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SADMR-ITERATIVE PROCEDURE WITH AN APPLICATION

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Abstract *In this study, we initiate a novel SADMR-iteration to approximate fixed points of mappings in extended metric spaces. Building upon recent advancements in fixed point approximation techniques, we establish convergence results for mappings satisfying condition (B) [7] within the framework of the SADMR-iteration. Through rigorous theoretical calculations, we demonstrate the superior convergence speed and stability of the SADMR-iteration compared to existing methods, including the classical Picard iteration and the recently proposed JK-iteration and HR*-iteration. Additionally, we provide empirical evidence of the efficacy of SADMR–iteration through numerical experiments in various settings. Our findings highlight the potential of SADMR-iteration as a versatile and effective tool for fixed point approximation in diverse mathematical and applied domains.*

1 Introduction

In the realm of iterative procedures for approximating fixed points, recent advancements have brought forth innovative methodologies with promising applications across various domains (see, [4],[5],[21],[25]-[27],[31],[32],[34]). Fixed point theory, a fundamental branch in mathematics, performs a crucial role in comprehending the behavior of iterative algorithms and finding solutions to a variety of issues in mathematics, engineering, and beyond. The Picard iteration [24], a classical method dating back to the late 19th century, stands as one of the foundational techniques for approximating fixed points. Named after the French mathematician Émile Picard, it forms the bedrock of many iterative algorithms used in numerical analysis and optimization. It involves a mathematical technique used to approximate solutions to problems by repeatedly applying a specific algorithm or operation. The basic idea is to start with an initial guess and successively refine it until convergence to the fixed point is achieved. Iterative methods are commonly used to approximate fixed points. Approximating fixed points is a common problem in mathematics, and various iterative methods have been developed to address this challenge. Although the Picard iteration exhibits simplicity and elegance, its convergence properties may vary depending on the characteristics of the underlying mapping function and the choice of initial guess.

Several authors developed distinct iterative schemes for approximating fixed points, for example, Abbas and Nazir [1], Agarwal et al. [2], Chugh et al. [12], Ishikawa [18], Mann [19], Noor [20], Sahu and Petrusel [28], Thakur et al. [33] and Ullah and Arshad [36]. Recently, Ahmad et al. [6] initiated the JK-iteration, which demonstrated notable improvements in convergence speed and efficiency compared to traditional methods. Ostrowski [22] established the notion of stability of fixed point iterative procedure in 1967 and demonstrated that with respect to contractions, Picard's iteration process is stable. Harder [17] extended Ostrowski's work in 1987 in his thesis, by addressing more broadly applicable iteration procedures and contractive conditions. Due to the numerical approximation of functions, when approximating fixed points, *instead of the theoretical sequence $\{x_n\}$, we assume an approximate sequence $\{t_n\}$. Building upon recent studies in this path, Hammad et al. [16] initiated a four-step iteration procedure, coined as the HR*-iteration, with the aim of further enhancing the convergence behavior and applicability of fixed point approximation techniques. The HR*-iteration, with its structured approach and refined steps, emerged as a noteworthy addition to the repertoire of fixed point approximation techniques. Its formulation incorporates insights from classical methods like the Picard iteration while leveraging modern algorithmic principles to achieve enhanced performance. As such, exploring the intricacies of HR*-iteration and its relationship with classical methods such as the Picard iteration holds immense significance in advancing the field of iterative procedures for fixed point approximation. Iterative methods are mostly used in optimization problems, where finding the fixed points of a particular function is essential. In numerical analysis, these methods are used to approximate solutions to equations arising in different scientific and engineering disciplines.*

In this study, we extend the HR-iteration to define SADMR-iteration, elucidate its underlying principles, analyze its convergence behavior, and assess its practical utility through theoretical investigations and empirical evaluations. We establish convergence results for mappings satisfying conditions (B) for the newly introduced SADMR-iteration process. Subsequently, we demonstrate that this new iteration not only converges more rapidly than all aforementioned methods but also exhibits stability. Furthermore, we showcase the versatility and efficacy of the SADMR-iteration through an application in solving integral equations, highlighting its potential to address real-world challenges and contribute to advancements in scientific and engineering disciplines. Through this exploration, we aim to pave the way for the adoption of SADMR-iteration and further refinement in the realm of fixed point approximation methodologies.*

2 Preliminaries

Throughout this work, U is a Banach space and C is a non-empty, closed and convex subset of U.

Definition 2.1. [14] U is considered uniformly convex if, for every ϵ in the interval $(0, 2]$, there is a $\delta > 0$ satisfying

$$\left.\begin{array}{l}\|u\| \leq 1,\\ \|v\| \leq 1,\\ \|u-v\| > \epsilon\end{array}\right\} \text{implies } \left\|\frac{u+v}{2}\right\| \leq \delta, \tag{2.1}$$

$u, v \in U$.

Definition 2.2. A self mapping T on C for $u \in C$ is
(i) a contraction [8] if $\alpha \in (0, 1)$ exists, satisfying

$$\|Tu - Tv\| \leq \alpha \|u - v\|, \ v \in C, \tag{2.2}$$

(ii) quasi non-expansive [13] if

$$\|Tu - v\| \leq \|u - v\|, \ v \in F(T), \tag{2.3}$$

(iii) satisfies condition (C) [31] if

$$\frac{1}{2}\|u - Tu\| \leq \|u - v\| \implies \|Tu - Tv\| \leq \|u - v\|,$$

$v \in C$.

Definition 2.3. [7] A self mapping T on U satisfies condition (B), if there exist $0 < \delta < 1$ and $L \geq 0$ satisfying

$$\|Tu - Tv\| \leq \delta \|u - v\| + L \min\{\|u - Tu\|, \|v - Tv\|, \|u - Tv\|, \|v - Tu\|\}, \tag{2.4}$$

$u, v \in U$.

Definition 2.4. [10] Consider the real sequences $\{u_n\}$ and $\{v_n\}$ converging to u and v, respectively. Suppose

$$\lim_{n\to\infty} \frac{\|u_n - u\|}{\|v_n - v\|} = s$$

Then,

(i) the sequence $\{u_n\}$ converges more rapidly to u than the sequence $\{v_n\}$ converges to v if $s = 0$,

(ii) $\{u_n\}$ and $\{v_n\}$ converge at the same rate, if $s \in (0, \infty)$.

Definition 2.5. [30] Consider a self mapping T on C with nonempty fixed point set F in C. Then, T is said to satisfy condition (I) if there is a nondecreasing function $h : [0, \infty) \to [0, \infty)$, with $h(\delta) > 0$ for all $\delta > 0$ and $h(0) = 0$, satisfying $\|q - Tq\| \geq h(d(q, F(T)))$, where $d(q, F(T)) = \inf_{q^* \in F(T)} \|q - q^*\|$, $q \in C$.

Definition 2.6. [17] Consider T to be a self mapping on a Banach space U. Consider $w_0 \in U$ and a sequence of points $w_n \in U$ described by the iteration $w_{n+1} = f(T, w_n)$, which converges to a fixed point p. Let $\{s_n\}$ and $\{\epsilon_n\}$ be sequences in U and $\mathbb{R}^+ = [0, \infty)$ respectively, such that by $\epsilon_n = \|s_{n+1} - f(T, s_n)\|$. Then the iteration process $w_{n+1} = f(T, w_n)$ is called stable in relation to T (T-stable) if $\lim_{n\to\infty} s_n = p$ if and only if $\lim_{n\to\infty} \epsilon_n = 0$.

