Gradient Boundary Stability Analysis of Nonlinear Systems on a Nanolayer

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Abstract. This study investigates the gradient stabilization of nonlinear systems within a nanolayer boundary region by considering a nonlinear boundary control condition. We aim to demonstrate the stabilization of the nanolayer's boundary gradient using the Lyapunov function approach, under specific regularity assumptions and control conditions. Additionally, we extend the stability analysis to more complex systems by examining the limit problem with interface conditions through the epi-convergence approach. The theoretical results presented in this paper are subsequently validated through numerical testing.

1 Introduction

The search for efficient methods to stabilize gradients in nonlinear systems remains a major challenge in control theory, particularly in nanolayer regions where physical properties are profoundly influenced by dimensional reduction. While some systems exhibit inherent instabilities, as noted in example 9.1.1 [1], they can nonetheless achieve gradient stability in specific regions like B_ε. This highlights the nuanced nature of stabilization strategies required in such complex environments. These variations can lead to unpredictable and unstable behavior, making gradient stabilization all the more crucial. Consider, by analogy, the application of an anti-inflammatory ointment to treat localized inflammation. The ointment contains an active ingredient whose concentration should not deviate too far from its intended gradient, similar to how a system's state variables in control theory should not stray from the desired trajectory. Ensuring the stability of this gradient is crucial to maintain the ointment's effectiveness throughout the treatment duration. Any disruption in this gradient could reduce the effectiveness of the treatment or cause undesirable side effects. This example underscores the importance of gradient stabilization in ensuring therapeutic efficacy and highlights the need for precise formulations in topical pharmaceutical products.

This approach illustrates the importance of gradient stabilization in nonlinear systems and highlights the challenges and opportunities associated with managing the unique properties of nanolayers. Understanding and controlling these phenomena is essential for developing advanced nanotechnology, biology, and chemistry technologies. Boundary stability is an essential feature of physical and biological systems, including diffusion and response phenomena. In this case, researchers in applied mathematics and engineering rely heavily on Lyapunov functions to prove stability. However, Lyapunov invented the direct Lyapunov technique in the 19th century, which is not character-specific. It determines a nonlinear system's stability properties through an energy function. This approach establishes the system's stability using a Lyapunov function, which is positive and decreasing along the system's trajectories. [2] examines how the long-term behavior of a system, the spectral aspects of its dynamic evolution, and the presence of a Lya-

punov function are interconnected. As in [2, 3, 4, 5, 6], the techniques used are predicated on these relationships, the Lyapunov function, or the Riccati equation. While the Riccati equation is used in [3] and [2] to research the asymptotic and exponential stabilizability, respectively, [6, 7] investigates the exponential stabilizability using an appropriate decomposition of the state space. Many studies have been conducted on the stability of dynamical systems, emphasizing the application of Lyapunov functions.

However, research into boundary stability has mainly focused on macroscopic or mesoscopic boundaries, and has not addressed the case of gradient boundary stability of nanolayers. To fill this gap, we address the problem of non-linear evolution in a $\Omega \subset \mathbb{R}^3$ domain with a Lipschitzian $\partial\Omega$ boundary and a Σ_ε surface located on the boundary and characterised by nanometric properties (see Figure 1), the last-mentioned body is subjected to an external temperature f, and cooled at the boundary $\partial\Omega$, and given a function f bounded on $L^{p'}\left(0, \infty; (W^{1,p})'(\Omega)\right)$. The domains are defined as follows: A nanolayer B_ε, and a domain $\Omega_\varepsilon = \Omega \backslash B_\varepsilon$, where ε is a parameter tending to 0. Furthermore, a surface $\Sigma_\varepsilon = \{x \in \partial\Omega \mid |x_3| \leq \varepsilon^2\}$ is positioned at a distance of ε^2 from both the upper and lower edges of $\partial\Omega$. Consequently, $\Gamma_\varepsilon = \partial\Omega \backslash \Sigma_\varepsilon$ represents the remaining segment of the boundary. The set of equations can be expressed as follows:

$$\begin{cases} \dot{z} - \operatorname{div}\left(|\nabla z|^{p-2}\nabla z\right) = f & \text{in } \Omega^\infty \\ |\nabla z|^{p-2}\frac{\partial z(t,x)}{\partial n} = \frac{1}{\varepsilon^\alpha}U(t) & \text{on } \Sigma_\varepsilon^\infty \\ z = 0 & \text{on } \Gamma_\varepsilon^\infty \\ z(t=0,x) = z_0 & \text{on } \Omega. \end{cases}$$

With, $\Omega^\infty = [0, \infty[\times \Omega,\ \Sigma_\varepsilon^\infty = [0, \infty[\times \Sigma_\varepsilon,\ \Gamma_\varepsilon^\infty = [0, \infty[\times \Gamma_\varepsilon,\ B_\varepsilon^\infty =]0, \infty[\times B_\varepsilon.$

Figure 1. Domain Ω,
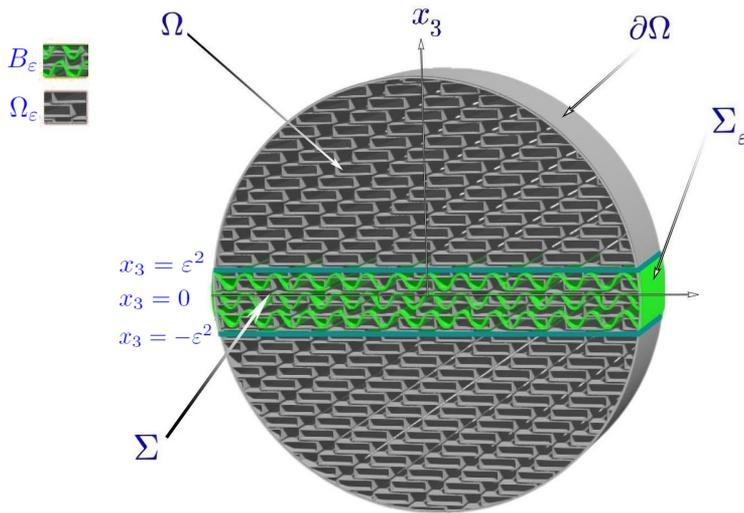
Given the control function $U(t)$, bounded in $L^2(0, \infty; L^2(\Sigma_\varepsilon))$, $3 \geq p > 1$, such as $\frac{1}{p} + \frac{1}{p'} = 1$. The nonlinear boundary control condition in diffusion-reaction systems with boundary gradient stability is now the main topic of this work. We show that nanolayer boundary stability may be achieved in these systems by applying Lyapunov functions. In addition, we extend our analysis of diffusion-reaction systems to more complex systems to further explore nanolayer boundary

stability. This study has demonstrated the utilization of Lyapunov functions to establish the stability of dynamical systems by analyzing the convergence of the system trajectories towards a stable equilibrium state. This study contributes significantly to our understanding of maintaining the stability of nanolayer edges in diffusion-reaction systems that have a nonlinear boundary control condition. In pursuit of addressing the limit problem highlighted in this article, our objective extends beyond conventional methods. We recognize that working with a region of nanostructure presents unique challenges during numerical resolution, particularly when creating an exceedingly fine mesh of the domain, which can lead to numerical explosions. To circumvent these issues, we aim to explore an alternative yet equivalent approximation model that enhances precision and accuracy within the finite element method framework. This innovative approach not only aligns with our goal to find a precise and accurate approximation model but also promises a more robust solution to the limit problem at hand.

The principal objective of this paper is to demonstrate the main result described below, which illustrates the limit behavior as specified in the forthcoming theorem:

We consider the energy operator.

$$F_\varepsilon\left(z_\varepsilon\right) = \frac{1}{p}\int_{\Omega^\infty}|\nabla z_\varepsilon|^p - \frac{1}{2\varepsilon^\alpha}\int_{\Sigma_\varepsilon^\infty}U(t)z_\varepsilon.$$

$$G\left(z_\varepsilon\right) = -\int_{\Omega^\infty}f(t)z_\varepsilon.$$

One denotes by τ_f the weak topology on $L^p(0,\infty;\mathbb{W}_\Sigma)$.

