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Multivalued $\mathcal{FG}$-contractive mappings on b-metric space with a graph structure

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Abstract In this work, we establish existence theorems for fixed points and best proximity points of multivalued $\mathcal{FG}$-contractive mappings. Our outcomes extend several recent results within the framework of complete b-metric spaces endowed with a graph. Additionally, we provide concrete examples demonstrating the role of graphs in these contractive conditions.

1 Introduction

The study of fixed points plays a key role in nonlinear analysis. Fixed point theorems focus on the existence of solutions to equations in the form of $\mathcal{S}(x) = x$, where $\mathcal{S}$ represents a mapping from a metric space $(\mathcal{W}, \omega)$ back to itself. If $\mathcal{S}$ is non-self mapping, then $\mathcal{S}$ may not possess a fixed point. Therefore, the minimization problem

$$\min_{x\in\mathcal{W}} \omega(x, \mathcal{S}(x))$$

yields the best proximity point of $\mathcal{S}$, representing the closest possible solution under the metric ω.

Recently, Wardowski [18] introduced the concept of $\mathcal{F}$-contraction. In 2016, Arvaneh et al. [13] introduced the concept of $\alpha\beta\mathcal{FG}$-contraction, which is a generalization of $\mathcal{F}$-contraction, and proved the existence of fixed points in b-metric spaces.

In 2008, Jachymski [10] extended the Ran-Reurings result to complete metric spaces endowed with a transitive directed graph. Indeed, the graph structure is another way to induce a partial order. The generalization of the Ran-Reurings fixed point theorems with a transitive binary relation can be found in H. Ben-El-Mechaiekh [5].

Acar [2] established fixed point theorems for multivalued $\mathcal{F}$-contraction mappings with a graph structure, and Kumar [11] established best proximity point theorems in partially ordered complete metric spaces for multivalued generalized contractions. Wangwe et al. [17] established fixed point theorems for multivalued non-self $\mathcal{F}$-contraction mappings in metrically convex partial metric spaces.

In this section, we present some fundamental definitions and notions.

Definition 1.1. *[8]* Let $\mathcal{W}$ be a nonempty set and $s \geq 1$ a real number. A mapping $\omega : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$ is said to be a b-metric on $\mathcal{W}$ if for all $x, y, z \in \mathcal{W}$, we have:

1. $\omega(x, y) = 0 \Leftrightarrow x = y$;
2. $\omega(x, y) = \omega(y, x)$;
3. $\omega(x, z) \leq s[\omega(x, y) + \omega(y, z)]$.

The pair $(\mathcal{W}, \omega)$ called a b-metric space.

Definition 1.2. *[7]* Let $(\mathcal{W}, \omega)$ be a b-metric space.

(i) We say that $\{x_n\}$ in $\mathcal{W}$ a b-convergent sequence if there exists $x \in \mathcal{W}$ such that $\omega(x_n, x) \to 0$ when $n \to \infty$ and we have $\lim_{n\to\infty} x_n = x$.

(ii) We also say $\{x_n\}$ in $\mathcal{W}$ a b-Cauchy if $\omega(x_n, x_m) \to 0$ when $n, m \to \infty$.

(iii) The b-metric space $(\mathcal{W}, \omega)$ is b-complete if every b-Cauchy sequence in $\mathcal{W}$ is b-convergent.

Let us recall some fundamental concepts and preliminary results to establish our existence theorems. We adopt standard notation from nonlinear analysis, where $\mathbb{N}$ denotes the set of positive integers. Let $(\mathcal{W}, \omega)$ be a b-metric space. We define the following collections of subsets of $\mathcal{W}$

- $P(\mathcal{W})$: denotes the collection of all nonempty subsets of $\mathcal{W}$;
- $CB(\mathcal{W})$: denotes the collection of all nonempty, closed, and bounded subsets of $\mathcal{W}$;
- $K(\mathcal{W})$: denotes the collection of all nonempty compact subsets of $\mathcal{W}$;

We define the Pompeiu-Hausdorff metric. $\mathcal{H} : CB(\mathcal{W}) \times CB(\mathcal{W}) \to [0, \infty)$ is defined by

$$\mathcal{H}(P, Q) = \max\{\sup_{x\in P} D(x, Q), \sup_{y\in Q} D(y, P)\}$$

such that $P,Q \in CB(\mathcal{W})$ and $D(x,P) = \inf_{y\in P} \omega(x,y)$. This $\mathcal{H}$ defines a valid metric on $CB(\mathcal{W})$. Further details about the metric $\mathcal{H}$ can be found in [4], [6].

A multivalued mapping $\mathcal{S} : \mathcal{W} \to CB(\mathcal{W})$ is called a contraction if there exists $k \in [0,1)$ such that for all $x,y \in \mathcal{W}$

$$\mathcal{H}(\mathcal{S}(x),\mathcal{S}(y)) \leq k\omega(x,y).$$

The proximal pair of nonempty subsets (P,Q) in a b-metric space $\mathcal{W},\omega$ is denoted by (P_0,Q_0) and defined as

$$P_0 = \{x \in P : \omega(x,y) = \omega(P,Q) \text{ for some } y \in Q\}$$
$$Q_0 = \{y \in Q : \omega(x,y) = \omega(P,Q) \text{ for some } x \in P\}$$

Definition 1.3. [10] Suppose we have two nonempty subsets P and Q within a b-metric space $(\mathcal{W},\omega)$ and $\mathcal{S} : P \to 2^Q$ represents a multivalued mapping. Then we denote a point $x \in \mathcal{W}$ as best proximity point for $\mathcal{S}$ if

$$D(x,\mathcal{S}(x)) = \omega(P,Q).$$

Definition 1.4. [15] Given a pair (P,Q) of $P(\mathcal{W})$ with P_0 being nonempty. Then the pair (P,Q) have the P-property if and only if

$$\left.\begin{array}{l}\omega(x_1,y_1) = \omega(P,Q)\\ \omega(x_2,y_2) = \omega(P,Q)\end{array}\right\} \Rightarrow \omega(x_1,x_2) = \omega(y_1,y_2),$$

where $x_1,x_2 \in P_0$ and $y_1,y_2 \in Q_0$.