Lemma 2.7. *[9] Consider a sequence of numbers $\{\epsilon_n\}$ which are all positive and a real number λ, $0 \leq \lambda < 1$, so that*

$$\lim_{n\to\infty} \epsilon_n = 0.$$

Then, for any other sequence of numbers $\{w_n\}$ which are all positive

$$w_{n+1} \leq \lambda w_n + \epsilon_n,$$

we get $\lim_{n\to\infty} w_n = 0$.

Lemma 2.8. *[29] Let $\{w_n\}$ be any real sequence in a uniformly convex Banach space U satisfying $0 < a \leq w_n \leq b < 1$, $n \geq 1$. Consider $\{u_n\}$ and $\{v_n\}$ to be any two sequences in U for which $\limsup_{n\to\infty} \|w_n u_n + (1 - w_n)v_n\| = r$, $r \geq 0$, $\limsup_{n\to\infty} \|u_n\| \leq r$ and $\limsup_{n\to\infty} \|v_n\| \leq r$. Then, $\limsup_{n\to\infty} \|u_n - v_n\| = 0$.*

Definition 2.9. [35] Assume that $\{w_n\}$ is a bounded sequence in U. For any $p \in U$, we fix $r(p, \{w_n\}) = \limsup_{n\to\infty} \|w_n - p\|$.
(i) The asymptotic radius of $\{w_n\}$ with respect to C is: $r(C, \{w_n\}) = \inf\{r(p, \{w_n\}) : p \in C\}$.
(ii) The asymptotic center of $\{w_n\}$ with respect to C is:

$$A(C, \{w_n\}) = \{p \in C : r(p, \{w_n\}) = r(C, \{w_n\})\}. \tag{2.5}$$

It is easy to see that, $A(C, \{w_n\})$ contains precisely one point within a uniformly convex Banach space.

Ahmad et al. [6] presented the JK-iteration, taking arbitrary $s_0 \in C$, as

$$\left.\begin{aligned} z_n &= (1 - \alpha_n)s_n + \alpha_n T s_n \\ y_n &= T z_n \\ s_{n+1} &= T((1 - \beta_n)T z_n + \beta_n T y_n) \end{aligned}\right\} \tag{2.6}$$

Later, Hammad et al. [15] presented the HR-iteration by taking arbitrary $s_0 \in C$ and generating the sequence $\{s_n\}$ iteratively as

$$\left.\begin{aligned} z_n &= (1 - \alpha_n)s_n + \alpha T s_n \\ y_n &= T((1 - \beta_n)z_n + \beta_n T z_n) \\ x_n &= T((1 - \gamma_n)y_n + \gamma_n T y_n) \\ s_{n+1} &= T x_n \end{aligned}\right\} \tag{2.7}$$

For obtaining an approximation to fixed points of almost contraction mappings [11] and Reich–Suzuki-type nonexpansive mappings [23], Hammad et al. [16] presented a four-step HR-iteration by taking arbitrary* $s_0 \in C$ *and generating the sequence* $\{s_n\}$ *iteratively as*

$$\left.\begin{aligned} z_n &= (1-\alpha_n)s_n + \alpha T s_n \\ y_n &= T((1-\beta_n)z_n + \beta_n T z_n) \\ x_n &= T(T y_n) \\ s_{n+1} &= (1-\gamma_n)x_n + \gamma_n T x_n \end{aligned}\right\} \tag{2.8}$$

3 Main Results

Following the direction of Hammad et al. [16] and [15], we introduce a novel iteration process by taking arbitrary $w_0 \in C$ *and generating the sequence* $\{w_n\}$ *as*

$$\left.\begin{aligned} z_n &= (1-\alpha_n)w_n + \alpha_n T w_n \\ y_n &= T(T z_n) \\ x_n &= T((1-\beta_n)y_n + \beta_n T y_n) \\ w_{n+1} &= T x_n \end{aligned}\right\} \tag{3.1}$$

$n \geq 0$, *where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequences in the interval* $(0, 1)$. *We call it the SADMR-iteration.*

We establish the convergence results for mappings satisfying condition (B) *for the SADMR-iteration process* (3.1). *We also show that the SADMR-iteration scheme converges more rapidly than some recently developed iterative schemes and is stable.*

Theorem 3.1. *Consider a self mapping* T *on* C *satisfying condition* (B) *in a uniformly convex Banach space* U. *Let* $\{w_n\}$ *be a sequence generated by the SADMR-iteration* (3.1). *Then* $\{w_n\} \to w^*$, *where* w^* *is a unique fixed point of* T.

Proof. Consider $x^* \in Fix(T)$. Using condition (B) given in equation (2.4) and the first step of SADMR-iteration (3.1), we have

$$\begin{aligned} \|z_n - w^*\| &= \|(1-\alpha_n)w_n + \alpha_n T w_n - T w^*\| \\ &\leq (1-\alpha_n)\|w_n - w^*\| + \alpha_n \|T w_n - T w^*\| \\ &\leq (1-\alpha_n)\|w_n - w^*\| + \alpha_n[\delta\|w_n - w^*\| + \\ &\qquad\qquad L \min\{\|w_n - T w_n\|, \|w^* - T w^*\|, \|w_n - T w^*\|, \|w^* - T w_n\|\}] \\ &= (1-\alpha_n(1-\delta))\|w_n - w^*\|. \end{aligned} \tag{3.2}$$

From the second step of SADMR-iteration (3.1) and inequality (3.2), we have

$$\begin{aligned} \|y_n - w^*\| &= \|T(T z_n) - T w^*\| \\ &\leq \delta\|T z_n - w^*\| \\ &= \delta\|T z_n - T w^*\| \\ &\leq \delta[\delta\|z_n - w^*\| + L \min\{\|z_n - T z_n\|, \|w^* - T w^*\|, \|z_n - T w^*\|, \|w^* - T z_n\|\}] \\ &= \delta^2\|z_n - w^*\| \\ &\leq \delta^2(1-\alpha_n(1-\delta))\|w_n - w^*\| \end{aligned} \tag{3.3}$$

From the third step of SADMR-iteration (3.1) and inequality (3.3), we obtain

$$\begin{aligned}\|x_n - w^*\| &= \|T((1-\beta_n)y_n + \beta_n T y_n) - Tw^*\| \\ &\le \delta\|(1-\beta_n)y_n + \beta_n T y_n - w^*\| \\ &\le \delta[(1-\beta_n)\|y_n - w^*\| + \beta_n\|Ty_n - Tw^*\|] \\ &\le \delta[(1-\beta_n)\|y_n - w^*\| + \beta_n[\delta\|y_n - w^*\| + \\ &\quad L\min\{\|y_n - Ty_n\|, \|w^* - Tw^*\|, \|y_n - Tw^*\|, \|w^* - Ty_n\|\}]] \\ &\le \delta[(1-\beta_n)\|y_n - w^*\| + \delta\beta_n\|y_n - w^*\|] \\ &\le \delta(1-\beta_n(1-\delta))\|y_n - w^*\| \\ &= \delta^3(1-\alpha_n(1-\delta))(1-\beta_n(1-\delta))\|w_n - w^*\| \qquad (3.4)\end{aligned}$$

