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THE EXISTENCE AND UNIQUENESS OF HASHIGUCHI CONNECTION IN KG-APPROACH

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Abstract. In this study, we treat intrinsic Finsler geometry using the Klein-Grifone approach (KG-approach). An existence and uniqueness theorem for the Hashiguchi connection on a Finsler manifold is investigated intrinsically (in coordinate-free fashion). Calculations are made for the Hashiguchi connection's torsion and curvature tensors. Some properties are examined, together with the Bianchi identities of the associated curvature and torsion tensors. An overview of the four fundamental linear connections in Finsler geometry in the KG-approach is provided globally for comparison's sake and completeness.

1 Introduction

Linear connections are fundamental tools in various areas of mathematics and physics, especially, in Finsler geometry and general relativity. In Finsler geometry, the theory of linear connections provide a framework for studying the geometry of spaces where the length of a curve depends not only on position but also on direction. In general relativity, the Levi-Civita connection is a specific type of linear connection that describes the curvature of spacetime, which is essential for understanding gravity.

The most popular and commonly used approaches to intrinsic Finsler geometry are and the pullback (PB-) approach (cf. [1, 2, 7, 11]) and the Klein-Grifone (KG-) approach (cf. [4, 5, 6, 9, 15]. Even though there are certain relations between the two approaches, each has its own geometry that is very different from the other.

There exists a canonical linear connection in Riemannian geometry on a manifold M, and a similar canonical linear connection in Finsler geometry exists due to E. Cartan. Nevertheless, this Cartan connection is defined on double tangent bundle of M (in the KG-approach) or on the pullback bundle of M (in the PB-approach).

By the four fundamental Finsler connections, we mean the connections that are introduced by Berwald, Cartan, Chern, and Hashiguchi in local Finsler geometry. Soleiman [8] has studied the four fundamental linear connections on a Finsler manifold in the PB-approach of global Finsler geometry. Namely, he established the existence and uniqueness theorems for these connections and calculated their associated torsion and curvature tensors. Also, he investigated some applications and changes of a Finsler manifold. On the other hand, Grifone [5] investigated the Cartan and Berwald connections in the KG-approach. In [15], N. L. Youssef and Elgendi addressed the existence and uniqueness theorem of the Chern connection employing the KG-approach, as well as the related curvature and torsion tensors and their properties. The Chern and Hasiguchi connections were studied by Szilasi and Vincze [10], although they did so by lifting vector fields to the tangent bundle. The KG-approach has not, to the best of our knowledge, established the existence and uniqueness theorems for the Hashiguchi connection from a purely global standpoint.

In this investigation, we study the existence and uniqueness of Hashiguchi connection's theorem on a Finsler manifold, applying the KG-approach to Finsler geometry. The formulae for this connection's curvature and torsion tensors are established. Furthermore, we prove certain properties and the Bianchi identities of the curvature and torsion tensors.

The following is the paper's structure. We provide the material that will be needed for the duration of the current work in the first section. We provide a brief overview of the Frölicher-Nijenhuis formalism pertaining to the vector forms and derivation, as well as the basics of KG-approach to intrinsic Finsler geometry. Aside from certain fundamentals regarding Berwald and Chern connections, we focus on the most significant features and equations related to the curvature tensors of Cartan connection in the second section. We establish the Hashiguchi connection's existence and uniqueness theorem in the third section. We obtain the equations for this connection's curvature and torsion tensors. We further investigate specific features of the curvature tensors of the Hashiguchi connection as well as the Bianchi identities.

To provide a complete picture, we summarize the four essential linear connections in Finsler geometry as defined in the KG-approach. Additionally, an appendix offers detailed local formulas and comparisons with the PB-approach, aiding in a deeper understanding of these connections.

2 Preliminaries

This section provides a basic introduction to the KG-approach, a framework for studying global Finsler geometry. For a more comprehensive understanding, please consult [4, 5, 6]. Throughout this paper, we will assume that all geometric objects are infinitely differentiable.

The n-dimensional smooth manifold M is denoted throughout, and the $\mathbb{R}$-algebra of C^∞-functions on the manifold M is denoted by $C^\infty(M)$. Moreover, the $C^\infty(M)$-module of vector fields on M is denoted as $\mathfrak{X}(M)$. The subbundle of nonzero vectors tangent to M is $\mathcal{T}M$, and TM is the tangent bundle of M. We denote the vertical subbundle by $V(TM)$. The exterior derivative of f is df, while the derivative i_η is the interior product corresponding to $\eta \in \mathfrak{X}(TM)$. In addition, for a vector form K we have the derivative $d_K := [i_K, d]$. As a special case, $\mathcal{L}_\eta$ is the Lie derivative in direction of $\eta \in \mathfrak{X}(TM)$.

The double tangent bundle $T(\mathcal{T}M)$ and the pullback bundle $\pi^{-1}(TM)$ are related by short exact sequence:

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(\mathcal{T}M) \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

where $\rho := (\pi_{\mathcal{T}M}, \pi)$ and $\gamma(u, v) := j_u(v)$ define the bundle morphisms ρ and γ, respectively, and j_u is the natural isomorphism $j_u : T_{\pi_M(v)} \longrightarrow T_u(T_{\pi_M(v)}M)$. The natural almost tangent structure of TM is the vector 1-form J on TM defined by $J := \gamma \circ \rho$. The canonical (Liouville) vector field, or fundamental vector field on TM, is $C := \gamma \circ \overline{\eta}$, where $\overline{\eta}$ is the vector field on $\pi^{-1}(TM)$, defined by $\overline{\eta}(u) = (u, u)$.

We will require the Frölicher-Nijenhuis bracket evaluation in specific particular situations for this work [3]:

Let L be a vector ℓ-form, then for $\eta \in \mathfrak{X}(TM)$, and for all $\eta_1, ..., \eta_\ell \in \mathfrak{X}(TM)$, we have

$$[\eta, L](\eta_1, ..., \eta_\ell) = [\eta, L(\eta_1, ..., \eta_\ell)] - \sum_{i=1}^{\ell} L(\eta_1, ..., [\eta, \eta_i], ..., \eta_\ell).$$

Particularly, for a vector 1-form L, we get

$$[\zeta, L]\xi = [\zeta, L\xi] - L[\zeta, \xi].$$

In addition, for vector 1- forms K, L, then for all $\zeta, \eta \in \mathfrak{X}(TM)$,

$$\begin{aligned}[K, L](\zeta, \eta) &= [K\zeta, L\eta] + [L\zeta, K\eta] + KL[\zeta, \eta] + LK[\zeta, \eta] \\ &\quad - K[L\zeta, \eta] - K[\zeta, L\eta] - L[K\zeta, \eta] - L[\zeta, K\eta].\end{aligned}$$

Now, for a vector 1-form K, the Nijenhuis torsion of K is the vector 2-form $N_K := \frac{1}{2}[K, K]$, given by

$$N_K := \frac{1}{2}[K, K](\zeta, \xi) = [K\zeta, K\xi] + K^2[\zeta, \xi] - K[K\zeta, \xi] - K[\zeta, K\xi]. \tag{2.1}$$

The following features can be demonstrated for the natural almost tangent structure J:

$$[J, J] = 0, \;\; J^2 = 0 \;\; [C, J] = -J, \;\; \mathrm{Im}(J) = Ker(J) = V(TM), \tag{2.2}$$

If $i_{J\eta}\omega = 0$ for all $\eta \in \mathfrak{X}(TM)$, then this indicates that a scalar p-form ω is semi-basic. Also, a vector ℓ-form K is semi-basic if and only if $JK = 0$ and $i_{J\eta}K = 0$, for all $\eta \in \mathfrak{X}(TM)$. In the case where $\mathcal{L}_C\omega = r\omega$, a scalar ℓ-form ω is homogeneous of degree r. For every vector ℓ-form L such that $[C, L] = (r-1)L$, the vector is homogeneous of degree r, or $h(r)$. That is, J is $h(0)$.

A semispray S on M is a vector field on TM with the properties that S is C^∞ on $\mathcal{T}M$, C^1 on TM, and $JS = C$. A spray is a semispray S as well as it is homogeneous of degree 2, i.e, $[C, S] = S$.

If vector 1-form Γ on TM satisfies that $J\Gamma = J$, $\Gamma J = -J$, smooth on $\mathcal{T}M$, and C^0 on TM, then Γ constitutes a nonlinear connection on the manifold M. The definitions of the vertical projection v and the horizontal projection h attached to Γ are provided as follows:

$$v := \frac{1}{2}(I - \Gamma), \; h := \frac{1}{2}(I + \Gamma).$$

As a result, we have the direct sum decomposition, generated by Γ, of the double tangent bundle TTM as follows

$$TTM = V(TM) \oplus H(TM),$$

where $H(TM)$ refers to the horizontal subbundle (or simply, bundle) $V(TM)$ the vertical bundle. Moreover, the horizontal bundle induced by Γ is given by $H(TM) := Im\, h = \ker\, v$ and, on the other hand, the vertical bundle is given by $V(TM) := Im\, v = Ker\, h$. We shall refer to the elements of $V(TM)$ (resp. $H(TM)$) as $v\zeta$ (resp. $h\zeta$). We have the properties $Jv = 0, \;\; vJ = J, \;\; Jh = J, \;\; hJ = 0$. Moreover, if $[C, \Gamma] = 0$, then Γ is homogeneous.

A semispray S that is horizontal with regard to Γ can be associated with each nonlinear connection Γ. This semispray is denoted by $S = hS'$, where S' is an arbitrary semispray. Furthermore, a semispray attached to a homogeneous Γ is a spray.

