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Minimal Extensions of Topologies: Characterizations and Structural Insights

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Abstract We investigate *minimal extensions* of topologies-pairs $\mathfrak{T} \subsetneq \mathfrak{U}$ such that no intermediate topology exists between them. Although a full characterization remains elusive in general, we establish precise criteria for minimality in several important settings. Specifically, we consider cases where $\mathfrak{U}$ is a chain topology, and where $\mathfrak{T}$ is clopen. These configurations reveal a surprising rigidity within the lattice of topologies and demonstrate how the existence of minimal refinements is intimately tied to the underlying structural properties of the space.

1 Introduction: Historical Background and Motivation

Let X be a fixed nonempty set. A topology $\mathfrak{T}$ on X is said to be *extended* by another topology $\mathfrak{U}$ if $\mathfrak{T} \subseteq \mathfrak{U}$, in which case $\mathfrak{U}$ is called a *topological extension* of $\mathfrak{T}$. The collection of all such intermediate topologies $\mathfrak{V}$ satisfying $\mathfrak{T} \subseteq \mathfrak{V} \subseteq \mathfrak{U}$ forms an interval $[\mathfrak{T}, \mathfrak{U}]$ in the lattice of topologies on X, ordered by inclusion.

Within this framework, the notion of *minimal extensions* plays a fundamental role. An extension $\mathfrak{T} \subsetneq \mathfrak{U}$ is said to be minimal if there exists no topology strictly between them-that is, if $[\mathfrak{T}, \mathfrak{U}] = \{\mathfrak{T}, \mathfrak{U}\}$. Equivalently, $\mathfrak{U}$ *covers* $\mathfrak{T}$ in the lattice $\mathbf{Top}(X)$. Such minimal steps-or adjacent topologies-serve as the atomic transitions within the lattice and offer insight into the structure of topological refinements.

The study of minimal extensions traces back to the foundational work of N. Levine. In his seminal 1964 paper [13], Levine introduced the notion of a *simple extension* $\mathfrak{T}(A)$: the smallest topology extending $\mathfrak{T}$ in which a prescribed subset $A \subseteq X$ becomes open. He investigated the impact of such extensions on the preservation of classical topological properties. In a later work [14], he further analyzed the conditions under which $\mathfrak{T}(A)$ constitutes a minimal extension of $\mathfrak{T}$, thereby connecting extension theory with questions of minimality.

Expanding on Levine's framework, Steiner [16] examined the global structure of the lattice $\mathbf{Top}(X)$, with a particular focus on *complementary topologies*-those whose intersection is the indiscrete topology and whose join yields the discrete topology. His work uncovered deep dualities and lattice-theoretic symmetries, highlighting the intrinsic algebraic richness of the space of all topologies on a set.

From a different angle, Borges [5] investigated general extensions of topologies, with emphasis on minimal and maximal extensions and their influence on topological properties such as compactness and separation. Continuing this line of inquiry, Larson and Thron [12] explored covering relations specifically in the lattice of T_1 topologies, providing sharp criteria for when an extension preserves the T_1 separation axiom while remaining minimal.

A particularly elegant concept in this context is that of *adjacency*, introduced by Agashe and Levine in [1]. Two topologies are said to be adjacent if one is an immediate successor or predecessor of the other. This idea characterizes minimal extensions in terms of minimal modifications of open sets and aligns closely with the combinatorial structure of the topology lattice.

The full collection $\mathbf{Top}(X)$ of topologies on a fixed set X forms a complete lattice, with the indiscrete topology $\{\emptyset, X\}$ as the least element and the discrete topology $\mathcal{P}(X)$ as the greatest.

As surveyed comprehensively by Larson and Andima [11], this lattice exhibits rich algebraic and combinatorial structure (see also [7, 10]). In particular, minimal extensions correspond to covering relations in the lattice and offer a key to understanding how topological properties emerge and interact under refinement.

The primary aim of this paper is to study the structure of the interval $[\mathfrak{T}, \mathfrak{U}]$ for topologies $\mathfrak{T} \subsetneq \mathfrak{U}$ on a fixed set X, with a particular focus on characterizing when such extensions are minimal. While the general problem is complicated by the immense variety of possible topologies, our approach is to focus on specific cases in which the nature of the topologies involved imposes strong constraints.

We investigate scenarios in which: - $\mathfrak{U}$ is a *chain topology*, and - $\mathfrak{T}$ is a *clopen topology*.

For these cases, we establish explicit necessary and sufficient conditions for minimality of the extension $\mathfrak{T} \subsetneq \mathfrak{U}$. These results illuminate how certain order-theoretic or algebraic features of the topologies involved can tightly restrict the possibility of intermediate structures, leading to a clear characterization of adjacency within the topology lattice.