Theorem 1.1. According to the values of α, there exists a functional $F^{\alpha,\beta}$ defined on $L^p(0,\infty;\mathbb{W}_\Sigma)$ with a value in $\mathbb{R}\cup\{+\infty\}$ such that $\tau_f - \lim_e F_\varepsilon = F^{\alpha,\beta}$ in $L^p(0,\infty;\mathbb{W}_\Sigma)$, where the functional $F^{\alpha,\beta}$ is given by;
  (1) If $\alpha > 2 + \beta$:

$$F^{\alpha,\beta}(z) = \frac{1}{p}\int_{]0,\infty[\times\Omega}|\nabla z|^p dx dt, \quad \forall z \in L^p(0,\infty;W^{1,p}\left(\Omega\right)).$$

  (2) If $\alpha \leq 2 + \beta$:

$$F^{\alpha,\beta}(z) = \frac{1}{p}\int_{\Omega^\infty}|\nabla z|^p dx dt + k\eta(\alpha,\beta)\oint_{\Sigma^\infty}u(t)z_{|\Sigma}d\sigma dt, \quad \forall z \in \mathbb{G} \subset L^p(0,\infty;W^{1,p}\left(\Omega\right)).$$

The document is organized as follows: In Section 2, we cover the foundational knowledge necessary for comprehending the subsequent sections. This part is crucial for establishing the context and preparing the groundwork for the subject under examination. Section 3 delves into the application of gradient stabilization to nonlinear systems within a nanolayer boundary region. This involves solving an approximate problem related to the initial issue using the Lyapunov method, which includes energy estimates or variational techniques. We also provide a demonstration of a priori estimates. Following the preliminary findings, definitions, and certain characteristics of the minimization problem, we progress to the next stage. To address the limit problem with interface conditions and gain a deeper understanding of the system's behavior near the nanolayer boundary, we consider the epi-convergence method. In Section 4, we validate the practicality and accuracy of the proposed technique through a numerical test that aligns with the theoretical results.

2 PRELIMINARIES

2.1 Notations

- Let us define the operator m^ε which transforms functions defined z on Σ_ε into functions defined on Σ by

$$m^\varepsilon z\left(t, x_1, x_2\right) = \frac{1}{2\varepsilon^2} \int_{-\varepsilon^2}^{\varepsilon^2} z\left(t, x_1, x_2, x_3\right) d\sigma.$$

- $d\sigma$: represents the surface measure on Σ_ε.

- $(t, x) = (t, x', x_3)$, where $x' = (x_1, x_2)$, $\nabla' = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$.

- $\eta(\alpha, \beta) = \lim_{\varepsilon \to 0} \varepsilon^{2+\beta-\alpha}$, with $2 + \beta \geq \alpha$.

- We define the operator :

$$\nabla_{B_\varepsilon} : H^1(B_\varepsilon) \longrightarrow \left(L^2(B_\varepsilon)\right)^n$$
$$z \longrightarrow \left(\chi_{B_\varepsilon} \frac{\partial z(x)}{\partial x_1}, \chi_{B_\varepsilon} \frac{\partial z(x)}{\partial x_2}, \cdots, \chi_{B_\varepsilon} \frac{\partial z(x)}{\partial x_n}\right)$$

In the following, C will denote any constant with respect to ε.

2.2 Functional setting.

$\mathbb{W}_{\varepsilon,0} = \left\{z \in W^{1,p}\left(\Omega\right) \mid z = 0 \,\text{on}\, \Gamma_\varepsilon\right\}$, sach that $W_0^{1,p}(\Omega) \subset \mathbb{W}_{\varepsilon,0} \subset W^{1,p}(\Omega)$.

Concerning the Hausdorff convergence of Σ_ε toward Σ. This is accomplished by defining a family of submanifolds $M_\varepsilon = \left\{x \in \partial\Omega \mid |x_3| \leq \varepsilon^2\right\}$, then calculating the Hausdorff distance between Σ_ε and Σ. Findings show that the distance to Σ for any x in Σ_ε is greater than ε^2, and the distance to Σ_ε for any y in Σ is equal to 0. For this reason, the Hausdorff distance is equal to ε^2. In conclusion, it is shown that while ε tends toward 0, so does the Hausdorff distance, indicating the convergence of Σ_ε toward Σ in the Hausdorff sense.

$$\mathbb{W}_\Sigma = \left\{z \in W^{1,p}\left(\Omega\right) \mid z = 0 \,\text{on}\, \partial\Omega \backslash \Sigma\right\}$$

$$\mathbb{G} = \begin{cases} z \in L^p(0, \infty; \mathbb{W}_\Sigma) : z_{|\Sigma} \in L^p(0, \infty; L^p(\Sigma)) & \text{if } \alpha \geq 2\beta - 2, \\ z \in L^p(0, \infty; \mathbb{W}_\Sigma) : z(t)|_\Sigma = C & \text{if } \alpha < 2\beta - 2. \end{cases}$$

$$\mathbb{D} = \begin{cases} \mathcal{D}(]0, \infty[\times\Omega) & \text{if } \alpha \geq 2\beta - 2, \\ \left\{z \in \mathcal{D}(]0, \infty[\times\Omega) : z(t)|_\Sigma = C\right\} & \text{if } \alpha < 2\beta - 2. \end{cases}$$

We know that $\overline{\mathbb{D}} = \mathbb{G}$.

2.3 Functional Framework

Remark 2.1. The regional stabilization problem can be seen as an output stabilization problem with a partial observation: $y = \nabla_{B_\varepsilon} z$.

We'll put out the epi-convergence notion of operator's sequence convergence;

Definition 2.1 ([8], Definition 1.9.). Let (X, τ) be a reflexive Banach space, $F_\varepsilon : X \to \mathbb{R} \cup +\infty$ a family of convex functionals, and $F : X \to \mathbb{R} \cup +\infty$ a convex functional. Suppose that
1. $\liminf_{\varepsilon \to 0} F_\varepsilon(x) \geq F(x)$ for all $x \in X$.
2. For any sequence $(x_\varepsilon) \subset X$ such that $x_\varepsilon \rightharpoonup x$ weakly in X, we have $\limsup_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon) \leq F(x)$. Then, we have $F_\varepsilon \xrightarrow{\tau-\text{epi}} F$.

3 Main Results

3.1 Stability study

We start with the following approximate problem:

$$
\begin{cases}
\dot{z}_\varepsilon - \operatorname{div}\left(|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\right) = f & \text{in } \Omega^\infty \\
|\nabla z_\varepsilon|^{p-2}\dfrac{\partial z_\varepsilon(t,x)}{\partial n} = \dfrac{1}{\varepsilon^\alpha}U(t) & \text{on } \Sigma_\varepsilon^\infty \\
z_\varepsilon = 0 & \text{on } \Gamma_\varepsilon^\infty \\
z_\varepsilon(t=0,x) = z_{0,\varepsilon} & \text{on } \Omega
\end{cases}
$$

To stabilize the gradient state on the boundary $\Sigma_\varepsilon^\infty$ using the Lyapunov method, we choose the Lyapunov function $V(\nabla z_\varepsilon, U)$ as follows:

$$
V(\nabla z_\varepsilon, U) = \frac{1}{p}\int_{B_\varepsilon^\infty} |\nabla z_\varepsilon|^p \, dx \, dt + \frac{1}{2}\int_{\Sigma_\varepsilon^\infty} U(t)^2 \, d\sigma \, dt
$$

Here are some key points regarding this Lyapunov function:

1. Since $|\nabla z_\varepsilon|^p \geq 0$ for all $z_\varepsilon \in L^p(0,\infty; W^{1,p}(\Omega))$ and $U(t)^2 \geq 0$ for all t, the Lyapunov function is non-negative:

$$
\frac{1}{p}\int_{B_\varepsilon^\infty} |\nabla z_\varepsilon|^p \, dx \, dt + \frac{1}{2}\int_{\Sigma_\varepsilon^\infty} U(t)^2 \, d\sigma \, dt \geq 0.
$$

2. If $\nabla z_\varepsilon = 0$ and $U(t) = 0$, then $V(\nabla z_\varepsilon, U) = 0$. Conversely, if $V(\nabla z_\varepsilon, U) = 0$, then $\nabla z_\varepsilon = 0$ almost everywhere in B_ε^∞ and $U(t) = 0$ almost everywhere on $\Sigma_\varepsilon^\infty$.