The torsion of a nonlinear connection Γ can defined by a vector 2-form on TM, precisely, $t := \frac{1}{2}[J, \Gamma]$. Moreover, the vector 2-form, denoted by $\mathfrak{R} := -\frac{1}{2}[h, h]$, gives the curvature of Γ. Now, the properties $FJ = h$ and $Fh = -J$ provide the so-called almost complex structure F ($F^2 = -I$) corresponding to Γ. For any $z \in TM$, this F provides an isomorphism of T_zTM.

Definition 2.1. *[6]* A Finsler manifold (space) of dimension n is the pair (M, E), where M is an n-dimensional smooth manifold, and E is the function (called the energy function)

$$E : TM \longrightarrow \mathbb{R}$$

with the conditions:

- **(a)** E is positive, i.e., $E(z) > 0$ for all $z \in \mathcal{T}M$ and $E(0) = 0$, $\mathcal{T}M = TM \backslash \{0\}$,
- **(b)** E is smooth on the slit tangent bundle $\mathcal{T}M$, and C^1 on TM,
- **(c)** E is positively homogeneous of degree 2 in the directional variable, i.e., $\mathcal{L}_C E = 2E$,
- **(d)** The fundamental 2-form $\Omega := dd_J E$ has maximal rank.

As shown in [6], we have the following result:

Theorem 2.2. [6] *Consider a Finsler space (M, E). The Euler-Lagrange equation $i_S\Omega = -dE$ determines a unique vector field $S \in \mathfrak{X}(TM)$ which is a spray. Furthermore, this spray is known as the canonical spray (or the geodesic spray) of the Finsler manifold (M, E).*

Building upon the work of [6], we provide an essential result concerning the existence and uniqueness of a particular nonlinear connection endowed with remarkable properties.

Theorem 2.3. [6] *There is a unique conservative and homogeneous nonlinear connection on a Finsler manifold (M,E), that is,$d_h E = 0$. Moreover, this connection has zero torsion, given by*

$$\Gamma = [J, S],$$

where S is the canonical spray of E.

The connection which we have discussed, is also known as the Cartan nonlinear connection, the canonical connection, or the Barthel connection, and is fundamental to the Finsler manifold (M,E). It's worth noting that the canonical spray is a specific type of spray associated with the Barthel connection, known as a semi-spray

3 Berwald, Cartan, and Chern connections

We present essential background information on the relationship between Berwald and Cartan connections, which is pertinent to the current investigation. For a more comprehensive treatment, the reader is referred to [5] and [14].

Theorem 3.1. *[5] On a Finsler manifold (M,E), we have a unique linear connection $\overset{\circ}{D}$ on TM that satisfies the facts:*

(a) $\overset{\circ}{D}J = 0.$

(b) $\overset{\circ}{D}C = v.$

(c) $\overset{\circ}{D}\Gamma = 0 \ (\Longleftrightarrow \overset{\circ}{D}h = \overset{\circ}{D}v = 0).$

(d) $\overset{\circ}{D}_{J\zeta}J\xi = J[J\zeta,\xi].$

(e) $\overset{\circ}{T}(J\zeta,\xi) = 0,$

where the Barthel connection's horizontal and vertical projections are denoted by h and v. The (classical) torsion of $\overset{\circ}{D}$ is $\overset{\circ}{T}$, and $\Gamma = [J,S]$. We refer to this connection as the Berwald connection.

The explicit formulae of Berwald connection $\overset{\circ}{D}$ are given as follows:

$$\left.\begin{array}{rcl} \overset{\circ}{D}_{J\zeta}J\xi & = & J[J\zeta,\xi], \\ \overset{\circ}{D}_{h\zeta}J\xi & = & v[h\zeta,J\xi], \\ \overset{\circ}{D}F & = & 0, \end{array}\right\} \tag{3.1}$$

where F is the corresponding almost complex structure to the Barthel connection Γ.

Lemma 3.2. *For the Berwald connection, we have the property*

$$\overset{\circ}{T}(h\zeta,h\xi) = \mathfrak{R}(\zeta,\xi),$$

where $\mathfrak{R}$ is the curvature of the Barthel connection.

Let (M,E) be a Finsler manifold equipped with the fundamental form $\Omega = dd_J E$. Then, the map $\overline{g}$ given by

$$\overline{g}(J\zeta,J\xi) := \Omega(J\zeta,\xi),\ \forall\,\zeta,\xi\in T(TM)$$

presents a metric on $V(TM)$. Moreover, the metric $\overline{g}$ can be extended to a metric g on $T(TM)$ by the formula:

$$g(\zeta,\xi) = \overline{g}(J\zeta,J\xi) + \overline{g}(v\zeta,v\xi) = \Omega(\zeta,F\xi). \tag{3.2}$$

According to [5], we have the following theorem which characterizes the Cartan connection on a Finsler manifold (M,E).

Theorem 3.3. *[5] Assume that (M,E) is a Finsler manifold. Then, we have a unique linear connection D on TM such that the following properties are satisfied:*

(a) $DJ = 0$.

(b) $DC = v$.

(c) $D\Gamma = 0$ ($\Longleftrightarrow Dh = Dv = 0$).

(d) $Dg = 0$.

(e) $T(J\zeta, J\eta) = 0$.

(f) $JT(h\zeta, h\xi) = 0$.

The connection D mentioned above is referred as the Cartan connection. Moreover, the explicit formulae of D are given by:

$$\left.\begin{array}{rcl} D_{J\zeta}J\xi &=& \overset{\circ}{D}_{J\zeta}J\xi + \mathcal{C}(\zeta,\xi), \\ D_{h\zeta}J\xi &=& \overset{\circ}{D}_{h\zeta}J\xi + \mathcal{C}'(\zeta,\xi), \\ DF &=& 0, \end{array}\right\} \tag{3.3}$$

where $\mathcal{C}$ and $\mathcal{C}'$ are scalar 2-forms on TM defined by the formulae

$$\Omega(\mathcal{C}(\zeta,\eta),\xi) = \frac{1}{2}(\mathcal{L}_{J\zeta}(J^*g))(\eta,\xi), \qquad \Omega(\mathcal{C}'(\zeta,\eta),\xi) = \frac{1}{2}(\mathcal{L}_{h\zeta}g)(J\eta, J\xi),$$

where we use $(J^*g)(\eta,\xi) = g(J\eta, J\xi)$. It should be noted that $\mathcal{C}$ and $\mathcal{C}'$ are the first and the second Cartan tensors respectively. Moreover, $\mathcal{C}$ and $\mathcal{C}'$ are semi-basics, symmetric, and (see [5])

$$\mathcal{C}(\eta, S) = \mathcal{C}'(\eta, S) = 0. \tag{3.4}$$

Recently, in [15], the Chern connection was studied and characterized by the following theorem.

Theorem 3.4. *[15] Suppose (M, E) is a Finsler manifold. Then, we have a unique linear connection, denoted by $\overset{*}{D}$, on TM such that the following facts are attained:*

(a) $\overset{*}{D}J = 0$.

(b) $\overset{*}{D}C = v$.

(c) $\overset{*}{D}\Gamma = 0$ ($\Longleftrightarrow \overset{*}{D}h = \overset{*}{D}v = 0$).

(d) $\overset{*}{D}_{h\zeta}g = 0$.

(e) $\overset{*}{T}(J\zeta, J\xi) = 0$.

(f) $J\overset{*}{T}(h\zeta, h\xi) = 0$.

The connection $\overset{*}{D}$ is called the Chern connection and its formulae are characterized by:

$$\left.\begin{array}{rcl} \overset{*}{D}_{J\zeta}J\xi &=& \overset{\circ}{D}_{J\zeta}J\xi, \\ \overset{*}{D}_{h\zeta}J\xi &=& \overset{\circ}{D}_{h\zeta}J\xi + \mathcal{C}'(\zeta,\xi), \\ \overset{*}{D}F &=& 0. \end{array}\right\} \tag{3.5}$$

For the purpose of our subsequent use, we provide the following lemmas:

Lemma 3.5. *For Cartan connection, the (h)h-torsion $T(h\zeta, h\xi)$ and (h)v-torsion $T(h\zeta, J\xi)$ can be calculated by*

$$T(h\zeta, h\xi) = \mathfrak{R}(\zeta,\xi), \quad T(h\zeta, J\xi) = (\mathcal{C}' - F\mathcal{C})(\zeta,\xi),$$

where $\mathfrak{R}$ is the curvature of the Barthel connection.

Lemma 3.6. *For Cartan connection, the h-curvature R, hv-curvature P, and v-curvature Q are calculated as follows:*

(a) $R(\eta,\kappa)\xi = \overset{\circ}{R}(\eta,\kappa)\xi + (D_{h\eta}\mathcal{C}')(\kappa,\xi) - (D_{h\kappa}\mathcal{C}')(\eta,\xi) + \mathcal{C}'(F\mathcal{C}'(\eta,\xi),\kappa)$
$\quad - \mathcal{C}'(F\mathcal{C}'(\kappa,\xi),\eta) + \mathcal{C}(F\mathfrak{R}(\eta,\kappa),\xi)$.

(b) $P(\eta,\kappa)\xi = \overset{\circ}{P}(\eta,\kappa)\xi + (D_{h\eta}\mathcal{C})(\kappa,\xi) - (D_{J\kappa}\mathcal{C}')(\eta,\xi) + \mathcal{C}(F\mathcal{C}'(\eta,\xi),\kappa)$
$\quad + \mathcal{C}(F\mathcal{C}'(\eta,\kappa),\xi) - \mathcal{C}'(F\mathcal{C}(\kappa,\xi),\eta) - \mathcal{C}'(F\mathcal{C}(\eta,\kappa),\xi)$.