From the fourth step of SADMR-iteration (3.1) and inequality (3.4), we obtain

$$\begin{aligned}\|w_{n+1} - w^*\| &= \|Tx_n - Tw^*\| \\ &\le \delta\|x_n - w^*\| + L\min\{\|x_n - Tx_n\|, \|w^* - Tw^*\|, \|x_n - Tw^*\|, \|w^* - Tx_n\|\} \\ &\le \delta^4(1-\alpha_n(1-\delta))(1-\beta_n(1-\delta))\|w_n - w^*\|\end{aligned}$$

As $\delta \in (0,1)$ and $0 < \alpha, \beta < 1$, it follows that $(1-\alpha(1-\delta)) < 1$ and $(1-\beta(1-\delta)) < 1$, so

$$\begin{aligned}\|w_{n+1} - w^*\| &\le \delta^4\|w_n - w^*\| \\ &\le \delta^8\|w_{n-1} - w^*\| \\ &\vdots \qquad (3.5) \\ &\le \delta^{4(n+1)}\|w_0 - w^*\| \qquad (3.6) \\ &\to 0 \text{ as } n \to \infty\end{aligned}$$

Hence, $\{w_n\} \to w^*$. Now, let w_1 and w_2 be two fixed points of T. Consider

$$\begin{aligned}\|Tw_1 - Tw_2\| &\le \delta\|w_1 - w_2\| + L\min\{\|w_1 - Tw_1\|, \|w_2 - Tw_2\|, \|w_1 - Tw_2\|, \|w_2 - Tw_1\|\} \\ &= \delta\|w_1 - w_2\|\end{aligned}$$

This implies that $\|w_1 - w_2\| < \|w_1 - w_2\|$, which is a contradiction. Hence a unique fixed point exists. □

Theorem 3.2. *Consider a self mapping T on C satisfying condition (B) in a uniformly convex Banach space U. If $\{w_n\}$ is a sequence generated by the SADMR-iteration (3.1). Then sequence $\{w_n\}$ converges more rapidly than the sequence $\{s_n\}$, which is generated by HR*-iteration (2.8).*

Proof. Using inequality (3.6) of Theorem 3.1,

$$\|w_{n+1} - w^*\| \le \delta^{4(n+1)}\|w_0 - w^*\|.$$

Additionally, from the first step of HR*-iteration (2.8) we obtain

$$\begin{aligned}\|z_n - w^*\| &= \|(1-\alpha_n)s_n + \alpha_n Ts_n - Tw^*\| \\ &\le (1-\alpha_n)\|s_n - w^*\| + \alpha_n\|Ts_n - Tw^*\| \\ &\le (1-\alpha_n)\|s_n - w^*\| + \alpha_n[\delta\|s_n - w^*\| + \\ &\qquad L\min\{\|s_n - Ts_n\|, \|w^* - Tw^*\|, \|s_n - Tw^*\|, \|w^* - Ts_n\|\}] \\ &= (1-\alpha_n(1-\delta))\|s_n - w^*\|, \qquad (3.7)\end{aligned}$$

From the second step of HR*-iteration (2.8) and inequality (3.7), we obtain

$$\begin{aligned}\|y_n - w^*\| &= \|T((1-\beta_n)z_n + \beta_n T z_n) - Tw^*\| \\ &\leq \delta\|(1-\beta_n)z_n + \beta_n T z_n - w^*\| \\ &= \delta[(1-\beta_n)\|z_n - w^*\| + \beta_n\|Tz_n - Tw^*\|] \\ &\leq \delta[(1-\beta_n)\|z_n - w^*\| + \beta_n[\delta\|z_n - w^*\| + \\ &\qquad L\min\{\|z_n - Tz_n\|, \|w^* - Tw^*\|, \|z_n - Tw^*\|, \|w^* - Tz_n\|\}]] \\ &\leq \delta(1-\beta_n(1-\delta))\|z_n - w^*\| \\ &\leq \delta(1-\alpha_n(1-\delta))(1-\beta_n(1-\delta))\|s_n - w^*\| \end{aligned} \tag{3.8}$$

From the third step of HR*-iteration (2.8) and inequality (3.8), we obtain

$$\begin{aligned}\|x_n - w^*\| &= \|T(Ty_n) - Tw^*\| \\ &\leq \delta\|Ty_n - w^*\| \\ &= \delta\|Ty_n - Tw^*\| \\ &\leq \delta[\delta\|y_n - w^*\| + L\min\{\|y_n - Ty_n\|, \|w^* - Tw^*\|, \|y_n - Tw^*\|, \|w^* - Ty_n\|\}] \\ &= \delta^2\|y_n - w^*\| \\ &\leq \delta^3(1-\alpha_n(1-\delta))(1-\beta_n(1-\delta))\|s_n - w^*\| \end{aligned} \tag{3.9}$$

From the fourth step of HR*-iteration (2.8) and inequality (3.9), we obtain

$$\begin{aligned}\|s_{n+1} - w^*\| &= \|(1-\gamma_n)x_n + \gamma_n T x_n) - Tw^*\| \\ &\leq (1-\gamma_n)\|x_n - w^*\| + \gamma_n\|Tx_n - Tw^*\| \\ &\leq (1-\gamma_n)\|x_n - w^*\| + \gamma_n[\delta\|x_n - w^*\| + \\ &\qquad L\min\{\|z_n - Tz_n\|, \|w^* - Tw^*\|, \|z_n - Tw^*\|, \|w^* - Tz_n\|\}] \\ &= (1-\gamma_n)\|x_n - w^*\| + \delta\gamma_n\|x_n - w^*\| \\ &= (1-\gamma_n(1-\delta))\|x_n - w^*\| \\ &\leq \delta^3(1-\alpha_n(1-\delta))(1-\beta_n(1-\delta))(1-\gamma_n(1-\delta))\|s_n - w^*\| \end{aligned} \tag{3.10}$$

Using the principle of mathematical induction,

$$\|s_{n+1} - w^*\| \leq \delta^{3(n+1)}\|s_0 - w^*\| \tag{3.11}$$

On dividing inequality (3.6) by inequality (3.11), we have

$$\frac{\|w_{n+1} - w^*\|}{\|s_{n+1} - w^*\|} \leq \frac{\delta^{4(n+1)}\|w_0 - w^*\|}{\delta^{3(n+1)}\|s_0 - w^*\|} = \delta^{n+1}\frac{\|w_0 - w^*\|}{\|s_0 - w^*\|} \to 0 \text{ as } n \to \infty,$$

which implies that $\{w_n\}$ converges more rapidly than $\{s_n\}$. □

Theorem 3.3. *Consider a self mapping T on C satisfying condition (B) in a uniformly convex Banach space U. Suppose $\{w_n\}$ is a sequence generated by the SADMR-iteration (3.1) and w^* is a fixed point of T. Then SADMR-iteration (3.1) is T-stable.*

Proof. Consider $\{q_n\} \subset C$ to be any sequence in C. Set $\epsilon_n = \|q_{n+1} - Tf_n\|$, where $f_n = T((1-t_n)g_n + t_n T g_n), g_n = T(Th_n), h_n = (1-s_n)q_n + s_n T q_n$. Assume that $\lim_{n\to\infty} \epsilon_n = 0$. From inequalities (2.4) and (3.1), we have

$$\begin{aligned}\|h_n - w^*\| &= \|(1-s_n)q_n + s_n T q_n - Tw^*\| \\ &\leq (1-s_n)\|q_n - w^*\| + s_n\|Tq_n - Tw^*\| \\ &\leq (1-s_n)\|q_n - w^*\| + \alpha_n[\delta\|q_n - w^*\| + \\ &\qquad L\min\{\|q_n - Tq_n\|, \|w^* - Tw^*\|, \|q_n - Tw^*\|, \|w^* - Tq_n\|\}] \\ &= (1-s_n(1-\delta))\|q_n - w^*\|, \end{aligned} \tag{3.12}$$