Now, we recall some definitions of the graph G

Definition 1.5. [10] Let $\mathcal{W}$ be a non-empty set and Δ denote the diagonal of the Cartesian product $\mathcal{W} \times \mathcal{W}$. A metric space $(\mathcal{W},\omega)$ is considered to be equipped with a directed graph G if G is defined as $(V(G),E(G))$ where $V(G)$ is a nonempty set representing its vertices and $E(G) \subset V(G) \times V(G)$ is a nonempty set of directed edges. The edge set $E(G)$ is said to possess the **transitivity property** if whenever $[(x,y) \in E(G)$ and $(y,z) \in E(G)] \Rightarrow (x,z) \in E(G)$, for any $x,y,z \in V(G)$.

For more details on graph theory see [12].

Definition 1.6. [2] Let $(\mathcal{W},\omega)$ be a b-metric space endowed with a graph G such that $V(G) = \mathcal{W}$. A multivalued mapping $\mathcal{S} : \mathcal{W} \to CB(\mathcal{W})$ is said to be a weakly graph preserving (WGP) property if whenever for each $x \in \mathcal{W}$, $y \in \mathcal{S}(x)$ and $(x,y) \in E(G)$ implies that $(y,z) \in E(G)$ for all $z \in \mathcal{S}(y)$.

In this work, we establish the concept of $\mathcal{FG}$-contraction and derive several fixed point and best proximity points results within b-metric spaces endowed with a transitive graph structure G.

Definition 1.7. [13] We denote $\Delta_{\mathcal{F}}$ the set of all functions $\mathcal{F} : \mathbb{R}_+ \to \mathbb{R}$ such that

(i) $\mathcal{F}$ is continuous and strictly non-decreasing;

(ii) for each sequence $\{x_n\} \subseteq \mathbb{R}_+$, $\lim_{n\to\infty} x_n = 0$ if and only if $\lim_{n\to\infty} \mathcal{F}(x_n) = -\infty$.

Note that condition (3) from [18, 19] will not be used.

Definition 1.8. [13] We denote $\Delta_{\mathcal{G},\phi}$ the set of pairs of functions $(\mathcal{G},\phi)$,

$$\mathcal{G} : \mathbb{R}_+ \to \mathbb{R}, \quad \phi : \mathbb{R}_+ \to [0,1),$$

such that

(i) for each sequence $\{x_n\} \subseteq \mathbb{R}_+$, $\limsup_{n\to\infty} \mathcal{G}(x_n) \geq 0$ if and only if $\limsup_{n\to\infty} x_n \geq 1$.

(ii) for each sequence $\{x_n\} \subseteq \mathbb{R}_+$, $\limsup_{n\to\infty} \phi(x_n) = 1$ implies $\lim_{n\to\infty} x_n = 0$;

(iii) for each sequence $\{x_n\} \subseteq \mathbb{R}_+$, $\sum_{n\geq 1} \mathcal{G}(\phi(x_n)) = -\infty$.

Example 1.9. Let $\mathcal{F}(\lambda) = \begin{cases} \frac{-1}{\sin(\lambda)} & \lambda \in (0,\pi/2] \\ \frac{2}{\pi}\lambda - 2 & \lambda \in [\pi/2,+\infty). \end{cases}$, $\mathcal{G}(\lambda) = \ln(\lambda)$, $\phi(\lambda) = \exp\left(\frac{-\lambda}{s^2}\right)$ for $\lambda > 0$ and $\phi(0) = 0$. Then $\mathcal{F} \in \Delta_{\mathcal{F}}$ and $(\mathcal{G},\phi) \in \Delta_{\mathcal{G},\phi}$. We choose the function $\mathcal{F}$ so that the condition (3) from [18, 19] is not verified.

Lemma 1.10. *[2] Consider a b-metric space $(\mathcal{W},\omega)$ and an upper semi-continuous mapping $\mathcal{S} : \mathcal{W} \to P(\mathcal{W})$, such that the set $\mathcal{S}(x)$ is closed for all $x \in \mathcal{W}$. If $x_n \to x_0$, $y_n \to y_0$, and $y_n \in \mathcal{S}(x_n)$, then $y_0 \in \mathcal{S}(x_0)$.*

2 Main results

Consider G as a directed graph on $(\mathcal{W},\omega)$, where $\mathcal{S} : \mathcal{W} \to CB(\mathcal{W})$ is a mapping. Define

$$A_G = \{(x,y) \in E(G) : \mathcal{H}(\mathcal{S}(x),\mathcal{S}(y)) > 0\}, \tag{2.1}$$

and

$$W_A = \{x \in \mathcal{W} : (x,y) \in E(G) \text{ for some } y \in \mathcal{S}(x)\}, \tag{2.2}$$