(c) $Q(\eta,\kappa)\xi = \mathcal{C}(F\mathcal{C}(\eta,\xi),\kappa) - \mathcal{C}(F\mathcal{C}(\kappa,\xi),\eta)$, *where* $\overset{\circ}{R}$ *and* $\overset{\circ}{P}$ *are the h-curvature and hv-curvature of Berwald connection, respectively.*

Lemma 3.7. *The curvatures of the Cartan connection D have the following properties:*

(a) $R(\eta, \kappa)S = \mathfrak{R}(\eta, \kappa)$.

(b) $P(\eta, \kappa)S = \mathcal{C}'(\eta, \kappa)$.

(c) $P(S, \eta)\kappa = P(\eta, S)\kappa = 0$.

(d) $Q(S, \eta)\kappa = Q(\eta, S)\kappa = Q(\eta, \kappa)S = 0$.

Lemma 3.8. *A semi spray S satisfies the following property*

$$J[J\eta, S] = J\eta, \ \forall \eta \in \mathfrak{X}(TM).$$

Lemma 3.9. *For a homogeneous connection $\mathbf{\Gamma}$, its horizontal projector $\boldsymbol{h}$ satisfies*

$$[C, \boldsymbol{h}\zeta] = \boldsymbol{h}[C, \zeta], \ \forall \zeta \in \mathfrak{X}(TM).$$

Proof. Since $\mathbf{\Gamma}$ is homogeneous, then $\mathbf{h}$ is h(1). Thus, $[C, \mathbf{h}] = 0$ and hence

$$\begin{aligned} 0 &= [C, \mathbf{h}]\zeta \\ &= [C, \mathbf{h}\zeta] - \mathbf{h}[C, \zeta]. \end{aligned}$$

□

4 Hashiguchi connection

This section is dedicated to the exploration of the existence, uniqueness, and properties of the Hashiguchi connection. Explicit formulas for the torsion and curvature tensors of this connection are derived, and Bianchi identities are investigated.

For this purpose, we list the following definitions which are essential in the KG-approach to the theory of connections in the study of intrinsic Finsler geometry.

Definition 4.1. A linear connection $\mathbf{D}$ on TM is called regular if $\mathbf{D}J = 0$ and the map

$$\varphi : V(TM) \to V(TM),$$

given by $\zeta \to \mathbf{D}_\zeta C$, defines an isomorphism on $V(TM)$.

The above map φ can be considered as a restriction to $V(TM)$ of a map $\widetilde{\varphi} = \mathbf{D}_\zeta C$. For a regular connection $\mathbf{D}$ on TM there is corresponding connection $\mathbf{\Gamma}$ on M given by

$$\mathbf{\Gamma} = \mathbf{I} - \mathbf{2}\varphi^{-\mathbf{1}} \circ \mathbf{D}C,$$

where $\mathbf{\Gamma}$ is a connection induced by $\mathbf{D}$.

Definition 4.2. Consider a regular connection $\mathbf{D}$ on TM and the connection $\mathbf{\Gamma}$ induced on M by $\mathbf{D}$. Then, $\mathbf{D}$ is called reducible if $\mathbf{D\Gamma} = \mathbf{0}$.

Definition 4.3. We call a linear connection $\mathbf{D}$ on TM almost-projectable if $\mathbf{D}J = 0$ and $\mathbf{D}_{J\eta}C = J\eta$, for all $\eta \in T(TM)$.

If we replace the axiom $\mathbf{D}_{J\zeta}C = J\zeta$ with the more general one, that is, $\mathbf{D}_{J\eta}J\xi = J[J\eta, \xi]$, for all η, $\xi \in \mathfrak{X}(TM)$, then the connection $\mathbf{D}$ is called normal almost-projectable.

We will refer to the connection $\mathbf{\Gamma}$ on M induced by the almost-projectable (resp. normal almost-projectable) connection $\mathbf{D}$ on TM as the projection of $\mathbf{D}$. More precisely, we will say that $\mathbf{D}$ projects (resp. projects normally) onto $\mathbf{\Gamma}$.

Definition 4.4. Assume that $\mathbf{\Gamma}$ is connection on M. Then, a reducible connection $\mathbf{D}$ on TM that projects on $\mathbf{\Gamma}$ is the lift of $\mathbf{\Gamma}$. If $\mathbf{D}$ is normal, then the lift of $\mathbf{\Gamma}$ is considered normal.

Definition 4.5. Let $\mathbf{D}$ be a linear connection on TM. $\mathbf{D}$ is said to be horizontally metric or (h-metrical) if it satisfies that $\mathbf{D}_{h\zeta}g = 0$, for all $\zeta \in \mathfrak{X}(TM)$.

Lemma 4.6. *Consider a reducible connection* $\boldsymbol{D}$ *with the almost-complex structure* $\boldsymbol{F}$ *associated to the connection* $\boldsymbol{\Gamma}$ *induced by* $\boldsymbol{D}$*. Then, we have* $\boldsymbol{DF} = 0$.

Proof. Since $\mathbf{D}$ is reducible, then we have $\mathbf{D\Gamma} = \mathbf{0}$. Thus, we get $\mathbf{Dh} = \mathbf{Dv} = 0$, where $\mathbf{h}$ and $\mathbf{v}$ are the associated horizontal vertical projectors to $\mathbf{\Gamma}$. Therefore, using the facts that $\mathbf{FJ} = \mathbf{h}$, $\mathbf{Fh} = -\mathbf{J}$, and $\mathbf{JF} = \mathbf{v}$, we obtain

$$\begin{aligned}
\mathbf{F}(\mathbf{D}\zeta) &= \mathbf{F}(\mathbf{Dh}\zeta + \mathbf{Dv}\zeta) \\
&= \mathbf{F}(\mathbf{hD}\zeta + \mathbf{JDF}\zeta) \\
&= -\mathbf{JD}\zeta + \mathbf{hDF}\zeta \\
&= \mathbf{D}(-\mathbf{J}\zeta) + \mathbf{DhF}\zeta \\
&= \mathbf{DFh}\zeta + \mathbf{DFv}\zeta \\
&= \mathbf{DF}(\mathbf{h}\zeta + \mathbf{v}\zeta) \\
&= \mathbf{DF}\zeta.
\end{aligned}$$

Hence, $\mathbf{DF} = 0$. □

Lemma 4.7. *The second Cartan tensor* $\mathcal{C}'_b(\zeta, \kappa, \xi) := g(\mathcal{C}'(\zeta, \kappa), J\xi)$ *is completely symmetric.*

We are now prepared to present the existence and uniqueness theorem for the Hashiguchi connection on TM.

Theorem 4.8. *Consider the Finsler manifold* (M, E)*, then, we have a unique lift* $\overset{\star}{D}$ *of the Barthel connection* $\Gamma = [J, S]$ *such that the following assertions are satisfied:*

(a) $\overset{\star}{D}$ *is vertically metric:* $\overset{\star}{D}_{J\eta}g = 0$*, for all* $\eta \in \mathfrak{X}(TM)$.

(b) *The classical torsion* $\overset{\star}{T}(J\zeta, J\eta)$ *satisfies that:* $\overset{\star}{T}(J\zeta, J\eta) = 0$*, for all* $\zeta, \eta \in \mathfrak{X}(TM)$.

(c) *The classical torsion* $\overset{\star}{T}(h\zeta, J\eta)$ *satisfies that:* $v\overset{\star}{T}(h\zeta, J\eta) = 0$*, for all* $\zeta, \eta \in \mathfrak{X}(TM)$.

The connection $\overset{\star}{D}$ mentioned above is called the Hashiguchi connection.

Proof. We start by proving the **uniqueness**. Let ζ, η and $\xi \in \mathfrak{X}(TM)$. Since $\overset{\star}{D}$ is a lift of the Barthel connection $\Gamma = [J, S]$, then $\overset{\star}{D}\Gamma = 0$, $\overset{\star}{D}J = 0$ and thus, by Lemma 4.6, we have

$$\overset{\star}{D}F = 0. \tag{4.1}$$

Making use of the conditions $(\overset{\star}{D}_{J\zeta}g)(J\eta, J\xi) = 0$ and $\overset{\star}{T}(J\zeta, J\eta) = 0$, we get:

$$\begin{aligned}
J\zeta \cdot g(J\eta, J\xi) &= g(\overset{\star}{D}_{J\zeta}J\eta, J\xi) + g(J\eta, \overset{\star}{D}_{J\zeta}J\xi) && (4.2)\\
J\eta \cdot g(J\xi, J\zeta) &= g(\overset{\star}{D}_{J\eta}J\xi, J\zeta) + g(J\xi, \overset{\star}{D}_{J\eta}J\zeta) && (4.3)\\
J\xi \cdot g(J\zeta, J\eta) &= g(\overset{\star}{D}_{J\xi}J\zeta, J\eta) + g(J\zeta, \overset{\star}{D}_{J\xi}J\eta). && (4.4)
\end{aligned}$$

By adding (4.2), (4.3) and subtracting (4.4), we get

$$\begin{aligned}
J\zeta \cdot g(J\eta, J\xi) \quad + \quad & J\eta \cdot g(J\xi, J\zeta) - J\eta \cdot g(J\xi, J\zeta) = g(\overset{\star}{D}_{J\zeta}J\eta + \overset{\star}{D}_{J\eta}J\zeta, J\xi) \\
& + g(J\eta, \overset{\star}{D}_{J\zeta}J\xi - \overset{\star}{D}_{J\xi}J\zeta) + g(\overset{\star}{D}_{J\eta}J\xi - \overset{\star}{D}_{J\xi}J\eta, J\zeta). \qquad (4.5)
\end{aligned}$$

By the condition $\overset{\star}{T}(J\zeta, J\eta) = 0$, we have

$$\overset{\star}{D}_{J\zeta}J\eta - \overset{\star}{D}_{J\eta}J\zeta = [J\zeta, J\eta]. \tag{4.6}$$