From inequalities (3.1) and (3.12), we obtain

$$\begin{aligned}\|g_n - w^*\| &= \|T(Th_n) - Tw^*\| \\ &\leq \delta\|Th_n - w^*\| \\ &= \delta\|Th_n - Tw^*\| \\ &\leq \delta[\delta\|h_n - w^*\| + L\min\{\|h_n - Th_n\|, \|w^* - Tw^*\|, \|h_n - Tw^*\|, \|w^* - Th_n\|\}] \\ &= \delta^2\|h_n - w^*\| \\ &\leq \delta^2(1 - s_n(1-\delta))\|q_n - w^*\|. \end{aligned} \tag{3.13}$$

Then, by triangle inequality, (2.4) and (3.1), we get

$$\begin{aligned}\|q_{n+1} - w^*\| &= \|q_{n+1} - Tf_n\| + \|Tf_n - w^*\| \\ &= \epsilon_n + \|Tf_n - Tw^*\| \\ &\leq \epsilon_n + \delta\|f_n - w^*\| + L\min\{\|f_n - Tf_n\|, \|w^* - Tw^*\|, \|f_n - Tw^*\|, \|w^* - Tf_n\|\} \\ &\leq \epsilon_n + \delta\|T((1-t_n)g_n + t_nTg_n) - w^*\| \\ &\leq \epsilon_n + \delta^2\|(1-t_n)g_n + t_nTg_n - w^*\| \\ &= \epsilon_n + \delta^2(1-t_n)\|g_n - w^*\| + \delta^2 t_n\|Tg_n - Tw^*\|\| \\ &\leq \epsilon_n + \delta^2(1-t_n)\|g_n - w^*\| + \delta^3 t_n\|g_n - w^*\|\| \\ &\leq \epsilon_n + \delta^2(1 - t_n(1-\delta))\|g_n - w^*\| \\ &\leq \epsilon_n + \delta^4(1 - t_n(1-\delta))(1 - s_n(1-\delta))\|q_n - w^*\|. \end{aligned}$$

Since $0 < \delta < 1$ and $s_n, t_n \in (0,1)$, we obtain

$$\delta^4(1 - t_n(1-\delta))(1 - s_n(1-\delta)) < 1.$$

From Lemma 2.7, we obtain

$$\lim_{n\to\infty} \|q_n - w^*\| = 0,$$

that gives $\{q_n\} \to w^*$. On the contrary, if $\lim_{n\to\infty} q_n = w^*$. Then

$$\begin{aligned}\epsilon_n &= \|q_{n+1} - Tf_n\| \\ &= \|q_{n+1} - w^* + w^* - Tf_n\| \\ &\leq \|q_{n+1} - w^*\| + \|Tf_n - Tw^*\| \\ &\leq \|q_{n+1} - w^*\| + \delta\|f_n - w^*\| + L\min\{\|f_n - Tf_n\|, \|w^* - Tw^*\|, \|f_n - Tw^*\|, \|w^* - Tf_n\|\} \\ &\leq \|q_{n+1} - w^*\| + \delta\|T((1-t_n)g_n + t_nTg_n) - w^*\| \\ &\leq \|q_{n+1} - w^*\| + \delta^2\|(1-t_n)g_n + t_nTg_n - w^*\| \\ &\leq \|q_{n+1} - w^*\| + \delta^2(1-t_n)\|g_n - w^*\| + \delta^2 t_n\|Tg_n - Tw^*\| \\ &\leq \|q_{n+1} - w^*\| + \delta^2(1 - t_n(1-\delta))\|g_n - w^*\| \\ &\leq \|q_{n+1} - w^*\| + \delta^4(1 - t_n(1-\delta))\|h_n - w^*\| \\ &\leq \|q_{n+1} - w^*\| + \delta^4(1 - t_n(1-\delta))(1 - s_n(1-\delta))\|q_n - w^*\|. \end{aligned} \tag{3.14}$$

Considering the limit as $n \to \infty$ in (3.14) gives $\lim_{n\to\infty} \epsilon_n = 0$. □

Lemma 3.4. *Assume that $T : C \to C$ is a given mapping. If $F(T) \neq \emptyset$ and T satisfies condition (B), then for arbitrary point $w^* \in F(T)$ and $v \in C$, $\|Tv - Tw^*\| \leq \|v - w^*\|$.*

Theorem 3.5. *Consider a self mapping T on C satisfying condition (B) in a uniformly convex Banach space U and $F(T) \neq \emptyset$. For $t_0 \in C$, the sequence $\{w_n\}$ is generated by the SADMR-iteration (3.1). Then $\lim_{n\to\infty} \|w_n - w^*\|$ exists, $w^* \in F(T)$.*

Proof. Suppose $w^* \in F(T)$. Using Lemma 3.4, we get

$$\begin{aligned}\|z_n - w^*\| &= \|(1-\alpha_n)w_n + \alpha_n T w_n) - w^*\| \\ &\le (1-\alpha_n)\|w_n - w^*\| + \alpha_n\|Tw_n - w^*\| \\ &\le (1-\alpha_n)\|w_n - w^*\| + \alpha_n\|w_n - w^*\| \\ &= \|w_n - w^*\|,\end{aligned} \tag{3.15}$$

$$\begin{aligned}\|y_n - w^*\| &= \|T(Tz_n) - w^*\| \\ &\le \|Tz_n - w^*\| \\ &\le \|z_n - w^*\| \le \|w_n - w^*\|,\end{aligned} \tag{3.16}$$

$$\begin{aligned}\|x_n - w^*\| &= \|T((1-\beta_n)y_n + \beta_n T y_n) - w^*\| \\ &\le \|(1-\beta_n)y_n + \beta_n T y_n - w^*\| \\ &\le (1-\beta_n)\|y_n - w^*\| + \beta_n\|Ty_n - w^*\| \\ &\le (1-\beta_n)\|y_n - w^*\| + \beta_n\|y_n - w^*\| \\ &\le (1-\beta_n)\|w_n - w^*\| + \beta_n\|w_n - w^*\| \\ &= \|w_n - w^*\|,\end{aligned} \tag{3.17}$$

$$\begin{aligned}\|w_{n+1} - w^*\| &= \|Tx_n - w^*\| \\ &\le \|x_n - w^*\| \\ &\le \|w_n - w^*\|.\end{aligned} \tag{3.18}$$

This proves that $\{\|w_n - w^*\|\}$ is a decreasing and bounded sequence. Thus $\lim_{n\to\infty}\|w_n - w^*\|$ exists, $w^* \in F(T)$. □

Lemma 3.6. *Consider T to be a self mapping on a non-empty subset C of a Banach space U. Suppose that T satisfies condition (B). Then*

$$\|u - Tv\| \le (1+L)\|Tu - u\| + \|u - v\|$$

holds for all $u, v \in C$.