Any terminology not explicitly defined here follows the conventions and definitions in [15].

2 Characterizations of Minimal Topological Extensions

As recalled from [13], if $(X, \mathfrak{T})$ is a topological space and $O \subseteq X$ with $O \notin \mathfrak{T}$, then the smallest topology on X containing $\mathfrak{T} \cup \{O\}$-that is, the topology generated by adjoining O to $\mathfrak{T}$-consists of all sets of the form $O_1 \cup (O_2 \cap O)$, where $O_1, O_2 \in \mathfrak{T}$. This topology is denoted $\mathfrak{T}(O)$ and is called a *simple extension* of $\mathfrak{T}$.

Let $(X, \mathfrak{T})$ be a topological space. The topology $\mathfrak{T}$, viewed as a family of subsets of X, is naturally partially ordered by inclusion. If, in addition, $\mathfrak{T}$ forms a chain under this ordering-that is, any two elements of $\mathfrak{T}$ are comparable-then $\mathfrak{T}$ is called a *chain topology*.

We begin with the following theorem, which characterizes minimal extensions in the case where the larger topology is a chain topology. It shows that such extensions occur precisely when a single open set outside the smaller topology determines the entire extension.

Theorem 2.1. *Let $\mathfrak{T} \subsetneq \mathfrak{U}$ be topologies on a nonempty set X, and suppose that $\mathfrak{U}$ is a chain topology. Then the following statements are equivalent:*

(1) The extension $\mathfrak{T} \subsetneq \mathfrak{U}$ is minimal.

(2) There exists an open set $O \in \mathfrak{U} \setminus \mathfrak{T}$ such that $\mathfrak{U} = \mathfrak{T}(O)$.

(3) There exist sets $O_1 \in \mathfrak{U}$ and $O_2 \in \mathfrak{U} \setminus \mathfrak{T}$ such that O_2 covers O_1, and $\mathfrak{U} = \mathfrak{T} \cup \{O_2\}$.

Proof. The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are immediate.

(2) $\Rightarrow$ (3): Assume there exists $O \in \mathfrak{U} \setminus \mathfrak{T}$ such that $\mathfrak{U} = \mathfrak{T}(O)$. Let $O_2 = O$ and define the collection

$$\Sigma := \{U \in \mathfrak{U} \mid U \subsetneq O_2\}.$$

Since $\emptyset \in \Sigma$, the set Σ is nonempty. Ordered by inclusion, $(\Sigma, \subseteq)$ is partially ordered. Let $(O_\lambda)_{\lambda \in \Lambda}$ be a chain in Σ and define $O' := \bigcup_{\lambda \in \Lambda} O_\lambda$. Clearly $O' \in \mathfrak{U}$ and $O_\lambda \subseteq O'$ for all λ.

We claim that $O' \in \Sigma$. Suppose, for contradiction, that $O' = O_2$. Then $O_\xi \notin \mathfrak{T}$ for some $\xi \in \Lambda$, since otherwise $\mathfrak{T}$ being closed under unions would imply $O_2 = O' \in \mathfrak{T}$, contradicting $O_2 \notin \mathfrak{T}$. Since $O_\xi \in \mathfrak{T}(O)$, there exist $U_1, U_2 \in \mathfrak{T}$ such that

$$O_\xi = U_1 \cup (U_2 \cap O).$$

As $\mathfrak{U}$ is a chain topology, either $U_2 \cap O = U_2$ or $U_2 \cap O = O$. In the first case, $O_\xi = U_1 \cup U_2 \in \mathfrak{T}$, a contradiction. In the second case, $O_\xi = U_1 \cup O = O$ or $O_\xi = U_1$, both of which contradict $O_\xi \in \Sigma$ and $O_\xi \notin \mathfrak{T}$. Thus, $O' \neq O_2$, and hence $O' \in \Sigma$.

By Zorn's Lemma, Σ has maximal element, say O_1. Then O_2 covers O_1 in $\mathfrak{U}$. Set $\mathfrak{V} := \mathfrak{U} \setminus \{O_2\}$. We claim $\mathfrak{V}$ is a topology on X.

Clearly, $\emptyset, X \in \mathfrak{V}$ (since $O_2 \notin \mathfrak{T}$ and hence cannot be $\emptyset$ or X). Let $V_1, V_2 \in \mathfrak{V}$ with $V_1 \subseteq V_2$. Then $V_1 \in \mathfrak{V}$. Now let $\{V_i \mid i \in I\} \subseteq \mathfrak{V}$. We show that $\bigcup_{i \in I} V_i \in \mathfrak{V}$. Consider two cases:

Case 1. $O_2 \subsetneq V_{i_0}$ for some $i_0 \in I$. Then $O_2 \subsetneq \bigcup_{i \in I} V_i \neq O_2$, so $\bigcup_{i \in I} V_i \in \mathfrak{V}$.