From (4.5) and (4.6), we get

$$\begin{aligned} g(2\overset{\star}{D}_{J\zeta}J\eta, J\xi) &= J\zeta \cdot g(J\eta, J\xi) + J\eta \cdot g(J\xi, J\zeta) - J\eta \cdot g(J\xi, J\zeta) \\ &\quad + g([J\zeta, J\eta], J\xi) - g([J\zeta, J\xi], J\eta) - g([J\eta, J\xi], J\zeta) \end{aligned} \tag{4.7}$$

By the facts that $\Omega(\zeta, \eta) = g(\zeta, J\eta) - g(J\zeta, \eta)$ and $J\mathcal{C} = 0$, we have

$$\frac{1}{2}(\mathcal{L}_{J\zeta}(J^*g))(\eta, \xi) = \Omega(\mathcal{C}(\zeta, \eta), \xi) = g(\mathcal{C}(\zeta, \eta), J\xi) = \mathcal{C}_b(\zeta, \eta, \xi),$$

which is completely symmetric. Now,

$$\begin{aligned} g(2\mathcal{C}(\zeta, \eta), J\xi) &= J\zeta \cdot g(J\eta, J\xi) - g(J[J\zeta, \eta], J\xi) - g(J\eta, J[J\zeta, \xi]), \\ g(2\mathcal{C}(\eta, \xi), J\zeta) &= J\eta \cdot g(J\xi, J\zeta) - g(J[J\eta, \xi], J\zeta) - g(J\xi, J[J\eta, \zeta]), \\ -g(2\mathcal{C}(\xi, \zeta), J\eta) &= -J\xi \cdot g(J\zeta, J\eta) + g(J[J\xi, \zeta], J\eta) + g(J\zeta, J[J\xi, \eta]). \end{aligned}$$

By adding the three equations above, we get

$$\begin{aligned} g(2\mathcal{C}(\zeta, \eta), J\xi) &= J\zeta \cdot g(J\eta, J\xi) + J\eta \cdot g(J\xi, J\zeta) - J\xi \cdot g(J\zeta, J\eta) \\ &\quad - g(J[J\zeta, h\eta] + J[J\eta, h\zeta], J\xi) + g(J[J\xi, h\zeta] - J[J\zeta, h\xi], J\eta) \\ &\quad + g(J[J\xi, h\eta] - J[J\eta, h\xi], J\zeta) \end{aligned} \tag{4.8}$$

One can see that (4.7) and (4.8) imply

$$\begin{aligned} g(2\overset{\star}{D}_{J\zeta}J\eta, J\xi) &= g(2\mathcal{C}(\zeta, \eta), J\xi) - g([J\eta, J\xi] + J[J\xi, h\eta] - J[J\eta, h\xi], J\zeta) \\ &\quad - g([J\zeta, J\xi] + J[J\xi, h\zeta] - J[J\zeta, h\xi], J\eta) \\ &\quad + g([J\zeta, J\eta] + J[J\zeta, h\eta] + J[J\eta, h\zeta], J\xi). \end{aligned} \tag{4.9}$$

Since J satisfy the property

$$[J\zeta, J\eta] = J[J\zeta, h\eta] + J[h\zeta, J\eta],$$

then, we have

$$\overset{\star}{D}_{J\zeta}J\eta = J[J\zeta, \eta] + \mathcal{C}(\zeta, \eta). \tag{4.10}$$

Now, using the condition $v\overset{\star}{T}(h\zeta, J\eta) = 0$, we have

$$v\overset{\star}{D}_{h\zeta}J\eta - v\overset{\star}{D}_{J\eta}h\zeta - v[h\zeta, J\eta] = 0.$$

By the above equation and using that $vJ = J$ and $vh = 0$, then we get

$$\overset{\star}{D}_{h\zeta}J\eta = v[h\zeta, J\eta]. \tag{4.11}$$

The right hand side of (4.10) and (4.11) are unique and hence $\overset{\star}{D}_{\zeta}\eta$ is uniquely determined by (4.1), (4.10) and (4.11).

To prove the **existence** of $\overset{\star}{D}$, we define $\overset{\star}{D}$ by the requirement that (4.1), (4.10) and (4.11) hold for all $\zeta, \eta \in \mathfrak{X}(TM)$.

Now, we have to establish the following properties. $\overset{\star}{D}$ is lift of $\Gamma = [J, S]$: ($\overset{\star}{D}J = 0$, $\overset{\star}{D}C = v$, $\overset{\star}{D}\Gamma = 0$).

• $\overset{\star}{D}J = 0$, it is sufficient to show that $J\overset{\star}{D}_\zeta \eta = \overset{\star}{D}_\zeta J\eta$. From (4.1), (4.10) and (4.11), we have

$$\begin{aligned} J\overset{\star}{D}_{h\zeta}\eta &= J\overset{\star}{D}_{h\zeta}h\eta + J\overset{\star}{D}_{h\zeta}v\eta \\ &= J\overset{\star}{D}_{h\zeta}h\eta + J\overset{\star}{D}_{h\zeta}JF\eta \\ &= J\overset{\star}{D}_{h\zeta}h\eta + Jv[h\zeta, v\eta] \\ &= JFv[h\zeta, J\eta] \\ &= v[h\zeta, J\eta] \\ &= \overset{\star}{D}_{h\zeta}J\eta. \end{aligned}$$

Similarly, one can show that $J\overset{\star}{D}_{v\zeta}\eta = \overset{\star}{D}_{v\zeta}J\eta$.

• $\overset{\star}{D}C = v$, we have to show that $\overset{\star}{D}_{h\zeta}C = vh\zeta = 0$ and $\overset{\star}{D}_{J\zeta}C = vJ\zeta = J\zeta$ as follows. From (4.1), (4.10), (4.11) and Lemma 3.9, we get

$$\overset{\star}{D}_{h\zeta}JS = v[h\zeta, JS] = -v[C, h\zeta] = -vh[C, \zeta] = 0,$$

then by (3.4) and Lemma 3.8, we obtain

$$\overset{\star}{D}_{J\zeta}JS = J[J\zeta, S] + \mathcal{C}(\zeta, S) = J\zeta = vJ\zeta.$$

• $\overset{\star}{D}\Gamma = 0$ or equivalently $\overset{\star}{D}v = 0$ or $\overset{\star}{D}h = 0$. We will show only that $h\overset{\star}{D}_{h\zeta}\eta = \overset{\star}{D}_{h\zeta}h\eta$. By (4.1), (4.10) and (4.11), we get

$$\begin{aligned} h\overset{\star}{D}_{h\zeta}\eta &= h\overset{\star}{D}_{h\zeta}h\eta + h\overset{\star}{D}_{h\zeta}v\eta \\ &= h\overset{\star}{D}_{h\zeta}h\eta + h\overset{\star}{D}_{h\zeta}JF\eta \\ &= h\overset{\star}{D}_{h\zeta}h\eta + hv[h\zeta, v\eta] \\ &= hFv[h\zeta, J\eta] \\ &= Fv^2[h\zeta, J\eta] \\ &= Fv[h\zeta, J\eta] \\ &= \overset{\star}{D}_{h\zeta}h\eta. \end{aligned}$$

• $\overset{\star}{D}$ is h-metrical: g is vertically metric $(\overset{\star}{D}_{J\zeta}g)(J\eta, J\xi) = 0$. By (4.1), (4.10) and (4.11), we have

$$\begin{aligned} (\overset{\star}{D}_{J\zeta}g)(J\eta, J\xi) &= J\zeta.g(J\eta, J\xi) - g(\overset{\star}{D}_{J\zeta}J\eta, J\xi) - g(J\eta, \overset{\star}{D}_{J\zeta}J\xi) \\ &= J\zeta.g(J\eta, J\xi) - g(J[J\zeta, \eta] + \mathcal{C}(\zeta, \eta), J\xi) \\ &\quad -g(J\eta, J[J\zeta, \xi] + \mathcal{C}(\zeta, \xi)) \\ &= J\zeta.g(J\eta, J\xi) - g(J[J\zeta, \eta], J\xi) - g(J\eta, J[J\zeta, \xi]) \\ &\quad -2\mathcal{C}_b(\zeta, \eta, \xi) \\ &= 0 \end{aligned}$$

• $\overset{\star}{T}(J\zeta, J\eta) = 0$, by (4.1), (4.10) and (4.11), we have

$$\begin{aligned} \overset{\star}{T}(J\zeta, J\eta) &= \overset{\star}{D}_{J\zeta}J\eta - \overset{\star}{D}_{J\eta}J\zeta - [J\zeta, J\eta] \\ &= J[J\zeta, \eta] + \mathcal{C}(\zeta, \eta) - J[J\eta, \zeta] - \mathcal{C}(\eta, \zeta) - [J\zeta, J\eta] \\ &= 0. \end{aligned}$$

• $v\overset{\star}{T}(h\zeta, J\eta) = 0$, by (4.1), (4.10) and (4.11), we have

$$\begin{aligned} v\overset{\star}{T}(h\zeta, J\eta) &= v\overset{\star}{D}_{h\zeta}J\eta - v\overset{\star}{D}_{J\eta}h\zeta - v[h\zeta, J\eta] \\ &= \overset{\star}{D}_{h\zeta}J\eta - v[h\zeta, J\eta] \\ &= 0. \end{aligned}$$

Hence the proof is completed. □

Theorem 4.9. *The Hashiguchi connection $\overset{\star}{D}$ is uniquely determined by the following relations:*

(a) $\overset{\star}{D}_{J\zeta}J\eta = \overset{\circ}{D}_{J\zeta}J\eta + \mathcal{C}(\zeta, \eta)$.