Proof. Using condition (B), we get

$$\|Tu - Tv\| \le \delta\|u - v\| + L\|Tu - u\|, \tag{3.19}$$

$u, v \in C$. Consider

$$\begin{aligned}\|u - Tv\| &\le \|u - Tu\| + \|Tu - Tv\| \\ &\le \|u - Tu\| + \delta\|u - v\| + L\|Tu - u\| \\ &\le (1+L)\|Tu - u\| + \|u - v\|.\end{aligned}$$

□

Theorem 3.7. *Consider a self mapping T on C in a uniformly convex Banach space U, satisfying condition (B). Let $w_0 \in C$ be arbitrary and the sequence $\{w_n\}$ be generated by the SADMR-iteration (3.1). Then $\lim_{n\to\infty}\|Tw_n - w_n\| = 0$ if and only if $F(T) \neq \emptyset$.*

Proof. First, let us assume that $F(T) \neq \emptyset$ and $w^* \in F(T)$ be any element. Then utilizing Theorem 3.5, $\lim_{n\to\infty}\|w_n - w^*\|$ exists and (w_n) is bounded. Let

$$\lim_{n\to\infty}\|w_n - w^*\| = c. \tag{3.20}$$

From inequalities (3.15) and (3.20)

$$\limsup_{n\to\infty}\|z_n - w^*\| \le \limsup_{n\to\infty}\|w_n - w^*\| = c. \tag{3.21}$$

Using Lemma 3.4

$$\limsup_{n\to\infty}\|Tw_n - w^*\| \leq \limsup_{n\to\infty}\|w_n - w^*\| = c. \tag{3.22}$$

Also

$$\begin{aligned}\|w_{n+1} - w^*\| &= \|Tx_n - q\| \\ &\leq \|x_n - q\| \\ &\leq \|T((1-\beta_n)y_n + \beta_n Ty_n) - w^*\| \\ &\leq (1-\beta_n)\|y_n - w^*\| + \beta_n\|Ty_n - w^*\| \\ &\leq \|y_n - w^*\|.\end{aligned}$$

Using inequality (3.16), we obtain

$$\|w_{n+1} - w^*\| \leq \|z_n - w^*\|$$

Therefore,

$$c \leq \liminf_{n\to\infty}\|z_n - w^*\|. \tag{3.23}$$

Using inequalities (3.21) and (3.23), we get

$$\begin{aligned} c &= \lim_{n\to\infty}\|z_n - w^*\| \\ &= \lim_{n\to\infty}\|(1-\alpha_n)w_n + \alpha_n Tw_n - w^*\| \\ &= \lim_{n\to\infty}\|(1-\alpha_n)(w_n - w^*) + \alpha_n(Tw_n - w^*)\|. \end{aligned} \tag{3.24}$$

From inequalities (3.20), (3.22), (3.24) and Lemma 2.8, we obtain $\lim_{n\to\infty}\|Tw_n - w_n\| = 0$.

On the contrary, suppose that $\lim_{n\to\infty}\|Tw_n - w_n\| = 0$ and $\{w_n\}$ is bounded. Let $w^* \in A(C, \{w_n\})$. Utilizing Lemma 3.6,

$$\begin{aligned} r(\mathfrak{T}w^*, \{w_n\}) &= \limsup_{n\to\infty}\|w_n - Tw^*\| \\ &\leq \limsup_{n\to\infty}[(1+L)\|Tw_n - w_n\| + \|w_n - w^*\|] \\ &\leq \limsup_{n\to\infty}\|w_n - w^*\| \\ &= r(w^*, \{w_n\}), \end{aligned}$$

which implies that $Tw^* \in A(C, \{w_n\})$. As U is uniformly convex, so $A(C, \{w_n\})$ is a singleton set. Therefore, $Tw^* = w^*$, i.e., $F(T) \neq \emptyset$. □

Theorem 3.8. *Consider a self mapping T on C satisfying condition (B) with $F(T) \neq \emptyset$ in a uniformly convex Banach space U and $\{w_n\}$ to be defined by the iteration (3.1). Then $\lim_{n\to\infty}\inf d(w_n, F(T)) = 0$ if and only if the sequence $\{w_n\}$ converges to some point of $F(T)$.*

Proof. It is very clear that $\lim_{n\to\infty}\inf d(w_n, F(T)) = 0$ if $\{w_n\}$ converges to some fixed point of T since $d(w_n, F(T)) = \inf\|w_n - q\| : q \in F(T)$. Conversely, consider that $\lim_{n\to\infty}\inf d(w_n, F(T)) = 0$. Using Theorem 3.5, $\lim_{n\to\infty}\|w_n - q\|$ exists, $q \in F(T)$. Now, we have to demonstrate that $\{w_n\}$ is a Cauchy sequence. For any given $\epsilon > 0$, we have some $N \in \mathbb{N}$ so that for every $n > N$, $d(w_n, F(T)) < \frac{\epsilon}{2}$. To be specific $\inf\{\|w_N - q\| : q \in F(T)\} < \frac{\epsilon}{2}$.
Therefore, there exists some q^* which satisfies $\|w_N - q^*\| < \frac{\epsilon}{2}$. Furthermore, for any $m, n > N$,

$$\begin{aligned}\|w_{m+n} - w_n\| &\leq \|w_{m+n} - q^*\| + \|w_n - q^*\| \\ &\leq 2\|w_N - q^*\| \\ &< \epsilon.\end{aligned}$$

From here we conclude that the sequence is a Cauchy sequence in C. Using the closedness of C, an element w^* in C exists, satisfying $\lim_{n\to\infty} w_n = w^*$. Also $\lim_{n\to\infty}\inf d(w_n, F(T)) = 0$ leads to $\lim_{n\to\infty}\inf d(w^*, F(T)) = 0$, i.e., $w^* \in F(T)$. □

Theorem 3.9. *Consider a self mapping T on C, where C is a non-empty, convex and compact subset of a uniformly convex Banach space U, satisfying condition (B). Consider $\{w_n\}$ to be a sequence defined by the SADMR-iteration (3.1). Then $\{w_n\}$ converges strongly to a fixed point of T.*

Proof. From Theorem 3.7, $\lim_{n\to\infty} \|Tw_n - w_n\| = 0$. Using the compactness of C, we obtain a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ converging strongly to w^* in C. From Lemma 3.6, we have

$$\|w_{n_i} - Tw^*\| \leq (1+L)\|Tw_{n_i} - w_{n_i}\| + \|w_{n_i} - w^*\|.$$

Taking limit $i \to \infty$, $Tw^* = w^*$, that is, $w^* \in F(T)$. Using Theorem 3.5, we obtain $\lim_{n\to\infty} \|w_n - w^*\|$ exists, $w^* \in F(T)$, that is, $\{w_n\}$ converges strongly to $w^* \in F(T)$. □

Theorem 3.10. *Consider a self mapping T on C in a uniformly convex Banach space U, satisfying condition (B). Let $F(T) \neq \emptyset$ and consider $\{w_n\}$ to be a sequence defined by the SADMR-iteration (3.1). If T satisfies condition (I), the sequence $\{w_n\}$ converges to a point of $F(T)$.*

Proof. From Theorem 3.7, $\lim_{n\to\infty} \|Tw_n - w_n\| = 0$. Using condition (I), we have

$$0 \leq \lim_{n\to\infty} f(d(w_n, F(T))) \leq \lim_{n\to\infty} \|Tw_n - w_n\| = 0.$$

Thus, $\lim_{n\to\infty} f(d(w_n, F(T))) = 0$. As $f : (0,\infty) \to (\infty)$ is a nondecreasing function with $f(0) = 0$ and for all $w > 0$, $f(w) > 0$, we get $\lim_{n\to\infty} d(w_n, F(T)) = 0$. As postulates of Theorem 3.8 have been satisfied, so we conclude that $\{w_n\}$ converges to $w^* \in F(T)$. □

4 Numerical Examples

First we define a mapping T that does not satisfies condition (C) but satisfies condition (B).