Case 2. $V_i \subsetneq O_2$ for all $i \in I$. If some $V_{j_0} \supseteq O_1$, then maximality of O_1 implies $V_{j_0} = O_1$, so:

$$\bigcup_{i\in I} V_i = \left(\bigcup_{i,\ V_i = O_1} V_i\right) \cup \left(\bigcup_{i,\ V_i \subsetneq O_1} V_i\right) = O_1 \neq O_2.$$

Otherwise, $\bigcup_{i\in I} V_i \subseteq O_1 \subsetneq O_2$, so again $\bigcup_{i\in I} V_i \neq O_2$.

Hence $\mathfrak{V}$ is a topology, and $\mathfrak{U} = \mathfrak{V} \cup \{O_2\}$. Since $\mathfrak{T} \subseteq \mathfrak{V} \subsetneq \mathfrak{U}$ and the extension is minimal, we must have $\mathfrak{T} = \mathfrak{V}$, so $\mathfrak{U} = \mathfrak{T} \cup \{O_2\}$ with O_2 covering O_1. □

A subset of a topological space that is simultaneously open and closed is termed a clopen set. A topology in which every open set is closed-equivalently, every closed set is open-will be referred to as a *clopen topology* (cf. [3, 4, 7, 9]). Such topologies are also known in the literature as partition topologies. In the presence of a clopen topology, minimal extensions admit a particularly clean description. The next result shows that such an extension is completely determined by a single nontrivial open set and the behavior of its boundary.

Theorem 2.2. *Let $\mathfrak{T} \subsetneq \mathfrak{U}$ be topologies on a nonempty set X, and suppose that $\mathfrak{T}$ is a clopen topology. Then the following statements are equivalent:*

(1) The extension $\mathfrak{T} \subsetneq \mathfrak{U}$ is minimal.

(2) There exists a set $O \in \mathfrak{U} \setminus \mathfrak{T}$ such that $\mathfrak{U} = \mathfrak{T}(O)$ and the set difference $O \setminus \mathrm{int}_{\mathfrak{T}}(O)$ is endowed with the indiscrete topology.

Proof. (1) ⇒ (2): Since the extension is minimal, there exists $O \in \mathfrak{U} \setminus \mathfrak{T}$ such that $\mathfrak{U} = \mathfrak{T}(O)$. We claim that the set $O \setminus \mathrm{int}_{\mathfrak{T}}(O)$ is indiscrete.

Let $U \in \mathfrak{T}$ be such that

$$U \cap (O \setminus \mathrm{int}_{\mathfrak{T}}(O)) \neq \emptyset,$$

and pick $b \in U \cap (O \setminus \mathrm{int}_{\mathfrak{T}}(O))$. Suppose that $U \cap O \in \mathfrak{T}$. Then:

$$b \in U \cap O \subseteq \mathrm{int}_{\mathfrak{T}}(O),$$

contradicting the choice of b. Therefore, $U \cap O \notin \mathfrak{T}$. By minimality, $\mathfrak{T}(U \cap O) = \mathfrak{T}(O)$, so in particular $O \in \mathfrak{T}(U \cap O)$. Thus, there exist $O_1, O_2 \in \mathfrak{T}$ such that

$$O = O_1 \cup (O_2 \cap (U \cap O)).$$

Let $x \in O \setminus \mathrm{int}_{\mathfrak{T}}(O)$. If $x \notin U$, then $x \in O_1 \subseteq \mathrm{int}_{\mathfrak{T}}(O)$, which is again a contradiction. Therefore,

$$O \setminus \mathrm{int}_{\mathfrak{T}}(O) \subseteq U.$$

Since U was arbitrary, this proves that $O \setminus \mathrm{int}_{\mathfrak{T}}(O)$ is indiscrete.

(2) ⇒ (1): Suppose that $\mathfrak{T} \subsetneq \mathfrak{B} \subseteq \mathfrak{U} = \mathfrak{T}(O)$ is an intermediate topology. We show that $O \in \mathfrak{B}$, and hence that $\mathfrak{T} \subsetneq \mathfrak{U}$ is minimal.