(b) $\overset{\star}{D}_{h\zeta}J\eta = \overset{\circ}{D}_{h\zeta}J\eta$.

(c) $\overset{\star}{D}F = 0$.

Now, we have the enough information needed to find explicit expressions for the attached torsion and curvature tensors to Hashiguchi connection.

Lemma 4.10. *For the Hashiguchi connection $\overset{\star}{D}$, the (h)h-torsion $\overset{\star}{T}(h\zeta, h\xi)$, and the (h)v-torsion $\overset{\star}{T}(h\zeta, J\xi)$ are calculated as follows:*

(a) $\overset{\star}{T}(h\zeta, h\xi) = \mathfrak{R}(\zeta, \xi)$.

(b) $\overset{\star}{T}(h\zeta, J\xi) = -F\mathcal{C}(\zeta, \xi)$.

Proof.
(a) Making use of Theorem 4.9, and the vanishing property of the torsion $t(\zeta, \xi)$ of the connection Γ, that is, $0 = t(\zeta, \xi) = v[J\zeta, h\xi] + v[h\zeta, J\xi] - J[h\zeta, h\xi]$ together with the fact that $hF = Fv$, we have

$$\begin{aligned} \overset{\star}{T}(h\zeta, h\xi) &= \overset{\star}{D}_{h\zeta}h\xi - \overset{\star}{D}_{h\xi}h\zeta - [h\zeta, h\xi] \\ &= Fv[h\zeta, J\xi] - Fv[h\xi, J\zeta] - [h\zeta, h\xi] \\ &= FJ[h\zeta, h\xi] - Fv[J\zeta, h\xi] - hF[h\xi, J\zeta] - [h\zeta, h\xi] \\ &= h[h\zeta, h\xi] - [h\zeta, h\xi] \\ &= -v[h\zeta, h\xi] \\ &= \mathfrak{R}(\zeta, \xi). \end{aligned}$$

(b) By Theorem 4.9 and using that the property $h[J\zeta, v\xi] = 0$, we get

$$\begin{aligned} \overset{\star}{T}(h\zeta, J\xi) &= \overset{\star}{D}_{h\zeta}J\xi - \overset{\star}{D}_{J\xi}h\zeta - [h\zeta, J\xi] \\ &= v[h\zeta, J\xi] - h[J\xi, \zeta] - F\mathcal{C}(\xi, \zeta) - [h\zeta, J\xi] \\ &= v[h\zeta, J\xi] - h[J\xi, h\zeta] - h[h\zeta, J\xi] - v[h\zeta, J\xi] \\ &= -F\mathcal{C}(\zeta, \xi). \end{aligned}$$

□

Proposition 4.11. *For Hashiguchi connection, the h-curvature $\overset{\star}{R}$, the hv-curvature $\overset{\star}{P}$, and the v-curvature $\overset{\star}{Q}$, can be written in the following formulae:*

(a) $\overset{\star}{R}(\zeta, \eta)\xi = \overset{\circ}{R}(\zeta, \eta)\xi + \mathcal{C}(F\mathfrak{R}(\zeta, \eta), \xi)$.

(b) $\overset{\star}{P}(\zeta,\eta)\xi = \overset{\circ}{P}(\zeta,\eta)\xi + (\overset{\star}{D}_{h\zeta}\mathcal{C})(\eta,\xi)$.

(c) $\overset{\star}{Q}(\zeta,\eta)\xi = Q(\zeta,\eta)\xi = \mathcal{C}(F\mathcal{C}(\zeta,\xi),\eta) - \mathcal{C}(F\mathcal{C}(\eta,\xi),\zeta)$.

Proof.

(a) By the definition of the classical curvature K, the properties of F and the fact that $\overset{\star}{D}_{h\zeta}J\eta = \overset{\circ}{D}_{h\zeta}J\eta$, we have

$$\begin{aligned}
\overset{\star}{R}(\zeta,\eta)\xi &= K(h\zeta,h\eta)J\xi \\
&= \overset{\star}{D}_{h\zeta}\overset{\star}{D}_{h\eta}J\xi - \overset{\star}{D}_{h\eta}\overset{\star}{D}_{h\zeta}J\xi - \overset{\star}{D}_{[h\zeta,h\eta]}J\xi \\
&= \overset{\circ}{D}_{h\zeta}\overset{\circ}{D}_{h\eta}J\xi - \overset{\circ}{D}_{h\eta}\overset{\circ}{D}_{h\zeta}J\xi - \overset{\star}{D}_{[h\zeta,h\eta]}J\xi \\
&= \overset{\circ}{R}(\zeta,\eta)\xi + \overset{\circ}{D}_{[h\zeta,h\eta]}J\xi - \overset{\star}{D}_{[h\zeta,h\eta]}J\xi \\
&= \overset{\circ}{R}(\zeta,\eta)\xi + \overset{\circ}{D}_{JF[h\zeta,h\eta]}J\xi - \overset{\star}{D}_{JF[h\zeta,h\eta]}J\xi \\
&= \overset{\circ}{R}(\zeta,\eta)\xi - \mathcal{C}(F[h\zeta,h\eta],\xi) \\
&= \overset{\circ}{R}(\zeta,\eta)\xi + \mathcal{C}(F\mathfrak{R}(\zeta,\eta),\xi).
\end{aligned}$$

(b) By Theorem 4.9 and the fact that $\overset{\star}{D}_{h\zeta}J\eta = \overset{\circ}{D}_{h\zeta}J\eta$, we have

$$\begin{aligned}
\overset{\star}{P}(\zeta,\eta)\xi &= K(h\zeta,J\eta)J\xi \\
&= \overset{\star}{D}_{h\zeta}\overset{\star}{D}_{J\eta}J\xi - \overset{\star}{D}_{J\eta}\overset{\star}{D}_{h\zeta}J\xi - \overset{\star}{D}_{[h\zeta,J\eta]}J\xi \\
&= \overset{\circ}{D}_{h\zeta}(\overset{\circ}{D}_{J\eta}J\xi + \mathcal{C}(\eta,\xi)) - \overset{\circ}{D}_{J\eta}\overset{\circ}{D}_{h\zeta}J\xi - \mathcal{C}(\overset{\circ}{D}_{h\zeta}\xi,\eta) - \overset{\star}{D}_{[h\zeta,h\eta]}J\xi \\
&= \overset{\circ}{D}_{h\zeta}\overset{\circ}{D}_{J\eta}J\xi - \overset{\circ}{D}_{J\eta}\overset{\circ}{D}_{h\zeta}J\xi - \overset{\circ}{D}_{[h\zeta,h\eta]}J\xi + \overset{\circ}{D}_{h\zeta}\mathcal{C}(\eta,\xi) \\
&\quad - \mathcal{C}(\overset{\circ}{D}_{h\zeta}\xi,\eta) - \mathcal{C}(F[h\zeta,h\eta],\xi) \\
&= \overset{\circ}{P}(\zeta,\eta)\xi + \overset{\circ}{D}_{h\zeta}\mathcal{C}(\eta,\xi) - \mathcal{C}(\overset{\circ}{D}_{h\zeta}\eta,\xi) - \mathcal{C}(\eta,\overset{\circ}{D}_{h\zeta}\xi) \\
&= \overset{\circ}{P}(\zeta,\eta)\xi + (\overset{\star}{D}_{h\zeta}\mathcal{C})(\eta,\xi).
\end{aligned}$$

(c) By Theorem 4.9 and the properties $\overset{\star}{D}_{J\zeta}J\eta = D_{J\zeta}J\eta$, $hv = 0$, we get

$$\begin{aligned}
\overset{\star}{Q}(\zeta,\eta)\xi &= K(J\zeta,J\eta)J\xi \\
&= \overset{\star}{D}_{J\zeta}\overset{\star}{D}_{J\eta}J\xi - \overset{\star}{D}_{J\eta}\overset{\star}{D}_{J\zeta}J\xi - \overset{\star}{D}_{[J\zeta,J\eta]}J\xi \\
&= D_{J\zeta}D_{J\eta}J\xi - D_{J\eta}D_{J\zeta}J\xi - D_{[J\zeta,J\eta]}J\xi \\
&= Q(\zeta,\eta)\xi \\
&= \mathcal{C}(F\mathcal{C}(\zeta,\xi),\eta) - \mathcal{C}(F\mathcal{C}(\eta,\xi),\zeta).
\end{aligned}$$

□

Proposition 4.12. *The curvatures* $\overset{\star}{R}$, *and* $\overset{\star}{P}$ *of Hashiguchi connection satisfy the properties:*

(a) $\overset{\star}{R}(\eta,\xi)S = \mathfrak{R}(\eta,\xi)$.

(b) $\overset{\star}{P}(\eta,\xi)S = \overset{\star}{P}(\eta,S)\xi = \overset{\star}{P}(S,\eta)\xi = 0$.

(c) $\overset{\star}{Q}(\eta,\xi)S = \overset{\star}{Q}(\eta,S)\xi = \overset{\star}{Q}(S,\eta)\xi = 0$.

Proof.
(a) Follows from (3.4), Lemma 3.7 and Proposition 4.11.
(b) By making use of (3.4), the facts that $\overset{\circ}{P}(\eta,\xi)S = \overset{\circ}{P}(\eta,S)\xi = 0$ [14] and the bracket $[h\eta, C]$ is horizontal, we get:

$$\begin{aligned}\overset{\star}{P}(\eta,\xi)S &= (\overset{\star}{D}_{h\eta}\mathcal{C})(\xi,S)\\ &= -\mathcal{C}(\xi,\overset{\star}{D}_{h\eta}hS)\\ &= -\mathcal{C}(\xi,Fv[h\eta,C])\\ &= 0.\end{aligned}$$

And $\overset{\star}{P}(\eta,S)\xi = 0$ can be proved in a similar manner and hence $\overset{\star}{P}(S,\eta)\xi = 0$ by the antisymmetric property of $\overset{\star}{P}$.
(c) Follows from (3.4) and Proposition 4.11. □

To investigate the Bianchi identities for Hashiguchi connection, the following lemma is required.