Definition 2.1. Consider G as a directed graph on $(\mathcal{W},\omega)$ with parameter $s \geq 1$ and let $\mathcal{S}:\mathcal{W}\to CB(\mathcal{W})$ represent a multivalued mapping. We say that $\mathcal{S}$ is a $\mathcal{FG}$-contraction type 1 if there are $\mathcal{F}\in\Delta_{\mathcal{F}}, (\mathcal{G},\phi)\in\Delta_{\mathcal{G},\phi}$ such that for all $(x,y)\in A_G$, we have

$$\mathcal{F}\left(s\mathcal{H}(\mathcal{S}(x),\mathcal{S}(y))\right)\leq\mathcal{F}\left(\mathcal{M}_s(x,y)\right)+\mathcal{G}\left(\phi(\mathcal{M}_s(x,y))\right), \tag{2.3}$$

such that

$$\mathcal{M}_s(x,y)=\max\left\{\omega(x,y),D(x,\mathcal{S}(x)),D(y,\mathcal{S}(y)),\frac{D(x,\mathcal{S}(y))+D(y,\mathcal{S}(x))}{2s}\right\}. \tag{2.4}$$

Theorem 2.2. *Consider a complete b-metric space $(\mathcal{W},\omega)$ endowed with a transitive directed graph G. $\mathcal{S}:\mathcal{W}\to K(\mathcal{W})$ represents a multivalued $\mathcal{FG}$-contraction type 1. Suppose that the set W_A is nonempty, $\mathcal{S}$ is weakly graph preserving and upper semi-continuous map, then $\mathcal{S}$ possess a fixed point.*

Proof. Suppose that $\mathcal{S}$ did not possess a fixed point, then $D(x,\mathcal{S}(x))>0, \forall\, x\in\mathcal{W}$. Let x_0 be a given element in W_A. Then, $(x_0,x_1)\in E(G)$ for some $x_1\in\mathcal{S}(x_0)$. Then, we obtain

$$0<D(x_1,\mathcal{S}(x_1))\leq\mathcal{H}(\mathcal{S}(x_0),\mathcal{S}(x_1)).$$

Using condition (2.3) and the fact $(x_0,x_1)\in A_G$, we get

$$\mathcal{F}(D(x_1,\mathcal{S}(x_1)))\leq\mathcal{F}(s\mathcal{H}(\mathcal{S}(x_0),\mathcal{S}(x_1)))\leq\mathcal{F}(\mathcal{M}_s(x_0,x_1))+\mathcal{G}\left(\phi\left(\mathcal{M}_s(x_0,x_1)\right)\right),$$

where

$$\begin{aligned}\mathcal{M}_s(x_0,x_1) &= \max\left\{\omega(x_0,x_1),D(x_0,\mathcal{S}(x_0)),D(x_1,\mathcal{S}(x_1)),\tfrac{D(x_0,\mathcal{S}(x_1))+D(x_1,\mathcal{S}(x_0))}{2s}\right\}\\ &= \max\left\{\omega(x_0,x_1),D(x_1,\mathcal{S}(x_1)),\frac{D(x_0,\mathcal{S}(x_1))}{2s}\right\}\\ &\leq \max\left\{\omega(x_0,x_1),D(x_1,\mathcal{S}(x_1)),\frac{s\omega(x_0,x_1)+sD(x_1,\mathcal{S}(x_1))}{2s}\right\}\\ &= \max\left\{\omega(x_0,x_1),D(x_1,\mathcal{S}(x_1)),\frac{\omega(x_0,x_1)+D(x_1,\mathcal{S}(x_1))}{2}\right\}\\ &= \max\left\{\omega(x_0,x_1),D(x_1,\mathcal{S}(x_1))\right\},\end{aligned}$$

which implies

$$\mathcal{F}\Big(\mathcal{H}\big(\mathcal{S}(x_0),\mathcal{S}(x_1)\big)\Big)\leq\mathcal{F}\Big(\max\left\{\omega(x_0,x_1),D(x_1,\mathcal{S}(x_1))\right\}\Big)+\mathcal{G}\Big(\phi\big(\mathcal{M}_s(x_0,x_1)\big)\Big).$$

If

$$\max\left\{\omega(x_0,x_1),D(x_1,\mathcal{S}(x_1))\right\}=D(x_1,\mathcal{S}(x_1)).$$

So we have

$$\mathcal{F}\Big(D(x_1,\mathcal{S}(x_1))\Big)\leq\mathcal{F}\Big(s\mathcal{H}\big(\mathcal{S}(x_0),\mathcal{S}(x_1)\big)\Big)\leq\mathcal{F}\Big(D(x_1,\mathcal{S}(x_1))\Big)+\mathcal{G}\Big(\phi\big(D(x_1,\mathcal{S}(x_1))\big)\Big).$$

So $G\Big(\phi\big(D(x_1,\mathcal{S}(x_1))\big)\Big)\geq 0$, so $\phi\big(D(x_1,\mathcal{S}(x_1))\big)\geq 1$, this leads to a contradiction with the definition of ϕ. Therefore,

$$\max\left\{\omega(x_0,x_1),D(x_1,\mathcal{S}(x_1))\right\}=\omega(x_0,x_1).$$

And we have

$$\mathcal{F}\Big(s\mathcal{H}\big(\mathcal{S}(x_0),\mathcal{S}(x_1)\big)\Big)\leq\mathcal{F}\Big(\omega(x_0,x_1)\Big)+\mathcal{G}\Big(\phi\big(\omega(x_0,x_1)\big)\Big).$$