3. Let's calculate the time derivative of V:

$$
\dot{V} = \frac{d}{dt}\left(\frac{1}{p}\int_{B_\varepsilon^\infty} |\nabla z_\varepsilon|^p \, dx \, dt + \frac{1}{2}\int_{\Sigma_\varepsilon^\infty} U(t)^2 \, d\sigma \, dt\right)
$$

$$
= \int_{B_\varepsilon^\infty} |\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon \cdot \nabla \dot{z}_\varepsilon \, dx \, dt + \int_{\Sigma_\varepsilon^\infty} U(t)\dot{U}(t) \, d\sigma \, dt.
$$

Let's continue the derivation of $(\dot{V})$ by replacing $(\nabla \dot{z}_\varepsilon)$ in the previous expression:

$$
\dot{V} = \int_{B_\varepsilon^\infty} |\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon \nabla(\operatorname{div}\left(|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\right) + f)dxdt + \int_{\Sigma_\varepsilon^\infty} U(t)\dot{U}(t) \, d\sigma \, dt
$$

$$
= \int_{B_\varepsilon^\infty} |\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon \cdot \nabla(\operatorname{div}\left(|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\right))dxdt + \int_{B_\varepsilon^\infty} |\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon \cdot \nabla f dxdt + \int_{\Sigma_\varepsilon^\infty} U(t)\dot{U}(t) \, d\sigma \, dt.
$$

To compute the term

$$
\int_{B_\varepsilon^\infty} |\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon \cdot \nabla\left[\operatorname{div}(|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon)\right] dx \, dt,
$$

Let

$$
A := |\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon, \qquad \phi := \operatorname{div}(A).
$$

Then, for each fixed time t, the divergence theorem gives

$$
\int_{B_\varepsilon^\infty} A \cdot \nabla\phi \, dxdt = -\int_{B_\varepsilon^\infty} (\operatorname{div} A)\,\phi \, dxdt + \int_{\partial B_\varepsilon^\infty} (A \cdot n)\,\phi \, d\sigma dt.
$$

Equivalently,

$$
\int_{B_\varepsilon^\infty} A \cdot \nabla(\operatorname{div} A) \, dx \, dt = -\int_{B_\varepsilon^\infty} (\operatorname{div} A)^2 \, dx \, dt + \int_{\partial B_\varepsilon^\infty} (A \cdot n)\operatorname{div} A \, d\sigma \, dt.
$$

Boundary conditions.   Assuming on $\partial B_\varepsilon^\infty$ the Neumann-type condition

$$A \cdot n = |\nabla z_\varepsilon|^{p-2}\, \partial_n z_\varepsilon = \frac{1}{\varepsilon^\alpha}\, U(t),$$

the boundary integral becomes

$$\int_{\partial B_\varepsilon^\infty} (A \cdot n)\, \mathrm{div}\, A\, d\sigma\, dt = \frac{1}{\varepsilon^\alpha} \int_{\partial B_\varepsilon^\infty} U(t)\, \mathrm{div}(|\nabla z_\varepsilon|^{p-2}\, \nabla z_\varepsilon)\, d\sigma\, dt.$$

Conclusion.   Combining the above results, we obtain the final expression

$$\int_{B_\varepsilon^\infty} |\nabla z_\varepsilon|^{p-2}\, \nabla z_\varepsilon \cdot \nabla\big[\mathrm{div}(|\nabla z_\varepsilon|^{p-2}\, \nabla z_\varepsilon)\big]\, dx\, dt = - \int_{B_\varepsilon^\infty} (\mathrm{div}\, A)^2\, dx\, dt + \frac{1}{\varepsilon^\alpha} \int_{\partial B_\varepsilon^\infty} U(t)\, \mathrm{div}(A)\, d\sigma\, dt.$$

The demonstration explains that to achieve a negative value for the integral $\int_{\Sigma_\varepsilon^\infty} U(t) \cdot \mathrm{div}(|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon) \cdot$ $\mathbf{n}\, d\sigma$, the control $U(t)$ must be chosen such that the dot product $U(t) \cdot \mathrm{div}(|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon)$ is negative on the boundary $\Sigma_\varepsilon^\infty$.

A proposition for $U(t)$ could be:

$$U(t) = -k\varepsilon^\beta u(t)$$

Here, $u(t) \in L^2(0, \infty)$, and $\beta \geq \alpha$ and k is a positive constant. Looking at the expression for $\dot{V}$:

$$\dot{V} = - \int_{B_\varepsilon^\infty} (\mathrm{div}\, A)^2\, dx\, dt + \frac{1}{\varepsilon^\alpha} \int_{\Sigma_\varepsilon^\infty} U(t)\, \mathrm{div}(A)\, d\sigma\, dt + \int_{B_\varepsilon^\infty} |\nabla z_\varepsilon|^{p-2}\, \nabla z_\varepsilon \cdot \nabla f\, dx\, dt + \frac{d}{dt} \int_{\Sigma_\varepsilon^\infty} U(t)^2\, d\sigma\, dt.$$

Since $U(t)$ is a bounded control operator in the space $L^2(0, \infty; L^2(\Sigma_\varepsilon))$, the integral $\frac{d}{dt}\int_{\Sigma_\varepsilon^\infty} U(t)^2 d\sigma dt$ is well-defined and finite.

When ε tends to 0, the sign of $\dot{V}$ corresponds to the dominant behavior of the integral term implying $|\mathrm{div}(|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon)|^2$ and the term contains ∇f. The other integral terms, which do not contain $U(t)$, are finite and their influence diminishes in comparison as ε decreases.

By appropriately selecting f, we ensure that $\dot{V}$ is negative, which leads to a decrease in the Lyapunov function V and guarantees the gradient boundary stability on Σ_ε of the system.

Exemple 3.1. To ensure the integral involving the gradient of z_ε is negative using the redefined function $f(x) = -\lambda z_\varepsilon(x)$, where λ is a positive constant, we analyze its influence on the integral.

The gradient of f is calculated as:

$$\nabla f(x) = -\lambda \nabla z_\varepsilon(x)$$

This indicates that ∇f is directly proportional and opposed to ∇z_ε.

Substituting ∇f into the integral, we get:

$$\int_{B_\varepsilon^\infty} |\nabla z_\varepsilon|^{p-2}\, \nabla z_\varepsilon \cdot (-\lambda \nabla z_\varepsilon)\, dx\, dt = -\lambda \int_{B_\varepsilon^\infty} |\nabla z_\varepsilon|^p\, dx\, dt$$

3.2   Asymptotic Behavior of the Solution

The space $W = W^{1,p}(\Omega)$ is a Banach space that is also reflexive. This space, equipped with the norm $|\cdot|_{W^{1,p}(\Omega)}$, is separable, thus possessing a countable basis $w_1, w_2, w_3, \ldots$, where each $w_i \in W$. For any finite subset $w_1, w_2, w_3, \ldots, w_n$, these elements form a linearly independent set, and the space $H = \mathrm{Span}\, w_1, w_2, w_3, \ldots$ is dense in W.

Consider the following approximate problem in the subspaces $W_m = \text{Span} w_1, w_2, w_3, \ldots, w_m$:

$$\text{We put } z_\varepsilon(t) = \sum_{i=1}^{m} h_{i\varepsilon}(t) w_i \in W_m.$$

$$\begin{cases} \dot{z}_\varepsilon - \text{div}\left(|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\right) = f & \text{in } \Omega^\infty \\ |\nabla z_\varepsilon|^{p-2}\frac{\partial z_\varepsilon(t,x)}{\partial n} = \frac{1}{\varepsilon^\alpha}U(t) & \text{on } \Sigma_\varepsilon^\infty \\ z_\varepsilon = 0 & \text{on } \Gamma_\varepsilon^\infty \\ z_\varepsilon(t=0,x) = z_{0,\varepsilon} & \text{on } \Omega \end{cases}$$

Existence of the solution

To address this issue, we focus on identifying a solution by minimizing the energy functional $J(z_\varepsilon)$ given by:

Let $J(z_\varepsilon)$ be the energy functional given by:

$$J(z_\varepsilon) = \frac{1}{p}\int_{\Omega^\infty}|\nabla z_\varepsilon|^p \, dx \, dt - \int_{\Omega^\infty} f z_\varepsilon \, dx \, dt - \frac{1}{2\varepsilon^\alpha}\int_{\Sigma_\varepsilon^\infty} U(t) z_\varepsilon \, d\sigma \, dt \qquad (3.1)$$

Our goal is to identify a solution that minimizes the energy functional $J(z_\varepsilon)$.