Lemma 4.13. *Consider a linear Finsler connection $\boldsymbol{D}$ on TM with the classical torsion tensor $\boldsymbol{T}$ and the classical curvature tensor $\boldsymbol{K}$. Then, for all $\zeta,\eta,\xi\in\mathfrak{X}(TM)$, we have the following identities:*

(a) $\mathfrak{S}_{\zeta,\eta,\xi}\{\boldsymbol{K}(\zeta,\eta)\xi\} = \mathfrak{S}_{\zeta,\eta,\xi}\{\boldsymbol{T}(\boldsymbol{T}(\zeta,\eta),\xi) + (\boldsymbol{D}_\zeta T)(\eta,\xi)\}$,

(b) $\mathfrak{S}_{\zeta,\eta,\xi}\{\boldsymbol{K}(T(\zeta,\eta),\xi) + (\boldsymbol{D}_\zeta\boldsymbol{K})(\eta,\xi)\} = 0$,

where the notation $\mathfrak{S}_{\zeta,\eta,\xi}$ refers to the cyclic sum over ζ,η and ξ.

We are now ready to explore the Bianchi identities for the Hashiguchi connection, as detailed in the following proposition.

Proposition 4.14. *For Hashiguchi connection on a Finsler manifold (M,E), we have the following Bianchi identities:*

(a) $\mathfrak{S}_{\zeta,\eta,\xi}\{\overset{\star}{R}(\zeta,\eta)\xi\} = \mathfrak{S}_{\zeta,\eta,\xi}\{\mathcal{C}(F\mathfrak{R}(\zeta,\eta),\xi)\}$.

(b) $\mathfrak{S}_{\zeta,\eta,\xi}\{\overset{\star}{Q}(\zeta,\eta)\xi\} = 0$.

(c) $\mathcal{C}(F\mathfrak{R}(\zeta,\eta),\xi) = \mathfrak{R}(F\mathcal{C}(\zeta,\xi),\eta) - \mathfrak{R}(F\mathcal{C}(\eta,\xi),\zeta)$.

(d) $\mathfrak{S}_{\zeta,\eta,\xi}\{(\overset{\star}{D}_{h\zeta}\mathfrak{R})(\eta,\xi)\} = 0$.

(e) $\mathfrak{S}_{\zeta,\eta,\xi}\,\{(\overset{\star}{D}_{h\zeta}\overset{\star}{R})(\eta,\xi)\} = \mathfrak{S}_{\zeta,\eta,\xi}\,\{\overset{\star}{P}(\zeta,F\mathfrak{R}(\eta,\xi))\}$.

(f) $(\overset{\star}{D}_{h\zeta}\overset{\star}{P})(\eta,\xi) - (\overset{\star}{D}_{h\eta}\overset{\star}{P})(\zeta,\xi) + (\overset{\star}{D}_{J\xi}\overset{\star}{R})(\zeta,\eta) = \overset{\star}{R}(F\mathcal{C}(\eta,\xi),\zeta)$
$- \overset{\star}{R}(F\mathcal{C}(\zeta,\xi),\eta) - \overset{\star}{Q}(F\mathfrak{R}(\zeta,\eta),\xi)$.

(g) $(\overset{\star}{D}_{h\zeta}\overset{\star}{Q})(\eta,\xi) - (\overset{\star}{D}_{J\eta}\overset{\star}{P})(\zeta,\xi) + (\overset{\star}{D}_{J\xi}\overset{\star}{P})(\zeta,\eta) = \overset{\star}{P}(F\mathcal{C}(\zeta,\eta),\xi)$
$- \overset{\star}{P}(F\mathcal{C}(\xi,\zeta),\eta)$.

(h) $\mathfrak{S}_{\zeta,\eta,\xi}\{(\overset{\star}{D}_{J\zeta}\overset{\star}{Q})(\eta,\xi)\} = 0$,

Proof.
(a) Making use of Lemma 4.13 **(a)** for $h\zeta$, $h\eta$ and $h\xi$, we get

$$F\mathfrak{S}_{\zeta,\eta,\xi}\{\overset{\star}{R}(\zeta,\eta)\xi\} = \mathfrak{S}_{\zeta,\eta,\xi}\{(\overset{\star}{D}_{h\zeta}\mathfrak{R})(\eta,\xi)\}. \tag{4.12}$$

Since $Fv = hF$ and $\overset{\star}{R}$, $\mathfrak{R}$ are verticals, by comparing the horizontal parts taking the fact that F is an isomorphism into account, we have

$$\mathfrak{S}_{\zeta,\eta,\xi}\{\overset{\star}{R}(\zeta,\eta)\xi\} = 0.$$

(b) It follows by employing Lemma 4.13 **(a)** on $J\zeta$, $J\eta$ and $J\xi$.
(c) Follows by applying Lemma 4.13 **(a)** on $h\zeta$, $h\eta$ and $J\xi$ and comparing the vertical parts.
(d) The result follows by comparing the vertical parts in (4.12).
(e) By using Lemma 4.13 **(b)** on $h\zeta$, $h\eta$ and $h\xi$.
(f) The result comes by implementing Lemma 4.13 **(b)** on $h\zeta$, $h\eta$ and $J\xi$.
(g) Follows by considering Lemma 4.13 **(b)** on $h\zeta$, $J\eta$ and $J\xi$.
(h) It follows by using Lemma 4.13 **(b)** on $J\zeta$, $J\eta$ and $J\xi$. □

Proposition 4.15. *The hv-curvature of the Hashiguchi connection* $\overset{\star}{P}$ *satisfies*

$$\overset{\star}{P}(\zeta,\kappa)\xi = \overset{\star}{P}(\zeta,\xi)\kappa.$$

Proof. Since $\overset{\circ}{P}$ is totally symmetric [14] and the symmetric property of $\mathcal{C}$ [5], then the result follows by Proposition 4.11. □

Lemma 4.16. *The hv-curvature* $\overset{\star}{P}$ *and the v-curvature* $\overset{\star}{Q}$ *of the Hashiguchi connection satisfy the following properties:*

$$(\overset{\star}{D}_{h\zeta}\overset{\star}{P})(\eta,S) = 0, \qquad (\overset{\star}{D}_{J\zeta}\overset{\star}{P})(\eta,S) = \overset{\star}{P}(\eta,\zeta),$$

$$(\overset{\star}{D}_{h\zeta}\overset{\star}{Q})(\eta,S) = 0, \qquad (\overset{\star}{D}_{J\zeta}\overset{\star}{Q})(\eta,S) = \overset{\star}{Q}(\eta,\zeta).$$

Proof. By Theorem 4.9 and since $[h\zeta, C]$ is horizontal, we get $\overset{\star}{D}_{h\zeta}hS = Fv[h\zeta, JS] = Fv[h\zeta, C] = 0$. Hence, by Proposition 4.12, we have $(\overset{\star}{D}_{h\zeta}\overset{\star}{P})(\eta,S) = (\overset{\star}{D}_{h\zeta}\overset{\star}{Q})(\eta,S) = 0$. Making use of (3.4) and Lemma 3.8, we get $\overset{\star}{D}_{J\zeta}hS = h[J\zeta,S] + F\mathcal{C}(\zeta,S) = h\zeta$. Thus, $(\overset{\star}{D}_{J\zeta}\overset{\star}{P})(\eta,S) = \overset{\star}{P}(\eta,\zeta)$ and $(\overset{\star}{D}_{J\zeta}\overset{\star}{Q})(\eta,S) = \overset{\star}{Q}(\eta,\zeta)$. □

Proposition 4.17. *The curvatures* $\overset{\star}{R}$, $\overset{\star}{P}$, *and* $\overset{\star}{Q}$ *of the Hashiguchi connection satisfy the following facts:*

$$\overset{\star}{D}_C\overset{\star}{R} = 0, \qquad \overset{\star}{D}_C\overset{\star}{P} = -\overset{\star}{P}, \qquad \overset{\star}{D}_C\overset{\star}{Q} = -2\overset{\star}{Q}.$$

Proof. The result follows from Lemma 4.14 **(d)** and **(e)**, by putting $\xi = S$. □

We have the following formulae of Lie brackets based on the Hashiguchi connection $\overset{\star}{D}$.

Proposition 4.18. *For the Hashiguchi connection* $\overset{\star}{D}$ *and for all* $\eta,\xi \in \mathfrak{X}(TM)$, *we have the following properties:*

(a) $[J\eta, J\xi] = J(\overset{\star}{D}_{J\eta}\xi - \overset{\star}{D}_{J\xi}\eta)$.

(b) $[h\eta, J\xi] = J(\overset{\star}{D}_{h\eta})\xi - h(\overset{\star}{D}_{J\xi}\eta) + F\mathcal{C}(\eta,\xi)$.

(c) $[h\eta, h\xi] = h(\overset{\star}{D}_{h\eta}\xi - \overset{\star}{D}_{h\xi}\eta) - \mathfrak{R}(\eta,\xi)$.