Because of the compactness of $\mathcal{S}(x_1)$, there exists $x_2\in\mathcal{S}(x_1)$ such that $\omega(x_1,x_2)=D(x_1,\mathcal{S}(x_1))$. So, we get

$$\mathcal{F}\Big(\omega(x_1,x_2)\Big)\leq\mathcal{F}\Big(\omega(x_0,x_1)\Big)+\mathcal{G}\Big(\phi\big(\mathcal{M}_s(x_0,x_1)\big)\Big).$$

Since $(x_0,x_1)\in E(G), x_1\in\mathcal{S}(x_0)$, and $x_2\in\mathcal{S}(x_1)$, by the WGP property, we can write $(x_1,x_2)\in E(G)$. In view of $0<D(x_2,\mathcal{S}(x_2))\leq\mathcal{H}(\mathcal{S}(x_1),\mathcal{S}(x_2))$, we obtain that $(x_1,x_2)\in A_G$. Then,

$$\mathcal{F}(D(x_2,\mathcal{S}(x_2)))\leq\mathcal{F}(s\mathcal{H}(\mathcal{S}(x_1),\mathcal{S}(x_2)))\leq\mathcal{F}(\mathcal{M}_s(x_1,x_2))+\mathcal{G}(\phi(\mathcal{M}_s(x_1,x_2))). \tag{2.5}$$

By following a similar way, we can obtain

$$\mathcal{M}_s(x_1,x_2)\leq\max\{\omega(x_1,x_2),D(x_2,\mathcal{S}(x_2))\}.$$

By (2.5), we obtain

$$\mathcal{F}(D(x_2,\mathcal{S}(x_2)))\leq\mathcal{F}(\omega(x_1,x_2))+\mathcal{G}(\phi(\omega(x_1,x_2))).$$

Once more, the compactness of $\mathcal{S}(x_2)$ implies the existence of $x_3\in\mathcal{S}(x_2)$ such that $\omega(x_2,x_3)=D(x_2,\mathcal{S}(x_2))$. We get

$$\mathcal{F}(\omega(x_2,x_3))\leq\mathcal{F}(\omega(x_1,x_2))+\mathcal{G}(\phi(\omega(x_1,x_2))).$$

In the same way, we construct the sequence $\{x_n\}$ in $\mathcal{W}$ where $x_{n+1}\in\mathcal{S}(x_n)$, $(x_n,x_{n+1})\in A_G$ and

$$\mathcal{F}(\omega(x_n,x_{n+1}))\leq\mathcal{F}(\omega(x_{n-1},x_n))+\mathcal{G}(\phi(\omega(x_{n-1},x_n))),\ \text{for all } n\in\mathbb{N}.$$

Consequently, we deduce that

$$\begin{aligned}\mathcal{F}(\omega(x_n,x_{n+1})) &\leq \mathcal{F}(\omega(x_{n-2},x_{n-1}))+\mathcal{G}(\phi(\omega(x_{n-2},x_{n-1})))\\ &\quad +\mathcal{G}(\phi(\omega(x_{n-1},x_n))).\end{aligned}$$

Continuing in this manner, we obtain

$$\mathcal{F}(\omega(x_n,x_{n+1}))\leq\mathcal{F}(\omega(x_0,x_1))+\sum_{i=1}^{n}\mathcal{G}(\phi(\omega(x_{i-1},x_i))). \tag{2.6}$$

Taking the limit as $n \to \infty$ in the precedent inequality (2.6), as $(\mathcal{G}, \phi) \in \Delta_{\mathcal{G},\phi}$, we get

$$\lim_{n\to\infty} \mathcal{F}(\omega(x_n, x_{n+1})) = -\infty,$$

and because, $\mathcal{F} \in \Delta_F$ we get

$$\lim_{n\to\infty} \omega(x_n, x_{n+1}) = 0. \tag{2.7}$$

We will demonstrate that $\{x_n\}$ is a b-Cauchy sequence. Assume the contrary that $\{x_n\}$ is not a b-cauchy sequence. Then, there is $\varepsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $m_k > n_k > k$ such that

$$\omega(x_{m_k}, x_{n_k}) \geq \varepsilon. \tag{2.8}$$

Additionally, for each m_k, we may choose n_k so that it is the smallest integer satisfying the last inequality and $m_k > n_k > k$. Then we have

$$\omega(x_{m_k}, x_{n_k-1}) < \varepsilon. \tag{2.9}$$

We use the triangle inequality, and we obtain

$$\varepsilon \leq \omega(x_{m_k}, x_{n_k}) \leq s\omega(x_{m_k}, x_{m_k+1}) + s\omega(x_{m_k+1}, x_{n_k}).$$

by taking the upper limit as $k \to \infty$, we obtain

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} \omega(x_{m_k+1}, x_{n_k}). \tag{2.10}$$

Also, from (2.9),

$$\limsup_{k\to\infty} \omega(x_{m_k}, x_{n_k-1}) \leq \varepsilon. \tag{2.11}$$

On the other hand, we have

$$\omega(x_{m_k}, x_{n_k}) \leq s\omega(x_{m_k}, x_{n_k-1}) + s\omega(x_{n_k-1}, x_{n_k}).$$

Using (2.7), (2.11) and taking the upper limit as $k \to \infty$, we obtain

$$\limsup_{k\to\infty} \omega(x_{m_k}, x_{n_k}) \leq s\varepsilon. \tag{2.12}$$

Again, using the triangular inequality, we have

$$\omega(x_{m_k+1}, x_{n_k-1}) \leq s\omega(x_{m_k+1}, x_{m_k}) + s\omega(x_{m_k}, x_{n_k-1}). \tag{2.13}$$