For a fixed $\varepsilon > 0$, we consider a sequence of finite dimensional approximations $\{z_{\varepsilon,k}\}_{k=1}^{\infty}$ obtained through a Galerkin method or similar approximation scheme.

- Convergence of $\{z_{\varepsilon,k}\}_{k=1}^{\infty}$

  The sequence $\{z_{\varepsilon,k}\}_{k=1}^{\infty}$ is sequentially bounded in the reflexive space $L^p(0,\infty;W^{1,p}(\Omega))$. Consequently, there exists a subsequence (still denoted $z_{\varepsilon,k}$) such that $z_{\varepsilon,k} \rightharpoonup z_\varepsilon$ weakly in $L^p(0,\infty;W^{1,p}(\Omega))$ as $k \to \infty$, where z_ε depends on the parameter ε.

- Boundedness of $J(z_{\varepsilon,k})$

  Since $\{z_{\varepsilon,k}\}_{k=1}^{\infty}$ is bounded in $L^p(0,\infty;W^{1,p}(\Omega))$, we have $\|z_{\varepsilon,k}\|_{L^p(0,\infty;W^{1,p}(\Omega))} \leq C$ for a constant $C > 0$ independent of k.

  Using the boundedness of f in $L^{p'}(0,\infty;(W^{1,p})'(\Omega))$, we can estimate the second term of $J(z_{\varepsilon,k})$ as follows:

  $$\left|\int_{\Omega^\infty} f z_{\varepsilon,k} \, dx \, dt\right| \leq \|f\|_{L^{p'}(0,\infty;(W^{1,p})'(\Omega))}\|z_{\varepsilon,k}\|_{L^p(0,\infty;W^{1,p}(\Omega))} \leq C$$

  For the third term:

  $$\frac{1}{\varepsilon^\alpha}\left|\int_{\Sigma_\varepsilon^\infty} U(t) z_{\varepsilon,k} \, d\sigma \, dt\right| \leq \varepsilon^{\beta-\alpha}C$$

  Consequently, we can bound $J(z_{\varepsilon,k})$ as follows:

  $$J(z_{\varepsilon,k}) \leq \frac{1}{p}\int_0^\infty \|z_{\varepsilon,k}\|_{W^{1,p}(\Omega)}^p + C + \varepsilon^{\beta-\alpha}C \leq +\infty \qquad (3.2)$$

  This shows that $J(z_{\varepsilon,k})$ is bounded.

- Calculation of the Gâteaux derivative $\nabla J(z_{\varepsilon,k})$

  To establish the variational formulation, we need to compute the Gâteaux derivative of J. Consider the directional derivative of J with respect to a perturbation v in the Hilbert space $L^2(\Omega^\infty)$:

  $$J'(z_{\varepsilon,k};v) = \lim_{h\to 0}\frac{J(z_{\varepsilon,k}+hv) - J(z_{\varepsilon,k})}{h} = \frac{d}{dh}J(z_{\varepsilon,k}+hv)\Big|_{h=0} \qquad (3.3)$$

  Let's calculate the derived terms one by one:

1. DERIVATIVE OF THE FIRST TERM:

$$\frac{d}{dh}\left(\frac{1}{p}\int_{\Omega^\infty}|\nabla(z_{\varepsilon,k}+hv)|^p\,dx\,dt\right) \tag{3.4}$$

Using the chain rule and the fact that $\frac{d}{dh}|\nabla(z_{\varepsilon,k}+hv)|^p = p|\nabla(z_{\varepsilon,k}+hv)|^{p-2}\nabla(z_{\varepsilon,k}+hv)\cdot\nabla v$:

$$= \lim_{h\to 0}\frac{1}{h}\left(ph\int_{\Omega^\infty}|\nabla z_{\varepsilon,k}|^{p-1}\cdot\nabla v\,dx\,dt + O(h^2)\right) \tag{3.5}$$

$$= p\int_{\Omega^\infty}|\nabla z_{\varepsilon,k}|^{p-2}\nabla z_{\varepsilon,k}\cdot\nabla v\,dx\,dt \tag{3.6}$$

2. DERIVATIVE OF THE SECOND TERM:

$$\frac{d}{dh}\left(-\int_{\Omega^\infty}f(z_{\varepsilon,k}+hv)\,dx\,dt\right) = -\int_{\Omega^\infty}fv\,dx\,dt \tag{3.7}$$

3. DERIVATIVE OF THE THIRD TERM: Let's start by simplifying the expression inside the integral: $U(t)(z_{\varepsilon,k}+hv)$.

$$\frac{d}{dh}\left(-\frac{1}{2\varepsilon^\alpha}\int_{\Sigma_\varepsilon^\infty}U(t)(z_{\varepsilon,k}+hv)\,d\sigma\,dt\right) = -\frac{1}{2\varepsilon^\alpha}\int_{\Sigma_\varepsilon^\infty}U(t)v\,d\sigma\,dt \tag{3.8}$$

Combining the derivatives: We can rewrite the Gâteaux derivative as:

$$\nabla J(z_{\varepsilon,k})\cdot v = p\int_{\Omega^\infty}|\nabla z_{\varepsilon,k}|^{p-2}\nabla z_{\varepsilon,k}\cdot\nabla v\,dx\,dt \quad -\int_{\Omega^\infty}fv\,dx\,dt - \frac{1}{2\varepsilon^\alpha}\int_{\Sigma_\varepsilon^\infty}U(t)v\,d\sigma\,dt$$

Since $\{z_{\varepsilon,k}\}$ are approximate solutions constructed via Galerkin method, they satisfy the approximate variational equation, which means that $\nabla J(z_{\varepsilon,k}) \to 0$ weakly as $k \to \infty$. Therefore:

$$\int_{\Omega^\infty}|\nabla z_{\varepsilon,k}|^{p-2}\nabla z_{\varepsilon,k}\cdot\nabla v\,dx\,dt - \int_{\Omega^\infty}fv\,dx\,dt - \frac{1}{2\varepsilon^\alpha}\int_{\Sigma_\varepsilon^\infty}U(t)v\,d\sigma\,dt \to 0 \tag{3.9}$$

As k approaches infinity, this is valid for all test functions v within $L^{p'}(0,\infty;(W^{1,p})'(\Omega))$. By the principle of weak convergence, when $k \to +\infty$, referring to the classical diagonalization lemma, there exists a function $k(\varepsilon) : \mathbb{R}^+ \to \mathbb{N}$ that increases to infinity as $\varepsilon \to 0$. Therefore, it can be concluded that:

$$\text{div}(|\nabla z_{\varepsilon,k}|^{p-2}\nabla z_{\varepsilon,k}) + f - \frac{1}{\varepsilon^\alpha}U(t) \to 0 \tag{3.10}$$

weakly in $L^{p'}(0,\infty;(W^{1,p})'(\Omega))$ as $k \to \infty$.

- The $\{z_{\varepsilon,k}\}$ is a minimizing sequence

Since the Gâteaux derivative $\nabla J(z_{\varepsilon,k}) \to 0$ as $k \to \infty$, and each $z_{\varepsilon,k}$ is constructed to approximate the critical point of J, the sequence $\{z_{\varepsilon,k}\}_{k=1}^\infty$ constitutes a minimizing sequence for $J(z_\varepsilon)$, meaning:

$$\lim_{k\to\infty}J(z_{\varepsilon,k}) = \inf_{z\in L^p(0,\infty;W^{1,p}(\Omega))}J(z) \tag{3.11}$$

- The weak lower semi-continuity of J

We need to show that for the minimizing sequence $\{z_{\varepsilon,k}\}_{k=1}^\infty$ converging weakly to z_ε in energy space, we have:

$$\liminf_{k\to\infty}J(z_{\varepsilon,k}) \geq J(z_\varepsilon) \tag{3.12}$$

We begin by considering the first term of the functional: $\frac{1}{p}\int_{\Omega^\infty}|\nabla z_{\varepsilon,k}|^p\,dx\,dt$.