Proof.
(a) By Theorem 4.9, we get

$$\begin{aligned} J(\overset{\star}{D}_{J\eta}\xi - \overset{\star}{D}_{J\xi}\eta) &= \overset{\star}{D}_{J\eta}J\xi - \overset{\star}{D}_{J\xi}J\eta \\ &= J[J\eta,\xi] - J[J\xi,\eta] \\ &= [J\eta, J\xi]. \end{aligned}$$

(b) Since the tensor $\mathcal{C}$ is symmetric, we obtain

$$\begin{aligned} J(\overset{\star}{D}_{h\eta}\xi) - h(\overset{\star}{D}_{J\xi}\eta) &= \overset{\star}{D}_{h\eta}J\xi - \overset{\star}{D}_{J\xi}h\eta \\ &= v[h\eta, J\xi] - h[J\xi, \eta] - F\mathcal{C}(\xi, \eta) \\ &= v[h\eta, J\xi] - h[J\xi, \eta] - F\mathcal{C}(\xi, \eta) \\ &= v[h\eta, J\xi] + h[h\eta, J\xi] - F\mathcal{C}(\xi, \eta) \\ &= [h\eta, J\xi] - F\mathcal{C}(\eta, \xi). \end{aligned}$$

(c) Once again, by making use of the symmetry property of the tensor $\mathcal{C}$, we can write

$$\begin{aligned} h(\overset{\star}{D}_{h\eta}\xi - \overset{\star}{D}_{h\xi}\eta) &= \overset{\star}{D}_{h\eta}h\xi - \overset{\star}{D}_{h\xi}h\eta \\ &= Fv[h\eta, J\xi] + Fv[J\eta, h\xi]. \end{aligned}$$

Since the torsion of the connection Γ vanishes, we have

$$0 = t(\eta, \xi) = v[J\eta, h\xi] + v[h\eta, J\xi] - J[h\eta, h\xi].$$

Which implies $Fv[J\eta, h\xi] + Fv[h\eta, J\xi] = FJ[h\eta, h\xi] = h[h\eta, h\xi]$. Consequently, we have

$$h(\overset{\star}{D}_{h\eta}\xi - \overset{\star}{D}_{h\xi}\eta) = h[h\eta, h\xi] = [h\eta, h\xi] - v[h\eta, h\xi] = [h\eta, h\xi] + \mathfrak{R}(\eta, \xi).$$

It should be noted that we have used the fact $\mathfrak{R}(\eta, \xi) = -v[h\eta, h\xi]$, for example, see [13]. □

5 Concluding remarks

Let's end this work by the following comments and remarks:

- For a Finsler manifold (M, E), following the Klein-Grifone approach, we can associate four fundamental linear connections canonically: the Berwald connection $\overset{\circ}{D}$ (Theorem 3.1), the Cartan connection D (Theorem 3.3), the Chern connection $\overset{*}{D}$ (Theorem 3.4) and the Hashiguchi connection $\overset{\star}{D}$ (Theorem 4.8). For each of these connections, the underlying nonlinear connection is the Barthel connection.
- Define the $\mathcal{C}$-process by adding $\mathcal{C}$ to the vertical counterpart of the connection, and the $\mathcal{C}'$-process by adding $\mathcal{C}'$ to the horizontal counterpart. Now, we can convert one Finsler connection to some other connections, as follows: by Theorem 4.8, we observe that the Hashiguchi connection $\overset{\star}{D}$ is obtained from the Berwald connection $\overset{\circ}{D}$ by $\mathcal{C}$-process. Moreover, by Theorem 3.3, the Cartan connection D is obtained from the Hashiguchi connection by $\mathcal{C}'$-process. Now, applying the $\mathcal{C}'$-process on Hashiguchi connection $\overset{\star}{D}$, the Chern connection $\overset{*}{D}$ is obtained. Then applying $\mathcal{C}$-process on Chern connection $\overset{*}{D}$, the Cartan connection D is obtained. That is, we have

$$D \xleftarrow{\mathcal{C}\text{-process}} \overset{*}{D} \xleftarrow{\mathcal{C}'\text{-process}} \overset{\circ}{D} \xrightarrow{\mathcal{C}\text{-process}} \overset{\star}{D} \xrightarrow{\mathcal{C}'\text{-process}} D.$$

- We offer a comparative analysis of the four fundamental Finsler connections in the KG-approach, which provides an intrinsic perspective on global Finsler geometry. The following table summarizes the key differences between these connections, including their associated canonical linear connections and geometric objects. For a detailed treatment of the Chern connection, refer to [15].

Table 1: The four fundamental Finsler Linear connections

Connection	**Berwald:** $\overset{\circ}{D}$	**Cartan:** D	**Chern:** $\overset{*}{D}$	**Hashiguchi:** $\overset{\star}{D}$
Expressions **of connection**	$\overset{\circ}{D}_{J\zeta}J\eta = J[J\kappa, \eta]$ $\overset{\circ}{D}_{h\kappa}J\eta = v[h\kappa, J\eta]$ $\overset{\circ}{D}F = 0$	$D_{J\kappa}J\eta = \overset{\circ}{D}_{J\kappa}J\eta + \mathcal{C}(\kappa, \eta)$ $D_{h\kappa}J\eta = \overset{\circ}{D}_{h\kappa}J\eta + \mathcal{C}'(\kappa, \eta)$ $DF = 0$	$\overset{*}{D}_{J\kappa}J\eta = J[J\kappa, \eta]$ $\overset{*}{D}_{h\kappa}J\eta = v[h\kappa, J\eta] + \mathcal{C}'(\kappa, \eta)$ $\overset{*}{D}F = 0$	$\overset{\star}{D}_{J\kappa}J\eta = J[J\kappa, \eta] + \mathcal{C}(\kappa, \eta)$ $\overset{\star}{D}_{h\kappa}J\eta = v[h\kappa, J\eta]$ $\overset{\star}{D}F = 0$
hh-torsions hv-torsions **vv-torsions**	$\mathfrak{R}$ 0 0	$\mathfrak{R}$ $\mathcal{C}' - F\mathcal{C}$ 0	$\mathfrak{R}$ $\mathcal{C}'$ 0	$\mathfrak{R}$ $-F\mathcal{C}$ 0
h-curvature	$\overset{\circ}{R}$	$R(\kappa, \eta)\xi = \overset{\circ}{R}(\kappa, \eta)\xi + (D_{h\kappa}\mathcal{C}')(\eta, \xi)$ $-(D_{h\eta}\mathcal{C}')(\kappa, \xi) + \mathcal{C}'(F\mathcal{C}'(\kappa, \xi), \eta)$ $-\mathcal{C}'(F\mathcal{C}'(\eta, \xi), \kappa) + \mathcal{C}(F\mathfrak{R}(\kappa, \eta), \xi)$	$\overset{*}{R}(\kappa, \eta)\xi = R(\kappa, \eta)\xi$ $-\mathcal{C}(F\mathfrak{R}(\kappa, \eta), \xi)$	$\overset{\star}{R}(\kappa, \eta)\xi = \overset{\circ}{R}(\kappa, \eta)\xi$ $+\mathcal{C}(F\mathfrak{R}(\kappa, \eta), \xi)$
hv-curvature	$\overset{\circ}{P}$	$P(\kappa, \eta)\xi = \overset{\circ}{P}(\kappa, \eta)\xi + (D_{h\kappa}\mathcal{C})(\eta, \xi)$ $-(D_{J\eta}\mathcal{C}')(\kappa, \xi) + \mathcal{C}(F\mathcal{C}'(\kappa, \xi), \eta)$ $+\mathcal{C}(F\mathcal{C}'(\kappa, \eta), \xi) - \mathcal{C}'(F\mathcal{C}(\eta, \xi), \kappa)$ $-\mathcal{C}'(F\mathcal{C}(\kappa, \eta), \xi)$	$\overset{*}{P}(\kappa, \eta)\xi = \overset{\circ}{P}(\kappa, \eta)\xi$ $-(\overset{*}{D}_{J\eta}\mathcal{C}')(\kappa, \xi)$	$\overset{\star}{P}(\kappa, \eta)\xi = \overset{\circ}{P}(\kappa, \eta)\xi$ $+(\overset{\star}{D}_{h\kappa}\mathcal{C})(\eta, \xi)$
v-curvature	0	$Q(\kappa, \eta)\xi = \mathcal{C}(F\mathcal{C}(\kappa, \xi), \eta)$ $-\mathcal{C}(F\mathcal{C}(\eta, \xi), \kappa)$	0	$\overset{\star}{Q} = Q$
v-metricity **h-metricity**	not v-metrical not h-metrical	v-metrical h-metrical	not v-metrical h-metrical	v-metrical not h-metrical

Appendix: Some local formulas

For completeness, we provide a concise overview of essential geometric objects' local expressions in this appendix.

Let (M, E) be a Finsler manifold, $(U, (x^i))$ be a system of local coordinates on M, and $(\pi^{-1}(U), (x^i, y^i))$ the attached system of coordinates on TM.

We have the following objects:

$(\partial_i) := (\frac{\partial}{\partial x^i})$: the natural bases of $T_x M$, $x \in M$,
$(\dot{\partial}_i) := (\frac{\partial}{\partial y^i})$: the natural bases of $V_u(TM)$, $u \in TM$,
$(\partial_i, \dot{\partial}_i)$: the natural bases of $T_u(TM)$,
$(\overline{\partial}_i)$: the natural bases of the fiber over u in $\pi^{-1}(TM)$.

$g_{ij} := \dot{\partial}_i \dot{\partial}_j E$: the components of the metric tensor, where $E = \frac{1}{2}\mathbf{L}^2$ is the energy function, and $\mathbf{L}$ is the Finsler structure, the Finsler metric tensor,
g^{hr}: the components of the inverse metric tensor,
$G^h := \frac{1}{2} g^{hr}(y^r \partial_r \dot{\partial}_i E - \partial_i E)$: the components of the canonical spray,
$N^h_i := \dot{\partial}_i \, G^h$: the components of the canonical nonlinear connection,
$G^h_{ij} := \dot{\partial}_j \, N^h_i = \dot{\partial}_j \dot{\partial}_i \, G^h$: the coefficients of Berwald connection,
$(\delta_i) := (\partial_i - N^h_i \, \dot{\partial}_h)$: the adapted bases of $H_u(TM)$,
$(\delta_i, \dot{\partial}_i)$: the bases of $T_u(TM) = H_u(TM) \oplus V_u(TM)$.