Example 4.1. Consider $(\mathbb{R}, \|.\|)$ to be a Banach space, endowed with a usual norm. Let $C = [3,7]$. Consider a mapping $T : C \to C$ as

$$Tw = \begin{cases} \frac{w+10}{3} & \text{if } w \in [3,7) \\ \frac{10}{3} & \text{if } w = 7. \end{cases}$$

Clearly, T does not satisfy condition (C) because if we consider $w = 6$ and $v = 7$.

$$\frac{1}{2}|w - Tw| = \frac{1}{2}|6 - T6| = \frac{1}{2}.\frac{2}{3} = \frac{2}{6} < 1 = |w - v|.$$

But

$$|Tw - Tv| = \left|\frac{16}{3} - \frac{10}{3}\right| = 2 > 1 = |w - v|.$$

Now we have subsequent cases:

(i) If $w, v < 7$

$$|Tw - Tv| = \left|\frac{w+10}{3} - \frac{v+10}{3}\right| = \frac{1}{3}|w - v| < \delta|w - v| + L\min\{|w - Tw|, |v - Tv|, |w - Tv|, |v - Tw|\},$$

for any $\frac{1}{3} < \delta < 1$ and any $L \geq 0$.

(ii) If $w < 7$ and $v = 7$

$$|Tw - Tv| = \left|\frac{w+10}{3} - \frac{10}{3}\right| = \left|\frac{w}{3}\right| < \delta|w - 7| + L\min\{|w - Tw|, \left|7 - \frac{10}{3}\right|, \left|w - \frac{10}{3}\right|, |7 - Tx|\},$$

Clearly, for all $w < 7$, we get some $0 < \delta < 1$ and some $L \geq 0$ such that the previous inequality is satisfied.

(iii) If $w = 7$ and $v < 7$, T satisfies condition (B) just like in Case (ii).

(iv) If $w = v = 7$, we have

$$|Tw - Tv| = \left|\frac{10}{3} - \frac{10}{3}\right| = 0 < L.\frac{11}{3} = \delta|7-7| + L\min\{\left|7 - \frac{10}{3}\right|, \left|7 - \frac{10}{3}\right|, \left|7 - \frac{10}{3}\right|, \left|7 - \frac{10}{3}\right|\},$$

for any $0 < \delta < 1$ and $L \geq 0$.

Hence, T satisfies condition (B) and admits a unique fixed point 5.

Now we verify numerically that the SADMR-iteration (3.1) *converges more quickly than each one of the HR***, HR, S****, JK and Thakur iteration processes. We have compared the results for three different initial guesses and different values of parameters* α_n, β_n *and* γ_n*. Figures 1, 3 and 5 are obtained corresponding to Tables 1, 2 and 3, respectively. The comparison of the execution time is shown in Figure 7.*

SADMR	HR*	S**	HR	JK	Thakur
6.0	6.0	6.0	6.0	6.0	6.0
5.003086419753	5.00462962963	5.006944444444	5.00462962963	5.027777777778	5.104166666667
5.000009525987	5.000021433471	5.000048225309	5.000021433471	5.000771604938	5.010850694444
5.000000029401	5.000000099229	5.000000334898	5.000000099229	5.000021433471	5.001130280671
5.000000000091	5.000000000459	5.000000002326	5.000000000459	5.000000595374	5.00011773757
5.0	5.000000000002	5.000000000016	5.000000000002	5.000000016538	5.00001226433
5.0	5.0	5.0	5.0	5.000000000459	5.000001277534
5.0	5.0	5.0	5.0	5.000000000013	5.000000133076
5.0	5.0	5.0	5.0	5.0	5.000000013862
5.0	5.0	5.0	5.0	5.0	5.000000001444
5.0	5.0	5.0	5.0	5.0	5.00000000015

Table 1. The rate of convergence with $w_0 = 6$ and $\alpha_n = \beta_n = \gamma_n = 3/4$.

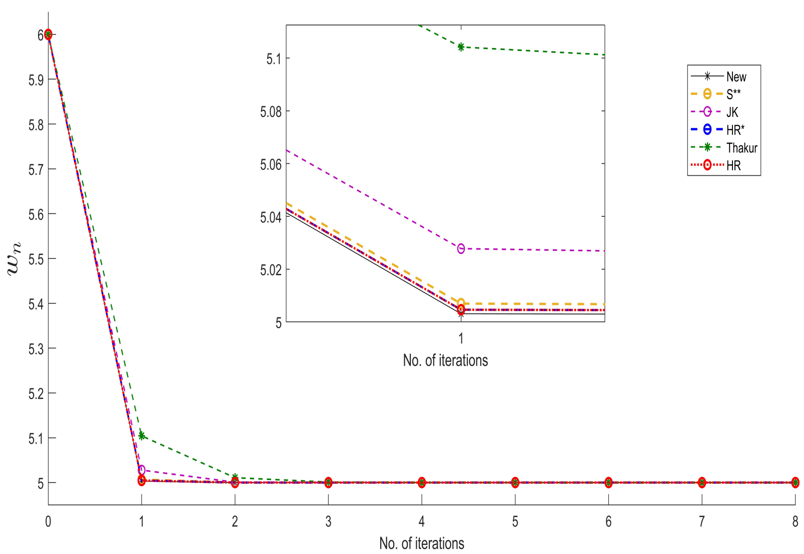

Figure 1. Graphical representation of convergence with $w_0 = 6$ and $\alpha_n = \beta_n = \gamma_n = 0.75$.

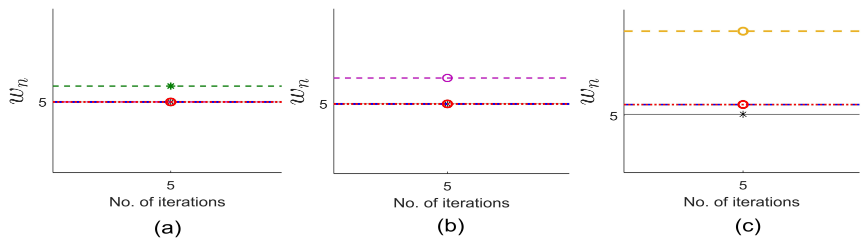

Figure 2. Zoomed Figure 1 at the fixed point.