Since $z_{\varepsilon,k}\rightharpoonup z_\varepsilon$ weakly, we have $\nabla z_{\varepsilon,k}\rightharpoonup\nabla z_\varepsilon$ weakly in $L^p(]0,\infty[\times\Omega)$. Using Fatou's lemma:

$$\frac{1}{p}\int_{\Omega^\infty}|\nabla z_\varepsilon|^p\,dx\,dt \le \liminf_{k\to\infty}\frac{1}{p}\int_{\Omega^\infty}|\nabla z_{\varepsilon,k}|^p\,dx\,dt \tag{3.13}$$

Next, let's analyze the second term of the functional: $-\int_{\Omega^\infty}fz_{\varepsilon,k}\,dx\,dt$.

As f is a bounded function, we can use the weak convergence of $z_{\varepsilon,k}$ to z_ε to obtain:

$$-\int_{\Omega^\infty}fz_\varepsilon\,dx\,dt = \lim_{k\to\infty}\left(-\int_{\Omega^\infty}fz_{\varepsilon,k}\,dx\,dt\right) \tag{3.14}$$

Finally, consider the third term of the functional: $-\frac{1}{2\varepsilon^\alpha}\int_{\Sigma_\varepsilon^\infty}U(t)\cdot z_{\varepsilon,k}\,d\sigma\,dt$.

Concerning the third term, it can be demonstrated, leveraging the proof of the epi-convergence theorem 1.1, that:

$$-\frac{1}{2\varepsilon^\alpha}\int_{\Sigma_\varepsilon^\infty}U(t)\cdot z_\varepsilon\,d\sigma\,dt = \lim_{k\to\infty}\left(-\frac{1}{2\varepsilon^\alpha}\int_{\Sigma_\varepsilon^\infty}U(t)\cdot z_{\varepsilon,k}\,d\sigma\,dt\right) \tag{3.15}$$

By combining these inequalities, we derive:

$$J(z_\varepsilon)\le\liminf_{k\to\infty}J(z_{\varepsilon,k}) \tag{3.16}$$

- Convergence of $J(z_{\varepsilon,k})$ towards its infimum

  It has been determined that $\{z_{\varepsilon,k}\}_{k=1}^\infty$ constitutes a minimizing sequence for $J(z_\varepsilon)$. The objective is to prove that $J(z_{\varepsilon,k})$ converges to $\inf_{z\in L^p(0,\infty;W^{1,p}(\Omega))}J(z)$ as $k\to\infty$.

  To establish this result, we utilize the property of weak lower semi-continuity of J, as previously demonstrated. Given that $\{z_{\varepsilon,k}\}_{k=1}^\infty$ is a minimizing sequence, we have:

  $$\inf_{z\in L^p(0,\infty;W^{1,p}(\Omega))}J(z) = \lim_{k\to\infty}J(z_{\varepsilon,k}) \tag{3.17}$$

  By the weak lower semi-continuity property:

  $$J(z_\varepsilon)\le\liminf_{k\to\infty}J(z_{\varepsilon,k}) = \lim_{k\to\infty}J(z_{\varepsilon,k}) = \inf_{z\in L^p(0,\infty;W^{1,p}(\Omega))}J(z) \tag{3.18}$$

  Since z_ε is in the admissible space, we also have:

  $$\inf_{z\in L^p(0,\infty;W^{1,p}(\Omega))}J(z)\le J(z_\varepsilon) \tag{3.19}$$

  Therefore:

  $$\lim_{k\to\infty}J(z_{\varepsilon,k}) = \inf_{z\in L^p(0,\infty;W^{1,p}(\Omega))}J(z) = J(z_\varepsilon) \tag{3.20}$$

  Therefore, we have demonstrated the convergence of $J(z_{\varepsilon,k})$ towards its infimum, which completes the proof. The results confirm the existence of a solution z_ε to the posed problem. The convergence of the minimizing sequence and functional suggests stability and minimum energy representation.

Lemma 3.1. The family $(z_\varepsilon)_{\varepsilon>0}$ satisfies:

$$\|z_\varepsilon\|_{L^p(0,\infty;W^{1,p}(\Omega))}\le C \tag{3.21}$$

Proof. Let's examine the approximate problem by multiplying the equations defined on Ω^∞ by $h_{i\varepsilon}(t)$ and summing them from $i = 1$ to m for a given k. This process results in the variational formulation of our problem;

$$< \dot{z}_\varepsilon, z_\varepsilon >_\Omega = < \operatorname{div}\left(|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\right), z_\varepsilon >_\Omega + < f(t), z_\varepsilon >_\Omega$$

$$< \dot{z}_\varepsilon, z_\varepsilon >_\Omega + < |\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon, \nabla z_\varepsilon >_\Omega - < \frac{1}{\varepsilon^\alpha}U(t), z_\varepsilon >_{\Sigma_\varepsilon} = < f(t), z_\varepsilon >_\Omega$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |z_\varepsilon|^2 + \int_\Omega |\nabla z_\varepsilon|^p = \frac{1}{\varepsilon^\alpha}\int_{\Sigma_\varepsilon} U(t)z_\varepsilon + \int_\Omega f(t)z_\varepsilon.$$

And by integrating from 0 to T;

$$\frac{1}{2}\|z_\varepsilon(T,x)\|^2_{L^2(\Omega)} - \frac{1}{2}\|z_{0,\varepsilon}\|^2_{L^2(\Omega)} + \int_0^T\int_\Omega |\nabla z_\varepsilon|^p \leq \frac{1}{\varepsilon^\alpha}\int_0^T\int_{\Sigma_\varepsilon} U(t)z_\varepsilon d\sigma dt + \frac{1}{p'}\int_0^T \|f\|^{p'}_{-1,p'} + \frac{1}{p}\int_0^T \|z_\varepsilon\|^p_{1,p}.$$

$$\frac{1}{2}\|z_\varepsilon(T,x)\|^2_{L^2(\Omega)} - \frac{1}{2}\|z_{0,\varepsilon}\|^2_{L^2(\Omega)} + \int_0^T\int_\Omega |\nabla z_\varepsilon|^p + \frac{k\varepsilon^\beta}{\varepsilon^\alpha}\int_0^T\int_{\Sigma_\varepsilon} u(t)z_\varepsilon d\sigma dt \leq \frac{1}{p'}\int_0^T \|f\|^{p'}_{-1,p'} + \frac{1}{p}\int_0^T \|z_\varepsilon\|^p_{1,p}.$$

$$\frac{1}{2}\|z_\varepsilon(T,x)\|^2_{L^2(\Omega)} - \frac{1}{2}\|z_{0,\varepsilon}\|^2_{L^2(\Omega)} + \int_0^T\int_\Omega |\nabla z_\varepsilon|^p + kC\varepsilon^{\beta-\alpha} \leq \frac{1}{p'}\int_0^T \|f\|^{p'}_{-1,p'} + \frac{1}{p}\int_0^T \|z_\varepsilon\|^p_{1,p}.$$

Then, let's reduce $-\frac{1}{2}\|z_\varepsilon(T,x)\|^2_{L^2(\Omega)}$ by 0,

$$(1 - \frac{1}{p})\int_0^T \|\nabla z_\varepsilon\|^p_{L^p(\Omega)} \leq C.$$

Such that $T \to \infty$, we get;

$$\int_{\Omega^\infty} |\nabla z_\varepsilon|^p \leq C.$$

It is clear that for a sufficiently small ε, the solution (z_ε) is bounded in $L^p(0, \infty; W^{1,p}(\Omega))$. Since $L^p(0, \infty; W^{1,p}(\Omega))$ is a reflexive space, then there exists a subsequence of $(z_\varepsilon)_{\varepsilon>0}$, always denoted by z_ε, such that $z_\varepsilon \rightharpoonup z^*$ in $L^p(0, \infty; W^{1,p}(\Omega))$.