By taking the upper limit as $k \to \infty$ in (2.13), using (2.7) and (2.11), we obtain

$$\limsup_{k\to\infty} \omega(x_{m_k+1}, x_{n_k-1}) \leq s\varepsilon. \tag{2.14}$$

Since $(x_{n_k-1}, x_{n_k}), (x_{n_k}, x_{n_k+1}), (x_{n_k+1}, x_{n_k+2}), \cdots, (x_{m_k}, x_{m_{k+1}}) \in E(G)$ and a digraph G is transitive then (x_{n_k-1}, x_{m_k}) and $(x_{n_k}, x_{m_k+1}) \in E(G)$,
from (2.10) $\omega(x_{n_k}, x_{m_k+1}) > 0$ therefor we can utilize (2.3) to deduce that

$$\begin{aligned} \mathcal{F}(s\omega(x_{m_k+1}, x_{n_k})) &\leq \mathcal{F}(s\mathcal{H}(\mathcal{S}(x_{m_k}), \mathcal{S}(x_{n_k-1}))) \\ &\leq \mathcal{F}(\mathcal{M}_s(x_{m_k}, x_{n_k-1})) + \mathcal{G}\left(\phi\left(\mathcal{M}_s(x_{m_k}, x_{n_k-1})\right)\right) \end{aligned} \tag{2.15}$$

where,

$$\begin{aligned} &\mathcal{M}_s(x_{m_k}, x_{n_k-1}) \\ &= \max\Big\{\omega(x_{m_k}, x_{n_k-1}), D(x_{m_k}, \mathcal{S}(x_{m_k})), D(x_{n_k-1}, \mathcal{S}(x_{n_k-1})), \\ &\qquad \frac{D(x_{m_k}, \mathcal{S}(x_{n_k-1})) + D(\mathcal{S}(x_{m_k}), x_{n_k-1})}{2s}\Big\} \\ &= \max\Big\{\omega(x_{m_k}, x_{n_k-1}), \omega(x_{m_k}, x_{m_k+1}), \omega(x_{n_k-1}, x_{n_k}), \\ &\qquad \frac{\omega(x_{m_k}, x_{n_k}) + \omega(x_{m_k+1}, x_{n_k-1})}{2s}\Big\} \end{aligned} \tag{2.16}$$

Now, considering the upper limit as $k \to \infty$ in (2.15), using (2.10), (2.11), (2.14) and (2.16), we get

$$\begin{aligned} \mathcal{F}(s \cdot \frac{\varepsilon}{s}) &\leq \mathcal{F}\left(s \limsup_{k\to\infty} \omega(x_{m_k+1}, x_{n_k})\right) \\ &\leq \limsup_{k\to\infty} \mathcal{F}(\mathcal{M}_s(x_{m_k}, x_{n_k-1})) + \limsup_{k\to\infty} \mathcal{G}\left(\phi\left(\mathcal{M}_s(x_{m_k}, x_{n_k-1})\right)\right) \\ &\leq \mathcal{F}(\varepsilon) + \limsup_{k\to\infty} \mathcal{G}\left(\phi\left(\mathcal{M}_s(x_{m_k}, x_{n_k-1})\right)\right). \end{aligned}$$

This also implies that

$$\limsup_{k\to\infty} \mathcal{G}\left(\phi\left(\mathcal{M}_s(x_{m_k}, x_{n_k-1})\right)\right) \geq 0.$$

This leads to $\limsup_{k\to\infty} \phi\left(\mathcal{M}_s(x_{m_k}, x_{n_k-1})\right) \geq 1$, and since $\phi(\lambda) < 1$ for all $\lambda \geq 0$, we get

$$\limsup_{k\to\infty} \phi\left(\mathcal{M}_s(x_{m_k}, x_{n_k-1})\right) = 1.$$

Therefore,

$$\limsup_{k\to\infty} \mathcal{M}_s(x_{m_k}, x_{n_k-1}) = 0,$$

which is impossible because (2.8) and (2.16). Thus, we demonstrate the convergence of the sequence $\{x_n\}$ as a b-Cauchy sequence in the b-complete metric space $(\mathcal{W}, \omega)$. Consequently, $\{x_n\}$ b-converges to some $x \in \mathcal{W}$. By leveraging the upper semi-continuity of $\mathcal{S}$ and Lemma 1.10, we infer that $x \in \mathcal{S}(x)$, which contradicts our assumption. Therefore, $\mathcal{S}$ possesses a fixed point. □

Example 2.3. Let $\mathcal{W} = \left\{ \frac{1}{2^n} \cup \{0\}, n \in \mathbb{N} \right\}$. Define the b-metric $\omega(x, y) = (x - y)^2$ with $s = 2$. Consider a graph given by $V(G) = \mathcal{W}$, and

$$E(G) = \left\{ \left(\frac{1}{2^n}, \frac{1}{2^m} \right), n < m, (n, m) \in \mathbb{N} \times \mathbb{N} \right\}.$$

Let $\mathcal{F}(\lambda) = -\frac{1}{\sqrt{\lambda}}, \mathcal{G}(\lambda) = \ln(\lambda), \phi(\lambda) = e^{\frac{-\lambda}{2}}$ and $\phi(0) = 0$. Define $\mathcal{S} : \mathcal{W} \to K(\mathcal{W})$ by

$$\mathcal{S}(x) = \begin{cases} \left\{ \frac{1}{2^{2n+1}}, \frac{1}{2^{2n+2}} \right\} & \text{if } x = \frac{1}{2^n}, n \in \mathbb{N} \\ \{0\} & \text{if } x = 0 \end{cases}$$