SADMR	HR*	S**	HR	JK	Thakur
8.0	8.0	8.0	8.0	8.0	8.0
5.021769547325	5.050069958848	5.062989711934	5.050069958848	5.195925925926	5.704055555556
5.000157971064	5.000835666926	5.001322567937	5.000835666926	5.01279565615	5.165231408436
5.00000114632	5.00001394727	5.000027769391	5.00001394727	5.000835666926	5.038777363687
5.000000008318	5.00000023278	5.000000583062	5.00000023278	5.000054576272	5.009100472778
5.00000000006	5.000000003885	5.000000012242	5.000000003885	5.000003564302	5.002135746139
5.0	5.000000000065	5.000000000257	5.000000000065	5.00000023278	5.000501227978
5.0	5.000000000001	5.000000000005	5.000000000001	5.000000015203	5.000117630781
5.0	5.0	5.0	5.0	5.000000000993	5.000027606202
5.0	5.0	5.0	5.0	5.000000000065	5.000006478767
5.0	5.0	5.0	5.0	5.000000000004	5.000001520471

Table 2. The rate of convergence with $w_0 = 8$ and $\alpha_n = \beta_n = \gamma_n = 0.35$.

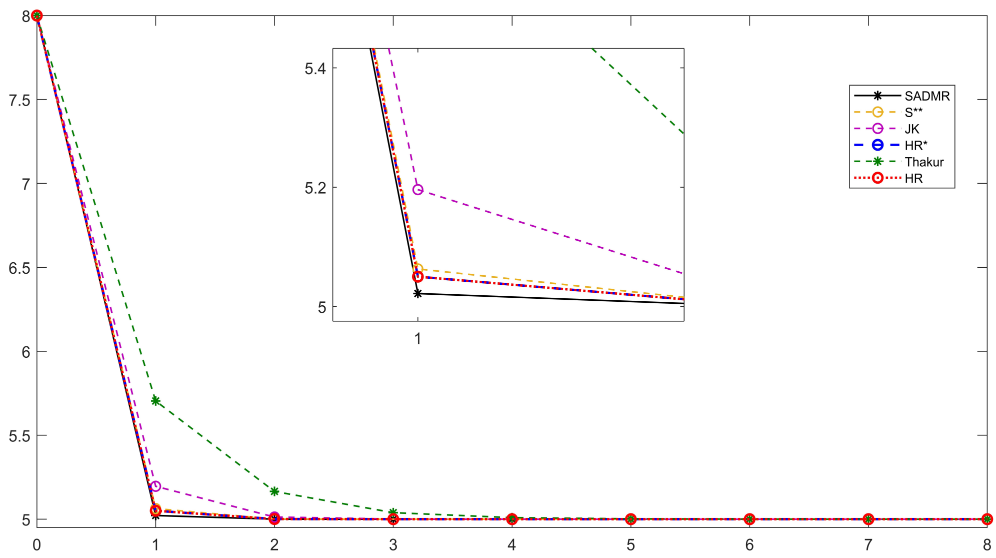

Figure 3. Graphical representation of convergence of different iterations with $w_0 = 8$ and $\alpha_n = \beta_n = \gamma_n = 0.35$.

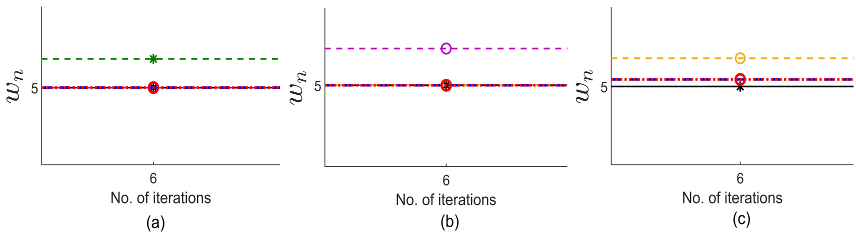

Figure 4. Zoomed Figure 3 at the fixed point.

SADMR	HR*	S**	HR	JK	Thakur
4.0	4.0	4.0	4.0	4.0	4.0
4.991426611797	4.978566529492	4.974708504801	4.978566529492	4.922839506173	4.733796296296
4.999926497015	4.999540606342	4.999360340271	4.999540606342	4.994046258192	4.929135588134
4.99999936983	4.9999901536	4.999983822049	4.9999901536	4.999540606342	4.981135631101
4.999999994597	4.999999788957	4.999999590835	4.999999788957	4.999964552958	4.994978235131
4.999999999954	4.999999995477	4.999999989652	4.999999995477	4.999997264889	4.998663187593
5.0	4.999999999903	4.999999999738	4.999999999903	4.999999788957	4.999644135586
5.0	4.999999999998	4.999999999993	4.999999999998	4.999999983716	4.999905267575
5.0	5.0	5.0	5.0	4.999999998744	4.999974781878
5.0	5.0	5.0	5.0	4.999999999903	4.999993286842
5.0	5.0	5.0	5.0	4.999999999993	4.999998212933

Table 3. The rate of convergence with $w_0 = 4$ and $\alpha_n = \beta_n = \gamma_n = 0.25$.

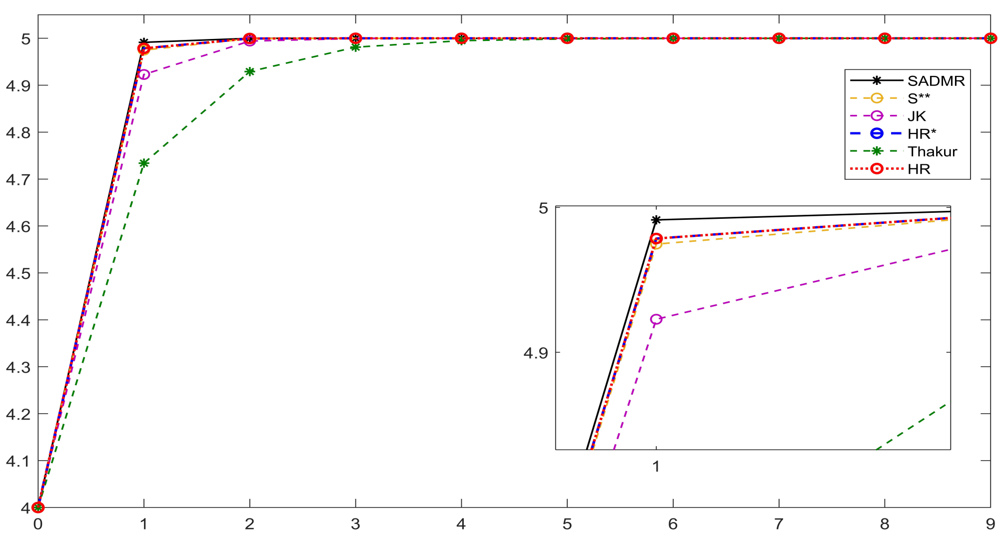

Figure 5. Graphical representation of convergence of different iterations with $w_0 = 4$ and $\alpha_n = \beta_n = \gamma_n = 0.25$.

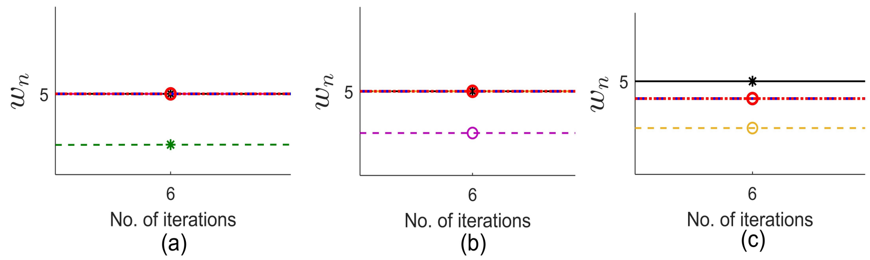

Figure 6. Zoomed Figure 5 at the fixed point.