$\square$

Remark 3.1.

$$\int_{\Sigma_\varepsilon^\infty} |\nabla z_\varepsilon|^p \, d\sigma dt = \int_{\Sigma_\varepsilon^\infty} |\nabla z_\varepsilon|^{(p-1)p'} \, d\sigma dt = \int_{\Sigma_\varepsilon^\infty} \left||\nabla z_\varepsilon|^{p-2}\frac{\partial z_\varepsilon(t,x)}{\partial n}\right|^{p'} \, d\sigma dt \qquad (3.22)$$

$$= \frac{1}{\varepsilon^\alpha}\int_{\Sigma_\varepsilon^\infty} |U(t)|^{p'} \, d\sigma dt \leq \frac{C}{\varepsilon^\alpha}\int_{\Sigma_\varepsilon^\infty} |U(t)|^2 \, d\sigma dt \leq \varepsilon^{2\beta-\alpha}C.$$

3.3 Limit behavior

We recall the energy operator of our problem

$$\inf_{z \in L^p(0,\infty;W^{1,p}(\Omega))} \left\{ \frac{1}{p}\int_{\Omega^\infty} |\nabla z|^p - \frac{1}{2\varepsilon^\alpha}\int_{\Sigma_\varepsilon^\infty} U(t)z - \int_{\Omega^\infty} f(t)z \right\} \qquad (\mathscr{P}_1)$$

In order to prove our theorem, we must establish the lemmas 3.2, 3.3 and the two propositions 3.1.

Lemma 3.2. The operator m^ε is linear and bounded of $L^p(0, \infty; L^p(\Sigma_\varepsilon))$(respectively $L^p(0, \infty; W^{1,p}(\Sigma_\varepsilon))$) in $L^p(0, \infty; L^p(\Sigma))$ (respectively $L^p(0, \infty; W^{1,p}(\Sigma))$), furthermore for all $z \in L^p(0, \infty; W^{1,p}(\Sigma_\varepsilon))$, we have

$$\left\| m^\varepsilon z - z_{|\Sigma} \right\|^p_{L^p(]0,\infty[\times\Sigma)} \leq C\varepsilon^{p-1} \int_0^\infty \int_{\Sigma_\varepsilon} |\nabla z|^p. \tag{3.23}$$

Proof. We have

$$\int_\Sigma |m^\varepsilon z|^p \, dx_1 dx_2 = \int_\Sigma \left(\frac{1}{2\varepsilon^2} \right)^p \left| \int_{-\varepsilon^2}^{\varepsilon^2} z \, dx_3 \right|^p \, dx_1 dx_2.$$

According to the Hölder inequality,

$$\int_\Sigma |m^\varepsilon z|^p \, dx_1 dx_2 \leq \int_\Sigma \frac{1}{2\varepsilon^2} \left(\int_{-\varepsilon^2}^{\varepsilon^2} |z|^p dx_3 \right) dx_1 dx_2 \leq \frac{1}{2\varepsilon^2} \int_\Sigma \left(\int_{-\varepsilon^2}^{\varepsilon^2} |z|^p dx_3 \right) dx_1 dx_2. \tag{3.24}$$

Since $z \in L^p(]0, \infty[\times\Sigma_\varepsilon)$ and (3.24), it follows that $m^\varepsilon z \in L^p(]0, \infty[\times\Sigma)$. Let $z \in \overline{\mathcal{D}}(]0, \infty[\times\Sigma_\varepsilon)$, we have

$$\frac{\partial}{\partial x_\lambda}(m^\varepsilon z)(t, x_1, x_2) = \frac{1}{2}\frac{\partial}{\partial x_\lambda}\left(\int_{-1}^{1} z(t, x_1, x_2, x_3\varepsilon^2) \, dx_3 \right)$$

$$= \frac{1}{2}\left(\int_{-1}^{1} \frac{\partial z}{\partial x_\lambda}(t, x_1, x_2, x_3\varepsilon^2) + \varepsilon^2 x_3 \frac{\partial z}{\partial x_3}(t, x_1, x_2, x_3\varepsilon^2) \, dx_3 \right)$$

$$= \frac{1}{2\varepsilon^2}\left(\int_{-\varepsilon^2}^{\varepsilon^2} \frac{\partial z}{\partial x_\lambda} + \varepsilon^2 x_3 \frac{\partial z}{\partial x_3} dx_3 \right).$$

So that,

$$\int_\Sigma \left| \frac{\partial}{\partial x_\lambda}(m^\varepsilon z) \right|^p = \int_\Sigma \left| \frac{1}{2\varepsilon^2}\left(\int_{-\varepsilon^2}^{\varepsilon^2} \frac{\partial z}{\partial x_\lambda} + \varepsilon^2 x_3 \frac{\partial z}{\partial x_3} dx_3 \right) \right|^p \leq \frac{1}{2\varepsilon^2} \int_\Sigma \left(\int_{-\varepsilon^2}^{\varepsilon^2} \left| \frac{\partial z}{\partial x_\lambda} + \varepsilon^2 x_3 \frac{\partial z}{\partial x_3} dx_3 \right|^p dx_3 \right).$$

However,

$$\int_\Sigma \left| \frac{\partial}{\partial x_\lambda}(m^\varepsilon z) \right|^p \leq \frac{C}{\varepsilon^2} \int_{\Sigma_\varepsilon} \left(\left| \frac{\partial z}{\partial x_\lambda} \right|^p + \left| \frac{\partial z}{\partial x_3} \right|^p \right) dx_3 \leq \frac{C}{\varepsilon^2} \int_{\Sigma_\varepsilon} |\nabla z|^p.$$

By density arguments, for any $z \in L^p(0, \infty; W^{1,p}(\Sigma_\varepsilon))$, we have

$$\int_0^\infty \int_\Sigma \left| \frac{\partial}{\partial x_\lambda}(m^\varepsilon z) \right|^p \leq \frac{C}{\varepsilon^2} \int_0^\infty \int_{\Sigma_\varepsilon} |\nabla z|^p.$$

Let $z \in \overline{\mathcal{D}}(]0, \infty[\times\Sigma_\varepsilon)$, so that

$$\left\| m^\varepsilon z - z_{|\Sigma} \right\|^p_{L^p(\Sigma)} = \int_\Sigma \left| \left(\frac{1}{2\varepsilon^2} \int_{-\varepsilon^2}^{\varepsilon^2} z(t, x_1, x_2, x_3) \, dx_3 \right) - z(t, x_1, x_2, 0) \right|^p dx_1 dx_2.$$

Using the Hölder inequality,

$$
\begin{aligned}
\left\| m^{\varepsilon} z - z_{|\Sigma} \right\|_{L^p(\Sigma)}^p &\le \frac{1}{2\varepsilon^2} \int_{\Sigma} \left(\int_{-\varepsilon^2}^{\varepsilon^2} |z(t, x_1, x_2, x_3) - z(t, x_1, x_2, 0)|^p \, dx_3 \right) dx_1 dx_2 \\
&\le \frac{C}{2\varepsilon^2} \int_{\Sigma} \left(\int_{-\varepsilon^2}^{\varepsilon^2} \left| \int_0^{x_3} \frac{\partial z}{\partial x_3}(t, x_1, x_2, w) \, dw \right|^p dx_3 \right) dx_1 dx_2 \\
&\le \frac{C}{2\varepsilon^2} \int_{\Sigma} \left(\int_{-\varepsilon^2}^{\varepsilon^2} |x_3|^{p-1} \left(\int_{-\varepsilon^2}^{\varepsilon^2} \left| \frac{\partial z}{\partial x_3}(t, x_1, x_2, w) \right|^p dw \right) dx_3 \right) dx_1 dx_2 \\
&\le C\varepsilon^{p-1} \int_{\Sigma} \left(\int_{-\varepsilon^2}^{\varepsilon^2} \left| \frac{\partial z}{\partial x_3} \right|^p dx_3 \right) dx_1 dx_2 \\
&\le C\varepsilon^{p-1} \int_{\Sigma_{\varepsilon}} |\nabla z|^p dx.
\end{aligned}
$$

By density arguments, we have for all $z \in L^p(0, \infty; W^{1,p}(\Sigma_{\varepsilon}))$

$$
\left\| m^{\varepsilon} z - z_{|\Sigma} \right\|_{L^p(]0,\infty[\times\Sigma)}^p \le C\varepsilon^{p-1} \int_0^{\infty} \int_{\Sigma_{\varepsilon}} |\nabla z|^p \, dx \, dt.
$$

Hence the result. $\square$

Lemma 3.3. Let $(z_{\varepsilon})_{\varepsilon > 0} \subset L^p(0, \infty; W^{1,p}(\Omega))$ which satisfy (3.21). Then

$$
\left\| \nabla'(m^{\varepsilon} z_{\varepsilon}) \right\|_{(L^p(]0,\infty[\times\Sigma))^2}^p \le C\varepsilon^{2\beta-\alpha-2}. \tag{3.25}
$$

In addition, $m^{\varepsilon} z_{\varepsilon}$ has a bounded subsequence in $L^p(]0,\infty[\times\Sigma)$.