Then it can be seen that $\mathcal{S}$ is WGP and upper semi-continuous mapping and $E(G)$ is a transitive directed graph.

$$\mathcal{M}_s(x, y) = \max \left\{ (x - y)^2, \left(x - \frac{x^2}{2} \right)^2 \right\},$$

for these choices of $E, \mathcal{S}, \omega$, all conditions of Theorem 2.2 are fulfilled and we have

$$\mathcal{H}(\mathcal{S}(x), \mathcal{S}(y)) \leq \frac{2\mathcal{M}_s(x, y)}{\left(2 + (\mathcal{M}_s(x, y))^{\frac{3}{2}} \right)^2}.$$

Then $\mathcal{S}$ is a multivalued $\mathcal{FG}$-contraction mapping. So $\mathcal{S}$ possesses a fixed point which is 0.

Taking $\mathcal{G}(t) = \ln t, \phi(t) = k$ and putting $\sigma = -\ln k$ where $k \in (0, 1)$, in Theorem 2.2, we obtain a generalization of our results.

Corollary 2.4. *Let $(\mathcal{W}, \omega)$ be a complete b-metric space with parameter $s \geq 1$ endowed with a transitive directed graph G and let $\mathcal{F} \in \Delta_{\mathcal{F}}$. Let $\mathcal{S} : \mathcal{W} \to K(\mathcal{W})$ be a multivalued mapping. If the set W_A is nonempty and $\mathcal{S}$ satisfies the conditions :*

(i) $\mathcal{S}$ is upper semi-continuous;

(ii) for some $\sigma > 0$ and for all $x, y \in W_A$ and $\mathcal{H}(\mathcal{S}(x), \mathcal{S}(y)) > 0$, where $\mathcal{M}_s$ is defined by (2.4) we have

$$\sigma + \mathcal{F}(s\mathcal{H}(\mathcal{S}(x), \mathcal{S}(y))) \leq \mathcal{F}(\mathcal{M}_s(x, y)).$$

Then $\mathcal{S}$ possesses a fixed point.

Definition 2.5. Consider G as a directed graph on a complete b-metric space $(\mathcal{W}, \omega)$ with parameter $s \geq 1$, let P and Q be two non-empty subsets of $(\mathcal{W}, \omega)$. A mapping $\mathcal{S} : P \to CB(Q)$ is a multivalued $\mathcal{FG}$-contraction type 2 if for all $x, y \in P, x \neq y$ with $(x, y) \in E(G)$:

$$\mathcal{F}\left(s\mathcal{H}(\mathcal{S}(x), \mathcal{S}(y))\right) \leq \mathcal{F}\left(\mathcal{N}_s(x, y) - \omega(P, Q)\right) + \mathcal{G}\left(\phi(\mathcal{N}_s(x, y))\right). \tag{2.17}$$

where $N_s(x, y) = \max \left\{ \omega(x, y), \frac{1}{s} D(x, \mathcal{S}(x)), \frac{1}{s} D(y, \mathcal{S}(y)), \frac{D(x, \mathcal{S}(y)) + D(y, \mathcal{S}(x))}{2s} \right\}$ for all $x, y \in P$.

Theorem 2.6. *Consider a complete b-metric space $(\mathcal{W}, \omega)$ with parameter $s \geq 1$ endowed with a transitive directed graph G. Let P, Q be two non-empty closed subsets of $(\mathcal{W}, \omega)$ where (P, Q) has the P-property. We define $\mathcal{S} : P \to CB(Q)$ a multivalued $\mathcal{FG}$-contraction type 2 such that $\mathcal{S}(P_0) \subseteq Q_0$. Suppose that the following conditions hold:*

1. *there exist two elements x_0, x_1 in P_0 and $y_0 \in \mathcal{S}(x_0)$ such that $\omega(x_1, y_0) = \omega(P, Q)$ and $(x_0, x_1) \in E(G)$;*
2. *for all $x, y \in P_0$, $(x, y) \in E(G)$ implies $\mathcal{S}(x) \subseteq \mathcal{S}(y)$;*
3. *$\mathcal{S}$ is upper semi-continuous.*

Hence there exists an element x in P where $D(x, \mathcal{S}(x)) = \omega(P, Q)$.

Proof. Let $x_0, x_1 \in P_0$. So according to condition (1), there exist $y_0 \in \mathcal{S}(x_0)$ such that $(x_0, x_1) \in E(G)$ and $\omega(x_1, y_0) = \omega(P, Q)$. Now from condition (2), $\mathcal{S}(x_0) \subseteq \mathcal{S}(x_1)$. So, there exist $y_1 \in \mathcal{S}(x_1)$ such that $(x_1, x_2) \in E(G)$ and $\omega(x_2, y_1) = \omega(P, Q)$. By continuing this process, we derive a sequence $\{x_n\}_{n \geq 0}$ in P_0 and $y_n \in \mathcal{S}(x_n)$ for any $n \geq 1$ such that $\omega(x_{n+1}, y_n) = \omega(P, Q)$, hence we obtain

$$\omega(x_{n+1}, y_n) = D(x_{n+1}, \mathcal{S}(x_n)) = \omega(P, Q)\ \forall n \in \mathbb{N}, \tag{2.18}$$

where $(x_n, x_{n+1}) \in E(G)\ \forall n \in \mathbb{N}$. If for some $n_0 \in \mathbb{N}$, $x_{n_0} = x_{n_0+1}$, then

$$\omega(x_{n_0+1}, y_{n_0}) = D(x_{n_0}, \mathcal{S}(x_{n_0})) = \omega(P, Q),$$