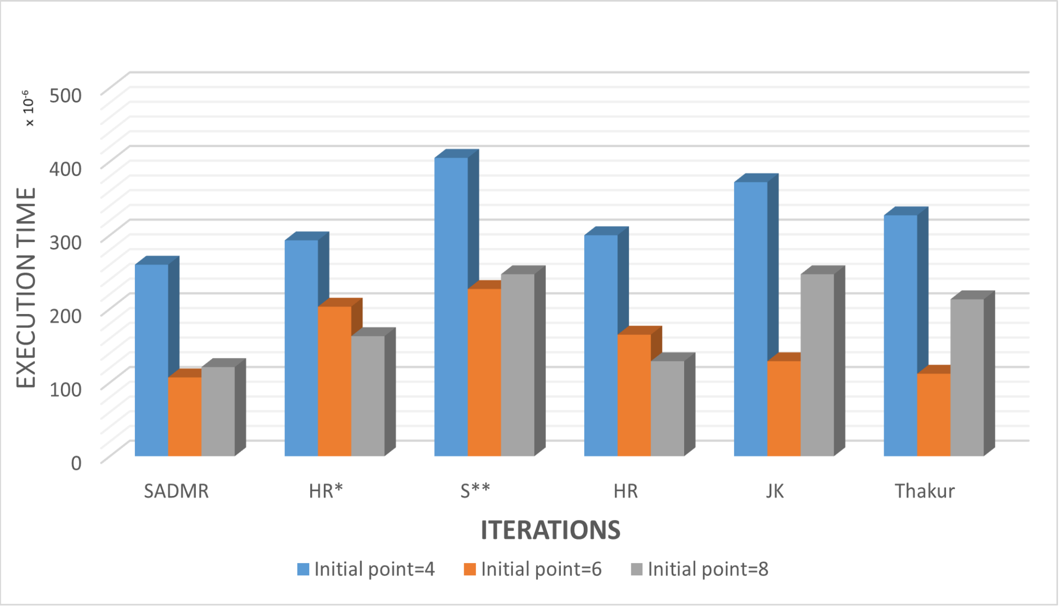

Figure 7. Time taken by the iterations to converge to the fixed point.

5 Application to Integral Equation

Let $F = C([0,a],\mathbb{R})$ be the set of all real-valued continuous functions with domain $[0,a]$. Consider the integral equation

$$u(t) = \int_0^a f(t,s)\phi(s,u(s))ds, \text{ where } t \in [0,a], \tag{5.1}$$

satisfying the following

(i) $\phi : [0,a] \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

(ii) For all $t \in [0,a]$, $f : [0,a] \times [0,a] \to \mathbb{R}^2$ *is continuous and measurable at* $s \in [0,a]$.

(iii) $f(t,s) \geq 0$, $t, s \in [0,a]$ *and* $\int_0^a f(t,s)ds \leq 1$.

Clearly, $F = C([0,a],\mathbb{R})$ *is a Banach space with the norm:* $\|u - v\|_\infty = \max_{t\in[0,a]} |u(t) - v(t)|$, *for all* $u, v \in C([0,a],\mathbb{R})$.

Theorem 5.1. *Assume that U is a non-empty, convex and closed subset of F and $T : U \to U$ is described as*

$$Tu(t) = \int_0^a f(t,s)\phi(s,u(s))ds, \text{ where } t \in [0,a],$$

Suppose that conditions 1-3 hold and there exists $L \geq 0$ *satisfying*

$$|\phi(t,u(t)) - \phi(t,v(t))| \leq L \min\left\{\left|u(t) - \int_0^a \phi(s,u(s))ds\right|, \left|v(t) - \int_0^a \phi(s,v(s))ds\right|,\right.$$
$$\left.\left|u(t) - \int_0^a \phi(s,v(s))ds\right|, \left|v(t) - \int_0^a \phi(s,u(s))ds\right|\right\},$$

$u, v \in C([0,a],\mathbb{R})$. *Then, the integral equation* (5.1) *has a unique solution in* $U \subseteq C([0,a],\mathbb{R})$ *provided that* T *admits a fixed point.*

Proof. Consider $\{w_n\}$ to be generated by the SADMR-iteration process (3.1). Let $(u,v) \in F \times F$, then

$$\begin{aligned}
\|Tu - Tv\| &= \max_{t\in[0,a]} |Tu(t) - Tv(t)| \\
&\leq \max_{t\in[0,a]} \left| \int_0^a f(t,s)\phi(s,x(s))ds - \int_0^a f(t,s)\phi(s,y(s))ds \right| \\
&\leq \max_{t\in[0,a]} \int_0^a f(t,s)|\phi(s,u(s))ds - \phi(s,v(s))|ds \\
&\leq \max_{t\in[0,a]} \int_0^a f(t,s)\left[L\min\left\{\left|u(t) - \int_0^a \phi(s,u(s))ds\right|, \left|v(t) - \int_0^a \phi(s,v(s))ds\right|,\right.\right. \\
&\qquad \left.\left.\left|u(t) - \int_0^a \phi(s,v(s))ds\right|, \left|v(t) - \int_0^a \phi(s,u(s))ds\right|\right\}\right] ds \\
&\leq \max_{t\in[0,a]} \int_0^a f(t,s)\left[L\min\left\{\left|u(t) - \int_0^a f(t,s)\phi(s,u(s))ds\right|,\right.\right. \\
&\qquad \left|v(t) - \int_0^a f(t,s)\phi(s,v(s))ds\right|, \left|u(t) - \int_0^a f(t,s)\phi(s,v(s))ds\right|, \\
&\qquad \left.\left.\left|v(t) - \int_0^a f(t,s)\phi(s,u(s))ds\right|\right\}\right] ds \\
&\leq \max_{t\in[0,a]} \int_0^a f(t,s)\left[L\min\{|u(t) - Tu(t)|, |v(t) - Tv(t)|, |u(t) - Tv(t)|,\right. \\
&\qquad \left. |v(t) - Tu(t)|\}\right] ds \\
&\leq L\min\left\{\max_{t\in[0,a]}|u(t) - Tu(t)|, \max_{t\in[0,a]}|v(t) - Tv(t)|, \max_{t\in[0,a]}|u(t) - Tv(t)|,\right. \\
&\qquad \left.\max_{t\in[0,a]}|v(t) - Tu(t)|\right\} \\
&\leq L\min\{\|u - Tu\|, \|v - Tv\|, \|u - Tv\|, \|v - Tu\|\} \\
&\leq \delta\|u - v\| + L\min\{\|u - Tu\|, \|v - Tv\|, \|u - Tv\|, \|v - Tu\|\}.
\end{aligned}$$

Therefore, T satisfies condition (B). Now by Theorem 3.1, as $\{w_n\}$ is generated by the SADMR-iterative procedure (3.1), then it converges to a unique fixed point of T, which is also a unique solution for the problem (5.1). □

6 Conclusion remarks

We have extended the HR-iteration [15] and HR-iteration [16] to define SADMR-iteration to approximate fixed points of mappings satisfying condition (B) and analyzed its convergence behavior to assess its practical utility through theoretical investigations and empirical evaluations. Further, we have demonstrated that this new iteration exhibits stability and converges more rapidly than the existing iteration schemes. Towards the end, we have utilized SADMR-iteration to solve integral equations to highlight its potential to address real-world challenges and contribute to advancements in scientific and engineering disciplines.*

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