Proof. By a consequence of lemma 3.2 we have

$$
\int_0^{\infty} \left\| \frac{\partial(m^{\varepsilon} z_{\varepsilon})}{\partial x_{\lambda}} \right\|_{L^p(\Sigma)^p}^p \le C\varepsilon^{-2} \int_0^{\infty} \int_{\Sigma_{\varepsilon}} |\nabla z_{\varepsilon}|^p \, dx.
$$

According to (3.21), we have

$$
\int_0^{\infty} \left\| \frac{\partial(m^{\varepsilon} z_{\varepsilon})}{\partial x_{\lambda}} \right\|_{L^p(\Sigma)^p}^p \le C\varepsilon^{2\beta-\alpha-2}.
$$

From the lemma 3.2, we get

$$
\left\| m^{\varepsilon} z - z_{|\Sigma} \right\|_{L^p(]0,\infty[\times\Sigma)}^p \le C\varepsilon^{p-1} \int_0^{\infty} \int_{\Sigma_{\varepsilon}} |\nabla z|^p \le C\varepsilon^{2\beta-\alpha+p-1}.
$$

z_{ε} is bounded in $L^p(0, \infty; W^{1,p}(\Omega))$, it follows that there exists $z^* \in L^p(0, \infty; W^{1,p}(\Omega))$ and a subsequence z_{ε}, always noted z_{ε}, such as $z_{\varepsilon} \rightharpoonup z^*$ in $L^p(0, \infty; W^{1,p}(\Omega))$, then $z_{\varepsilon|\Sigma}$ is a bounded sequence in $L^p(]0,\infty[\times\Sigma)$.
Since,

$$
\left\| m^{\varepsilon} z_{\varepsilon} \right\|_{L^p(]0,\infty[\times\Sigma)} \le \left\| m^{\varepsilon} z_{\varepsilon} - z_{\varepsilon|\Sigma} \right\|_{L^p(]0,\infty[\times\Sigma)} + \left\| z_{\varepsilon|\Sigma} \right\|_{L^p(]0,\infty[\times\Sigma)},
$$

then there exists C such that $\left\| m^{\varepsilon} z_{\varepsilon} \right\|_{L^p(]0,\infty[\times\Sigma)}^p \le C$.

                    $\square$

Proposition 3.1. $(z_{\varepsilon})_{\varepsilon}$, *has a weakly convergent sub-sequence to an element z^* in $L^p(0, \infty; W^{1,p}(\Omega))$ satisfactory, $z^*|_{\Sigma} \in L^p(0, \infty; L^p(\Sigma))$.*

Proof. The sequence z_ε is bounded in $L^p(0, \infty; W^{1,p}(\Omega))$, it follows that there is an element $z^* \in L^p(0, \infty; W^{1,p}(\Omega))$ and a sub-sequence of z_ε, always designated by z_ε such as $z_\varepsilon \rightharpoonup z^*$ in $L^p(0, \infty; W^{1,p}(\Omega))$. We have

$$\left\| m^\varepsilon z_\varepsilon - z_{\varepsilon|\Sigma} \right\|^p_{L^p(]0,\infty[\times\Sigma)} \le C\varepsilon^{2\beta - \alpha + p - 1} \text{ and } z_{\varepsilon|\Sigma} \rightharpoonup z^*_{|\Sigma} \text{ in } L^p(]0,\infty[\times\Sigma).$$

For $\alpha = 2\beta - 2$, According to the evaluation (3.25), the sequence $\nabla' m^\varepsilon z_\varepsilon$ has a sub-sequence, always denoted by $\nabla' m^\varepsilon z_\varepsilon$ weakly convergent to an element z_2 in $L^p(0, \infty; L^p(\Sigma))^2$, as $m^\varepsilon z_\varepsilon \rightharpoonup z^*_{|\Sigma}$ in $L^p(0, \infty; L^p(\Sigma))$ and that $\nabla' z^*_{|\Sigma} = z_2$. Hence $z^*_{|\Sigma} \in L^p(0, \infty; L^p(\Sigma))$.
For $2\beta - 2 > \alpha$, one shows, as in the case $\alpha = 2\beta - 2$ and taking $z_2 = 0$, that $z^*_{|\Sigma} = C$.
Hence the results.

$\square$

The prior results have been crucial in highlighting our core finding, as detailed in theorem 1.1.

3.4 Proof of theorem 1.1

Proof. First, we write the energy functional $F_\varepsilon(z_\varepsilon)$ associated with the problem as follows; let $z_\varepsilon \in L^p(0, \infty; \mathbb{W}_\Sigma)$, we have:

$$F_\varepsilon(z_\varepsilon) = \frac{1}{p} \int_{\Omega^\infty} |\nabla z_\varepsilon|^p - \frac{1}{2\varepsilon^\alpha} \int_{\Sigma_\varepsilon^\infty} U(t) z_\varepsilon.$$

Given $U(t) = -k\varepsilon^\beta u(t)$, We want to use the method of epi-convergence.
(A) We will determine the upper epi-limit:
From a density result, let $z \in \mathbb{G} \subset L^p(0, \infty; \mathbb{W}_\Sigma)$, there is a sequence (z_k) in $\mathbb{D}$ such that

$$z_k \to z \text{ in } \mathbb{G} \text{, as } k \to +\infty.$$

So that $z_k \to z$ in $L^p(0, \infty; \mathbb{W}_\Sigma)$.

Let θ be a smooth function verifying $\theta(x_3) = 1$ if $|x_3| \le 1$, $\theta(x_3) = 0$ if $|x_3| \ge 2$ and $|\theta'(x_3)| \le 2$, $\forall x \in \mathbb{R}$.
We define

$$\theta_\varepsilon(x) = \theta\left(\frac{x_3}{\varepsilon^2}\right).$$

And $z_{\varepsilon,k} = \theta_\varepsilon(x) z_{k|\Sigma} + (1 - \theta_\varepsilon(x)) z_k$.

It is easy to show that $z_{\varepsilon,k} \in L^p(0, \infty; \mathbb{W}_\Sigma)$ and $z_{\varepsilon,k} \to z_k$ in $\mathbb{G}$, when $\varepsilon \to 0$.

So that

$$F_\varepsilon(z_{\varepsilon,k}) = \frac{1}{p} \int_{]0,\infty[\times(|x_3|>2\varepsilon^2)} |\nabla z_{\varepsilon,k}|^p + \frac{1}{p} \int_{]0,\infty[\times(\varepsilon^2<|x_3|<2\varepsilon^2)} |\nabla z_{\varepsilon,k}|^p + \frac{k\varepsilon^\beta}{2\varepsilon^\alpha} \int_{\Sigma_\varepsilon^\infty} u(t) z_{\varepsilon,k} d\sigma dt$$

$$= \frac{1}{p} \int_{]0,\infty[\times(|x_3|>2\varepsilon^2)} |\nabla z_k|^p + \frac{1}{p} \int_{]0,\infty[\times(\varepsilon^2<|x_3|<2\varepsilon^2)} |\nabla z_{\varepsilon,k}|^p + k\varepsilon^{2-\alpha+\beta} \oint_{\Sigma^\infty} u(t) z_{k|\Sigma} d\sigma dt.$$

We can easily verify that

$$\lim_{\varepsilon \to 0} \left\{ \frac{1}{p} \int_{]0,\infty[\times(\varepsilon^2<|x_3|<2\varepsilon^2)} |\nabla z_{\varepsilon,k}|^p \right\} = 0.$$

By passage to the upper limit, one has

$$\limsup_{\varepsilon \to 0} F_\varepsilon(z_{\varepsilon,k}) \le \limsup_{\varepsilon \to 0} \left(\frac{1}{p} \int_{]0,\infty[\times(|x_3|>2\varepsilon^2)} |\nabla z_k|^p + k\varepsilon^{2-\alpha+\beta} \oint_{\Sigma^\infty} u(t) z_{k|\Sigma} d\sigma dt \right)$$

$$\le \frac{1}{p} \int_{\Omega^\infty} |\nabla z_k|^p dx dt + k\eta(\alpha, \beta) \oint_{\Sigma^\infty} u(t) z_{k|\Sigma} d\sigma dt.$$