So $x_{n_0} \in P_0$ would be a best proximity point of the mapping $\mathcal{S}$, and we are finished. Assume that $x_n \neq x_{n+1}\ \forall n \in \mathbb{N}$. Since $\omega(x_{n+1}, y_n) = \omega(P, Q)$ and $\omega(x_n, y_{n-1}) = \omega(P, Q)$ has the P-property

$$\omega(x_n, x_{n+1}) = \omega(y_{n-1}, y_n)\ \forall n \in \mathbb{N}. \tag{2.19}$$

Because $(x_{n-1}, x_n) \in E(G)$, so

$$\begin{aligned} \mathcal{F}(\omega(x_n, x_{n+1})) &= \mathcal{F}(\omega(y_{n-1}, y_n)) \\ &\leq \mathcal{F}(s\mathcal{H}(\mathcal{S}(x_{n-1}), \mathcal{S}(x_n)) \\ &\leq \mathcal{F}\left(\mathcal{N}_s(x_{n-1}, x_n) - \omega(P, Q)\right) + \mathcal{G}\left(\phi(\mathcal{N}_s(x_{n-1}, x_n))\right) \end{aligned} \tag{2.20}$$

$$
\begin{aligned}
&\mathcal{N}_s(x_{n-1}, x_n)\\
&= \max\left\{\omega(x_{n-1}, x_n), \frac{1}{s}D(x_{n-1}, \mathcal{S}(x_{n-1})), \frac{1}{s}D(x_n, \mathcal{S}(x_n)), \frac{D(x_{n-1}, \mathcal{S}(x_n)) + D(x_n, \mathcal{S}(x_{n-1}))}{2s}\right\}\\
&\leq \max\left\{\omega(x_{n-1}, x_n), \frac{1}{s}\omega(x_{n-1}, y_{n-1}), \frac{1}{s}\omega(x_n, y_n), \frac{\omega(x_{n-1}, y_n) + \omega(x_n, y_{n-1})}{2s}\right\}\\
&\leq \max\{\omega(x_{n-1}, x_n), \omega(x_{n-1}, y_{n-2}) + \omega(y_{n-2}, y_{n-1}), \omega(x_n, y_{n-1}) +\\
&\qquad \omega(y_{n-1}, y_n) + \frac{\omega(x_{n-1}, y_{n-1}) + \omega(y_{n-1}, y_n) + \omega(x_n, y_n) + \omega(y_n, y_{n-1})}{2}\Big\}\\
&\leq \max\{\omega(x_{n-1}, x_n), \omega(P, Q) + \omega(x_{n-1}, x_n), \omega(P, Q) + \omega(x_n, x_{n+1}),\\
&\qquad \frac{\omega(P, Q) + \omega(x_{n-1}, x_n) + \omega(P, Q) + \omega(x_n, x_{n+1})}{2}\Big\}\\
&\leq \max\{\omega(P, Q) + \omega(x_{n-1}, x_n), \omega(P, Q) + \omega(x_n, x_{n+1})\}.
\end{aligned}
$$

If $\omega(x_n, x_{n+1}) > \omega(x_{n-1}, x_n)$, from (2.20) we get

$$
\begin{aligned}
\mathcal{F}(\omega(x_n, x_{n+1})) &\leq \mathcal{F}(\omega(P, Q) + \omega(x_n, x_{n+1}) - \omega(P, Q)) + \mathcal{G}(\phi(\mathcal{N}_s(x_{n-1}, x_n)))\\
&= \mathcal{F}(\omega(x_n, x_{n+1})) + \mathcal{G}(\phi(\mathcal{N}_s(x_{n-1}, x_n))).
\end{aligned}
$$

So $\mathcal{G}(\phi(\mathcal{N}_s(x_{n-1}, x_n))) \geq 0$, which yields that $\phi(\mathcal{N}_s(x_{n-1}, x_n)) \geq 1$, a contradiction with definition of ϕ. Therefore,

$$\max\{\omega(x_{n-1}, x_n) + \omega(P, Q), \omega(x_n, x_{n+1}) + \omega(P, Q)\} = \omega(x_{n-1}, x_n) + \omega(P, Q).$$

So, we get

$$
\begin{aligned}
\mathcal{F}(\omega(x_n, x_{n-1}) &\leq \mathcal{F}(\omega(x_{n-1}, x_n)) + \mathcal{G}(\phi(\mathcal{N}_s(x_{n-1}, x_n)))\\
&\leq \mathcal{F}(\omega(x_{n-2}, x_{n-1})) + \mathcal{G}(\phi(\mathcal{N}_s(x_{n-2}, x_{n-1}))) + \mathcal{G}(\phi(\mathcal{N}_s(x_{n-1}, x_n)))\\
&\leq \mathcal{F}(\omega(x_0, x_1)) + \sum_{i=1}^{n} \mathcal{G}(\phi(\mathcal{N}_s(x_{i-1}, x_i))).
\end{aligned}
$$

