k+1) E_{n+k}(-y) = 0. \tag{4.3}$$

Equation (4.3) can be expressed as follows:

$$\begin{aligned} &(-1)^n \sum_{k=0}^{n} \binom{n+1}{k} (\ell+k+1) E_{\ell+k}(y) \\ &+ (-1)^\ell \sum_{k=0}^{\ell} \binom{\ell+1}{k} (n+k+1) E_{n+k}(-y) \\ &= (-1)^{n+1} 2(n+\ell+1+2)(E_{n+\ell+1}(y) - y^{n+\ell+1}). \end{aligned} \tag{4.4}$$

In 2004, Wu, Sun, and Pan [8] discovered Equations (4.2) and (4.4). For $r = 0$ and $\alpha = 1$, Equation (4.1) rewritten as

$$(-1)^n \sum_{k=0}^{n} x^{n-k} \binom{n}{k} E_{\ell+k}(y) = (-1)^\ell \sum_{k=0}^{\ell} x^{\ell-k} \binom{\ell}{k} E_{n+k}(1-x-y). \tag{4.5}$$

In 2003, Sun [7, Theorem. 1.2, Equation. (iii)] obtained Equation (4.5).

Corollary 4.2. *For every non-negative integers k, q, m, and n, we have*

$$\begin{aligned} &(-1)^m \sum_{i=0}^{m+q} \alpha^{n+q-i} \binom{m+q}{i} \binom{n+q+i}{k} E^{(\alpha)}_{n+q+i-k}(x) \\ &+ (-1)^{n+k+1} \sum_{j=0}^{n+q} \alpha^{n+q-j} \binom{n+q}{j} \binom{m+q+j}{j} E^{(\alpha)}_{m+q+j-k}(-x) = 0. \end{aligned} \tag{4.6}$$

Proof. For $s = 0$ and $\lambda = \alpha$, Theorem 3.2 leads us to

$$\begin{aligned} &\sum_{k=0}^{n+r} \alpha^{n+r-k} \binom{n+r}{k} \binom{\ell+k+r}{r} E^{(\alpha)}_{\ell+k}(x) \\ &+ (-1)^{\ell+n+r+1} \sum_{k=0}^{\ell+r} \alpha^{\ell+r-k} \binom{\ell+r}{k} \binom{n+k+r}{r} E^{(\alpha)}_{n+k}(-x) = 0. \end{aligned}$$

Changing n to $m+q-k$, ℓ to $n+q-k$, and r to k, we obtain Equation (4.6). This completes the proof of Corollary 4.2. □

Su Hu and Min-Soo Kim [3, Theorem. 1.1, p. 3] proved an equation using Corollary 4.2 for $\alpha = 1$ and k odd,

$$\begin{aligned} &(-1)^m \sum_{i=0}^{m+q} \binom{m+q}{i} \binom{n+q+i}{k} E_{n+q+i-k}(x) \\ &+ (-1)^n \sum_{j=0}^{\ell+r} \binom{n+q}{j} \binom{m+q+j}{k} E_{m+q+j-k}(-x) = 0. \end{aligned} \tag{4.7}$$

Corollary 4.3. *For every non-negative integers k, r, n, and ℓ, we have*

$$\begin{aligned} &(-1)^\ell \sum_{k=0}^{n+r} \binom{n+r}{k} \binom{\ell+k+r}{r} 2^{n+r-1-k} E_{\ell+k} \\ &+ (-1)^{n+r} \sum_{k=0}^{\ell+r} \binom{\ell+r}{k} \binom{n+k+r}{r} 2^{\ell+r-1-k} E_{n+k} \\ &= \sum_{j=0}^{r} (-1)^j \binom{n+r}{j} \binom{\ell+r}{r-j}. \end{aligned} \tag{4.8}$$

We obtain exactly [1, Equation. (4.3)(ii), p. 210].
The following equality can be deduced when r is an even integer.

$$\sum_{k=0}^{n+r} \binom{n+r}{k} \binom{n+r+k}{r} 2^{n+r-k} E_{n+k} = (-1)^{n+\frac{r}{2}} \binom{n+r}{\frac{r}{2}}, \tag{4.9}$$

as shown in [1, Equation. (4.4)(ii), p. 210].

Exploiting the relation $G_n(x) = nE_{n-1}(x)$, for $n \geq 1$, where $G_n(x)$ is a classical Genocchi polynomial defined by

$$\sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \frac{2t}{e^t+1} e^{tx}. \tag{4.10}$$

The classical Genocchi numbers form the sequence of numbers $(G_n)_{n\geq 1}$, verify

$$G_n = G_n(0),\ n \in \mathbb{N}.$$

Therefore, they are integers and appear in the OEIS (*On-Line Encyclopedia of Integer Sequences*) [6] as A036968. Knowing that $E_n = 2^n E_n(\frac{1}{2})$ with $(E_n)_{n\in\mathbb{N}}$ is the sequence of Euler numbers appear in the OEIS as A000364. So, Corollary 4.3 implies the following corollary:

Corollary 4.4. *For every non-negative integers k, r, n, and ℓ, we have*

$$\begin{aligned}(-1)^{\ell}\sum_{k=0}^{n+r}\binom{n+r}{k}\binom{\ell+k+r}{r-1}2^{n+r-1-k}G_{\ell+k+1}\\ +(-1)^{n+r}\sum_{k=0}^{\ell+r}\binom{\ell+r}{k}\binom{n+k+r}{r-1}2^{\ell+r-1-k}G_{n+k+1}\\ = r\sum_{j=0}^{r}(-1)^j\binom{n+r}{j}\binom{\ell+r}{r-j}.\end{aligned} \tag{4.11}$$

By applying Equality (4.9), we obtain the following equality:

$$\sum_{k=0}^{n+r}\binom{n+r}{k}\binom{n+r+k}{r-1}2^{n+r-k}G_{n+k+1} = r(-1)^{n+\frac{r}{2}}\binom{n+r}{\frac{r}{2}}. \tag{4.12}$$

Alternatively, substituting q with 0 in Corollary 4.2 yields the following equality:

$$\sum_{k=0}^{n}\binom{n}{k}G_{\ell+k}(x) + (-1)^{\ell+n}\sum_{k=0}^{\ell}\binom{\ell}{k}G_{n+k}(-x) = 0. \tag{4.13}$$

5 Conclusion

This paper presents a crucial theorem on generalized Euler polynomials using composition operators, promising for analogous theorems for specific Appell polynomial sequences.

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