Since $z_k \to z$ in $L^p(0, \infty; \mathbb{W}_\Sigma)$, if $k \to +\infty$. According to the classical result, the diagonalisation lemma [[8], Lemma 1.15], there exists a function $k(\varepsilon) : \mathbb{R}^+ \to \mathbb{N}$ which tends to $+\infty$ when $\varepsilon \to 0$, such as $z_{\varepsilon, k(\varepsilon)} \to z$ in $L^p(0, \infty; \mathbb{W}_\Sigma)$ when $\varepsilon \to 0$. As k approaches $+\infty$;

$$\limsup_{\varepsilon \to 0} F_\varepsilon(z_{\varepsilon, k(\varepsilon)}) \leq \lim_{k \to +\infty} \sup \limsup_{\varepsilon \to 0} F_\varepsilon\left(z_{\varepsilon, k(\varepsilon)}\right)$$

$$\leq \frac{1}{p} \int_{\Omega^\infty} |\nabla z|^p dx dt + k \eta(\alpha, \beta) \oint_{\Sigma^\infty} u(t) z_{|\Sigma} d\sigma dt.$$

(B) We will determine the lower epi-limit.
Let $z \in \mathbb{G}$ and (z_ε) be a sequence in $L^p(0, \infty; \mathbb{W}_\Sigma)$ such that $z_\varepsilon \rightharpoonup z$ in $L^p(0, \infty; \mathbb{W}_\Sigma)$ such that

$$\nabla z_\varepsilon \rightharpoonup \nabla z \quad \text{in } L^p(0, \infty; L^p(\Omega))^3. \tag{3.26}$$

Using Fatou's lemma and the fact that z_ε converges weakly to z in $L^p(0, \infty; \mathbb{W}_\Sigma)$, we get $\liminf_{\varepsilon \to 0} \frac{1}{p} \int_{\Omega^\infty} |\nabla z_\varepsilon|^p dx dt \geq \frac{1}{p} \int_{\Omega^\infty} |\nabla z|^p dx dt$.

If $\alpha \neq 2 + \beta$.

$$\liminf_{\varepsilon \to 0} F_\varepsilon(z_\varepsilon) \geq \frac{1}{p} \int_{\Omega^\infty} |\nabla z|^p dx dt.$$

If $\alpha = 2 + \beta$: If $\liminf_{\varepsilon \to 0} F_\varepsilon(z_\varepsilon) = +\infty$, there is nothing to prove, because

$$\frac{1}{p} \int_{\Omega^\infty} |\nabla z|^p dx dt + k \eta(\alpha, \beta) \oint_{\Sigma^\infty} u(t) z_{|\Sigma} d\sigma dt \leq +\infty.$$

Otherwise, $\liminf_{\varepsilon \to 0} F_\varepsilon(z_\varepsilon) < +\infty$, there is a subsequence of $F_\varepsilon(z_\varepsilon)$ still denoted by $F_\varepsilon(z_\varepsilon)$ and a constant $C > 0$, such as $F_\varepsilon(z_\varepsilon) \leq C$. which implies that $\frac{1}{p} \int_{\Omega^\infty} |\nabla z_\varepsilon|^p dx dt < C$.

Thus, z_ε satisfies the conditions of Lemma 3.3. According to this lemma, $\nabla' m^\varepsilon z_\varepsilon$ is bounded in $L^p(0, \infty; L^p(\Sigma))^2$. Consequently, there exists an element $z_1 \in L^p(0, \infty; L^p(\Sigma))^2$ and a subsequence of $\nabla' m^\varepsilon z_\varepsilon$, still denoted as $\nabla' m^\varepsilon z_\varepsilon$, which converges weakly to z_1 in $L^p(0, \infty; L^p(\Sigma))^2$. This weak convergence is supported by the fact that $z_{\varepsilon|\Sigma}$ converges weakly to $z_{|\Sigma}$ in $L^p(]0, \infty[\times\Sigma)$, as indicated by equations (3.23) and (3.22). Thus, $m^\varepsilon z_\varepsilon$ also converges weakly to $z_{|\Sigma}$ in $L^p(]0, \infty[\times\Sigma)$, and in $L^p(0, \infty; W_0^{1,p}(\Sigma))$.

$$F_\varepsilon(z_\varepsilon) = \frac{1}{p} \int_{]0,\infty[\times\Omega} |\nabla z_\varepsilon|^p + \frac{k\varepsilon^\beta}{2\varepsilon^\alpha} \int_{\Sigma_\varepsilon^\infty} u(t) z_\varepsilon d\sigma dt$$

$$\geq \frac{1}{p} \int_{\Omega^\infty} |\nabla z_\varepsilon|^p dx dt + k\varepsilon^{2-\alpha+\beta} \int_{\Sigma_\varepsilon^\infty} u(t) m^\varepsilon z_\varepsilon d\sigma dt.$$

Moving to the lower limit we obtain

$$\liminf_{\varepsilon \to 0} F_\varepsilon(z_\varepsilon) \geq \frac{1}{p} \int_{\Omega^\infty} |\nabla z|^p dx dt + k \eta(\alpha, \beta) \oint_{\Sigma^\infty} u(t) z_{|\Sigma} d\sigma dt.$$

Hence the result. □

In what follows, the focus is on limiting the problem determination partner to the original problem as ε nears zero. This is made possible by the results of epi-convergence (see [[8], Proposition p. 40]), and according to the τ_f-continuity of G in $L^p(0, \infty; \mathbb{W}_\Sigma)$, one has $F_\varepsilon + G$ τ_f-epi converges to $F^{\alpha,\beta} + G$ in $L^p(0, \infty; \mathbb{W}_\Sigma)$.

4 NUMERICAL TESTS

For sufficiently small ε values, the solution z_ε of the approximating problem converges towards the solution z of the limit problem. This section delves into the numerical analysis, focusing on the control's influence over the surface $\Sigma_\varepsilon^\infty$, defined as follows:

$$T = 10$$
$$\Omega = \{(x_1, x_2, x_3) \mid x_1 \in (-1, 1), x_2 \in (-1, 1), x_3 \in (-1, 1)\} \quad \left| \quad \begin{array}{l} \Sigma_\varepsilon = \{x \in \partial\Omega \mid |x_3| < \varepsilon^2\} \\ u(t) = \operatorname{div}(|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon) \end{array} \right.$$

Using Python, finite element methods, and Newton's method with parameters $p = 3$, $\alpha = 2$, and $\varepsilon = 1e - 7$, computations were executed on the HPC-MARWAN cluster. This platform was selected due to its exceptional computational capabilities, which facilitate complex and intensive simulations. The utilization of HPC-MARWAN not only reduces computational time but also enhances precision with resources that accommodate very stringent convergence criteria. Additionally, its scalability, specialized technical support, and robust security are crucial for generating reliable and pertinent results efficiently, particularly for large-scale, demanding projects.

The solution to the approximating problem converges towards the solution of the limit problem.

Initially, $U(t)$ does not stabilize the state across the entire domain Ω, which is expected since the control is defined solely on Σ_ε. Consequently, the control stabilizes the state only within Σ_ε.

Table 1. Numerical Tests for Stability - Σ_ε and Ω

t	$\|z_\varepsilon\|$	$\|z\|$	$\|\nabla z_\varepsilon\|$	$\|\nabla z\|$	$\|z_{\varepsilon\mid\Sigma}\|$	$\|z_{\mid\Sigma}\|$	$\|\nabla z_{\varepsilon\mid\Sigma}\|$	$\|\nabla z_{\mid\Sigma}\|$
t=0	0.0250367	0.0250367	21.459898	1.6561524	8.8388347e-05	8.8388347e-05	4.6878279e-14	2.9512632e-17
t= 3	0.001032	0.001032	8.099442e-13	0.004718	8.838834e-05	8.838834e-05	4.337786e-15	1.907434e-18
t= 6	0.010683	0.010687	3.525554	0.335823	8.838834e-05	8.838834e-05	8.331744e-15	5.829897e-18

TABLE 1 demonstrates that the approximating problem's solution converges to the limit problem's solution. It also indicates that $U(t)$ stabilizes both $\nabla z_{\varepsilon\mid\Sigma}$ and $z_{\varepsilon\mid\Sigma}$, as well as $\nabla z_{\mid\Sigma}$ and $z_{\mid\Sigma}$ within the nanolayer, validating the model's applicability for specialists in nanolayer control.

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