Proceeding as in Theorem 2.2, we can proved the convergence of the sequence $\{x_n\}$ as a b-Cauchy sequence in the b-complete metric space (P, ω), then there exist $x \in P$ such that

$$\lim_{n\to\infty} x_n = x.$$

Since $\omega(x_n, x_{n+1}) = \omega(y_{n-1}, y_n)$. The sequence $\{y_n\}$ in Q is b-Cauchy and then is convergent. Assume that $y_n \to y$. By the relation $\omega(x_{n+1}, y_n) = \omega(P, Q)$ for all n. By leveraging the upper semi-continuity of $\mathcal{S}$ and Lemma 1.10, we obtain $y \in \mathcal{S}(x)$. Thus $\omega(x, y) = D(x, \mathcal{S}(x)) = \omega(P, Q)$. Then x is a best proximity point of $\mathcal{S}$. □

Example 2.7. Let $\mathcal{W} = \mathbb{R}^2$ be endowed with the b-metric

$$\omega((x, x'), (y, y')) = \max\{(x - y)^2, (x' - y')^2\} \text{ with } s = 2.$$

Consider a graph given by $V(G) = \mathcal{W}$, and $E(G) = \mathcal{W} \times \mathcal{W}$.
Suppose that

$$P = \{(-4, 0), (0, 5), (6, 1)\}, Q = \{(1, 1), (0, -4), (1, -1)\}.$$

Clearly, $P_0 = \{(-4, 0), (0, 5)\}$ and $Q_0 = \{(1, 1), (0, -4)\}$. So $\omega(P, Q) = 16$. Let's consider a mapping $\mathcal{S} : P \to CB(Q)$ is defined as follows:

$$
\begin{aligned}
&\mathcal{S}(-4, 0) = \mathcal{S}(0, 5) = \{(1, -1), (1, 1)\},\\
&\mathcal{S}(6, 1) = \{(1, 1)\}.
\end{aligned}
$$

Then $\mathcal{S}$ is upper semi-continuous because the topology is discrete.

Condition one of Theorem 2.6 is holds true because there exist two elements $(0, 5), (-4, 0) \in P_0$ and $(1, 1) \in \mathcal{S}(x_0)$ such that

$$\omega((0, 5), (1, 1)) = 16 \text{ and } ((-4, 0), (0, 5)) \in E(G),$$

easily observe that $\mathcal{S}(0, 5) = \mathcal{S}(-4, 0)$ so condition two is also true. Now, we prove that $\mathcal{S}$ is multivalued $\mathcal{FG}-$contraction type two.

Let $\mathcal{F}(\lambda) = \begin{cases} \dfrac{-1}{\sin(\lambda)} & \lambda \in \left]0, \dfrac{\pi}{2}\right[\\ \frac{2}{\pi}\lambda - 2 & \lambda \in \left[\dfrac{\pi}{2}, +\infty\right[\end{cases}$, $\mathcal{G}(\lambda) = \ln(\lambda)$, $\phi(\lambda) = e^{\frac{-\lambda}{3s^4}}$ and $\phi(0) = 0$ Therefore for all $x, y \in P$, we have $\mathcal{H}(\mathcal{S}(x), \mathcal{S}(y)) = 4$ and $\mathcal{N}_s(x, y) = \omega(x, y) \in \{25, 36, 100\}$

$$\mathcal{F}(s\mathcal{H}(\mathcal{S}(x), \mathcal{S}(y)) = F(8) = \frac{16}{\pi} - 2 \leq \mathcal{F}(\mathcal{N}_s(x, y) - 16) - \frac{\mathcal{N}_s(x, y)}{48}$$

Then $(0, 5)$ is the best proximity point of $\mathcal{S}$, that is, $D((0, 5), \mathcal{S}(0, 5)) = \omega(P, Q) = 16$.

Taking $\mathcal{G}(\lambda) = \ln(\lambda), \phi(\lambda) = k$ and putting $\sigma = -\ln k$ where $k \in (0, 1)$, in the Theorem 2.6, we thereby establish a generalization of the results within the framework of b-metric spaces endowed with graph structures.

Corollary 2.8. *Let $(\mathcal{W}, \omega)$ be a complete b-metric space with parameter $s \geq 1$ endowed with a transitive directed graph G. Let P and Q be two non-empty closed subsets of $(\mathcal{W}, \omega)$ such that (P, Q) has the P-property. Let $\mathcal{S} : P \to CB(Q)$ be a multivalued mapping and let $\mathcal{F} \in \Delta_{\mathcal{F}}$. If $\mathcal{S}(P_0) \subseteq Q_0$ and $\mathcal{S}$ satisfies the conditions :*

1. *there exist two elements x_0, x_1 in P_0 and $y_0 \in \mathcal{S}(x_0)$ such that $\omega(x_1, y_0) = \omega(P, Q)$ and $(x_0, x_1) \in E(G)$;*
2. *for some $\sigma > 0$ and for all $(x, y) \in E(G)$ and $\omega(\mathcal{S}(x), \mathcal{S}(y)) > 0$, where $\mathcal{N}_s$ is defined as in Definition 2.5 we have*
$$\sigma + \mathcal{F}(s\mathcal{H}(\mathcal{S}(x), \mathcal{S}(y))) \leq \mathcal{F}(\mathcal{N}_s(x, y));$$
3. *for all $x, y \in P_0$, $(x, y) \in E(G)$ implies $\mathcal{S}(x) \subseteq \mathcal{S}(y)$;*
4. *$\mathcal{S}$ is upper semi-continuous.*

Then there exists an element x in P such that $D(x, \mathcal{S}(\mathcal{I})) = \omega(P, Q)$.

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