$\gamma^h_{ij} := \frac{1}{2} \, g^{h\ell}(\partial_i \, g_{\ell j} + \partial_j \, g_{i\ell} - \partial_\ell \, g_{ij})$,
$C^h_{ij} := \frac{1}{2} \, g^{h\ell}(\dot{\partial}_i \, g_{\ell j} + \dot{\partial}_j \, g_{i\ell} - \dot{\partial}_\ell \, g_{ij}) = \frac{1}{2} \, g^{h\ell} \, \dot{\partial}_i \, g_{\ell j}$,
$\Gamma^h_{ij} := \frac{1}{2} \, g^{h\ell}(\delta_i \, g_{\ell j} + \delta_j \, g_{i\ell} - \delta_\ell \, g_{ij})$: the coefficients of Caratn connection.

The bundle moriphisms γ and ρ are acting on the bases as follows:

$$\gamma(\overline{\partial}_i) = \dot{\partial}_i, \quad \rho(\partial_i) = \overline{\partial}_i, \quad \rho(\dot{\partial}_i) = 0.$$

The local formula of the almost tangent structure J is given by $J = \dot{\partial}_i \otimes dx^i$ and we have

$$J(\partial_i) = \dot{\partial}_i, \quad J(\dot{\partial}_i) = 0.$$

The horizontal and vertical projections are given, locally, by

$$h = dx^i \otimes \partial_i - N^i_j \, dx^j \otimes \dot{\partial}_i, \quad v = dy^i \otimes \dot{\partial}_i + N^i_j \, dx^j \otimes \dot{\partial}_i.$$

Moreover, we have

$$h(\partial_i) = \delta_i, \quad h(\dot{\partial}_i) = 0, \quad v(\partial_i) = 0, \quad v(\dot{\partial}_i) = \dot{\partial}_i.$$

The local formula of F can be found in [4, Eq. (I.70)], moreover, we have

$$F(\partial_i) = N^j_i \partial_j - N^h_i N^j_h \dot{\partial}_j - \dot{\partial}_i, \quad F(\dot{\partial}_i) = \delta_i, \quad F(\delta_i) = -\dot{\partial}_i.$$

Table 2: A comparison between the fundamental connections in KG- and PB-approachs in Finsler geoemtry

The object	**KG-approach**	**PB-approach**
The Universe	$(T\mathcal{T}M, \pi_{\mathcal{T}M}, \mathcal{T}M)$	$(\pi^{-1}(\mathcal{T}M), P, \mathcal{T}M)$
Fibers	$T_u\mathcal{T}M,\, u \in \mathcal{T}M$	$\{v\} \times T_xM,\, \pi(v) = x$
Berwald connection $\overset{\circ}{D}$	$\overset{\circ}{D}_{\dot\partial_j}\dot\partial_i = 0,$ $\overset{\circ}{D}_{\delta_j}\dot\partial_i = G^k_{ij}\dot\partial_k,$ $\overset{\circ}{D}_{\dot\partial_j}\delta_i = 0,$ $\overset{\circ}{D}_{\delta_j}\delta_i = G^k_{ij}\delta_k.$	$\overset{\circ}{D}_{\dot\partial_j}\overline{\partial}_i = 0,$ $\overset{\circ}{D}_{\delta_j}\overline{\partial}_i = G^k_{ij}\overline{\partial}_k.$
Cartan connection D	$D_{\dot\partial_j}\dot\partial_i = C^k_{ij}\dot\partial_k,$ $D_{\delta_j}\dot\partial_i = \Gamma^k_{ij}\dot\partial_k,$ $D_{\dot\partial_j}\delta_i = C^k_{ij}\delta_k,$ $D_{\delta_j}\delta_i = \Gamma^k_{ij}\delta_k.$	$D_{\dot\partial_j}\overline{\partial}_i = C^k_{ij}\overline{\partial}_k,$ $D_{\delta_j}\overline{\partial}_i = \Gamma^k_{ij}\overline{\partial}_k.$
Chern connection $\overset{*}{D}$	$\overset{*}{D}_{\dot\partial_j}\dot\partial_i = 0,$ $\overset{*}{D}_{\delta_j}\dot\partial_i = \Gamma^k_{ij}\dot\partial_k,$ $\overset{*}{D}_{\dot\partial_j}\delta_i = 0,$ $\overset{*}{D}_{\delta_j}\delta_i = \Gamma^k_{ij}\delta_k.$	$\overset{*}{D}_{\dot\partial_j}\overline{\partial}_i = 0,$ $\overset{*}{D}_{\delta_j}\overline{\partial}_i = \Gamma^k_{ij}\overline{\partial}_k.$
Hashiguchi connection $\overset{\star}{D}$	$\overset{\star}{D}_{\dot\partial_j}\dot\partial_i = C^k_{ij}\dot\partial_k,$ $\overset{\star}{D}_{\delta_j}\dot\partial_i = G^k_{ij}\dot\partial_k,$ $\overset{\star}{D}_{\dot\partial_j}\delta_i = C^k_{ij}\delta_k,$ $\overset{\star}{D}_{\delta_j}\delta_i = G^k_{ij}\delta_k.$	$\overset{\star}{D}_{\dot\partial_j}\overline{\partial}_i = C^k_{ij}\overline{\partial}_k,$ $\overset{\star}{D}_{\delta_j}\overline{\partial}_i = G^k_{ij}\overline{\partial}_k.$

Local formulas of the curvature tensors

The curvature tensors associated with the fundamental connections on the double tangent bundle annihilate vertical vectors. Consequently, their non-trivial components are confined to the horizontal bundle, as illustrated by the Berwald h-curvature:

$$\overset{\circ}{R}(\delta_i, \dot\partial_j)\delta_k = 0, \quad \overset{\circ}{R}(\delta_i, \dot\partial_j)\dot\partial_k = 0, \quad \overset{\circ}{R}(\delta_i, \delta_j)\delta_k = R^{\circ}{}^h_{ijk}\dot\partial_h.$$

In what follow, we list the formulas of the components of the torsion and curvature tensors of the four fundamental tensors on the double tangent bundle.

For the Berwald connection, we have:
$(v)h$-torsion: $R^{\circ}{}^i_{jk} = \delta_k G^i_j - \delta_j G^i_k,$
h-curvature: $R^{\circ}{}^i_{hjk} = \mathfrak{A}_{jk}\{\delta_k G^i_{hj} + G^m_{hj}G^i_{mk}\}$, where $\mathfrak{A}_{jk}X_{jk} = X_{jk} - X_{kj}$,
v-curvature: $S^{\circ}{}^i_{hjk} \equiv 0,$
hv-curvature: $P^{\circ}{}^i_{hjk} = \dot\partial_k G^i_{hj} =: G^i_{hjk}.$

For the Cartan connection, we have:
$(v)h$-torsion: $R^i_{jk} = \delta_k G^i_j - \delta_j G^i_k = R^{\circ}{}^i_{jk},$
$(v)hv$-torsion: $P^i_{jk} = G^i_{jk} - \Gamma^i_{jk},$
$(h)hv$-torsion: $C^i_{jk} = \frac{1}{2}g^{ri}\dot\partial_r g_{jk},$
h-curvature: $R^i_{hjk} = \mathfrak{A}_{jk}\{\delta_k\Gamma^i_{hj} + \Gamma^m_{hj}\Gamma^i_{mk}\} - C^i_{hm}R^m_{jk},$
hv-curvature: $P^i_{hjk} = \dot\partial_k\Gamma^i_{hj} - C^i_{hk|j} + C^i_{hm}P^m_{jk}$, where the symbol "|" refers to the horizontal covarient derivative with regard the Cartan connection,
v-curvature: $S^i_{hjk} = C^m_{hk}C^i_{mj} - C^m_{hj}C^i_{mk} = \mathfrak{A}_{jk}\{C^m_{hk}C^i_{mj}\}.$

For the Chern connection, we have:
$(v)h$-torsion: $R^{*}{}^i_{jk} = \delta_k\, G^i_j - \delta_j\, G^i_k = R^i_{jk},$

$(v)hv$-torsion: $P^{*i}_{jk} = G^i_{jk} - \Gamma^i_{jk}$,
h-curvature: $R^{*i}_{hjk} = \mathfrak{A}_{jk}\{\delta_k \Gamma^i_{hj} + \Gamma^m_{hj}\Gamma^i_{mk}\}$,
hv-curvature: $P^{*i}_{hjk} = \dot{\partial}_k \Gamma^i_{hj}$.

For the Hashiguchi connection, we have
$(v)h$-torsion $R^{\star i}_{jk} = \delta_k G^i_j - \delta_j G^i_k = R^i_{jk}$,
$(h)hv$-torsion $C^{\star i}_{jk} = 1/2\{g^{ri}\dot{\partial}_r g_{jk}\} = C^i_{jk}$,
h-curvature $R^{\star i}_{hjk} = R^{\circ i}_{hjk} + C^i_{hm} R^m_{jk}$,
hv-curvature $P^{\star i}_{hjk} = \dot{\partial}_k G^i_{hj} - C^i_{hk\overset{\star}{|}j}$, where the symbol "$\overset{\star}{|}$" refers to the horizontal covarient derivative with regard the Hashiguchi connection,

v-curvature $S^{\star i}_{hjk} = \mathfrak{A}_{jk}\{C^m_{hk} C^i_{mj}\} = S^i_{hjk}$.

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