Let $B \in \mathfrak{B} \setminus \mathfrak{T}$. Since $B \in \mathfrak{T}(O)$, we can write

$$B = O_1 \cup (O_2 \cap O),$$

for some $O_1, O_2 \in \mathfrak{T}$. Then:

$$B = O_1 \cup (O_2 \cap \mathrm{int}_{\mathfrak{T}}(O)) \cup (O_2 \cap (O \setminus \mathrm{int}_{\mathfrak{T}}(O))).$$

Since $B \notin \mathfrak{T}$, the third term must be nonempty:

$$O_2 \cap (O \setminus \mathrm{int}_{\mathfrak{T}}(O)) \neq \emptyset.$$

By the assumption that $O \setminus \mathrm{int}_{\mathfrak{T}}(O)$ is endowed with the indiscrete topology, it follows that:

$$O_2 \supseteq O \setminus \mathrm{int}_{\mathfrak{T}}(O).$$

Thus:

$$B = O_1 \cup (O_2 \cap \mathrm{int}_{\mathfrak{T}}(O)) \cup (O \setminus \mathrm{int}_{\mathfrak{T}}(O)).$$

Set $U := O_1 \cup (O_2 \cap \mathrm{int}_{\mathfrak{T}}(O)) \in \mathfrak{T}$, so $B = U \cup (O \setminus \mathrm{int}_{\mathfrak{T}}(O))$.

Since $\mathfrak{T}$ is clopen, we have:

$$\mathrm{cl}_{\mathfrak{T}}(U \setminus \mathrm{int}_{\mathfrak{T}}(O)) = U \setminus \mathrm{int}_{\mathfrak{T}}(O).$$

Now consider:

$$(O \setminus \mathrm{int}_{\mathfrak{T}}(O)) \cap \mathrm{cl}_{\mathfrak{T}}(U \setminus \mathrm{int}_{\mathfrak{T}}(O)) = (O \setminus \mathrm{int}_{\mathfrak{T}}(O)) \cap (U \setminus \mathrm{int}_{\mathfrak{T}}(O)).$$

If this intersection is nonempty, then the indiscreteness of $O \setminus \mathrm{int}_{\mathfrak{T}}(O)$ forces:

$$U \setminus \mathrm{int}_{\mathfrak{T}}(O) \supseteq O \setminus \mathrm{int}_{\mathfrak{T}}(O),$$

so $B = U \in \mathfrak{T}$, contradicting $B \notin \mathfrak{T}$.

Thus, the intersection above must be empty. Consider:

$$V := \mathrm{int}_{\mathfrak{T}}(O) \cup \left(B \cap (U \setminus \mathrm{int}_{\mathfrak{T}}(O))^c\right).$$

We compute:

$$B = U \cup (O \setminus \mathrm{int}_{\mathfrak{T}}(O)), \quad \text{so}$$

$$B \cap (U \setminus \mathrm{int}_{\mathfrak{T}}(O))^c = (U \cup (O \setminus \mathrm{int}_{\mathfrak{T}}(O))) \cap (U \setminus \mathrm{int}_{\mathfrak{T}}(O))^c.$$

This simplifies to:

$$(U \cap (U \setminus \mathrm{int}_{\mathfrak{T}}(O))^c) \cup (O \setminus \mathrm{int}_{\mathfrak{T}}(O)).$$

Since $U \cap (U \setminus \mathrm{int}_{\mathfrak{T}}(O))^c = U \cap \mathrm{int}_{\mathfrak{T}}(O)$, we finally get:

$$V = \mathrm{int}_{\mathfrak{T}}(O) \cup (U \cap \mathrm{int}_{\mathfrak{T}}(O)) \cup (O \setminus \mathrm{int}_{\mathfrak{T}}(O)) = O.$$

Hence, $O \in \mathfrak{B}$, and since $\mathfrak{T}(O) = \mathfrak{U}$, it follows that $\mathfrak{B} = \mathfrak{U}$. Therefore, the extension $\mathfrak{T} \subsetneq \mathfrak{U}$ is minimal. □

3 Conclusion and Perspectives

In this paper, we investigated the structure of intervals $[\mathfrak{T}, \mathfrak{U}]$ in the lattice $\mathbf{Top}(X)$, focusing on minimal extensions where $\mathfrak{T}$ is clopen and $\mathfrak{U}$ is a chain topology. The obtained characterizations of adjacency clarify how order-theoretic and algebraic constraints shape the refinement process between topologies.

This study naturally opens several directions for further research:

(i) Do similar minimal-extension criteria hold when $\mathfrak{T}$ or $\mathfrak{U}$ belong to more structured families, such as spectral (cf. [6]), metrizable, connected, or compact topologies?

(ii) Under what conditions is an extension of an S-Zariski or an S-quasi Zariski topology minimal (see, for example, [2, 8, 17])?

(iii) How do categorical or combinatorial invariants reflect adjacency relations between such algebraic or spectral topologies?

These questions lie at the intersection of general topology, lattice theory, and algebraic geometry. Exploring them further may reveal deeper correspondences between minimal topological refinements and the spectral structures that arise in algebraic and categorical settings.

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