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A Banach algebra with several properties

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Abstract In this paper we study some properties of $C(X)$, the space of all continuous complex valued functions on a compact Hausdorff space X. We show that $C(X)$ has no nontrivial idempotent if and only if X is connected, and it is generated by its idempotents if and only if X is totally disconnected. Moreover, several properties of mentioned Banach algebra are also investigated.

1 Introduction

Let $C(X)$ denote the space of all continuous complex-valued functions on a compact Hausdorff space X equipped with the supremum norm

$$\|f\| = \sup\{|f(x)| : \; x \in X\}.$$

It is known that $C(X)$ is a Banach algebra with the pointwise multiplication, in particular it is a C^*-algebra. The Banach algebra $C(X)$ is one of the famous example in harmonic and functional analysis which has many important and interesting properties.

For example, $C(X)$ is a unital commutative semisimple Banach algebra, which is neither reflexive nor ideal in its second dual.

If X is countable, then $C(X)$ is the unique function algebra on X [13], for X uncountable it is shown in [11] that if there exists a projection of $C(X)$ onto $\mathcal{A}$, where $\mathcal{A}$ is a function algebra on a compact metric space X, then $\mathcal{A}$ is linearly isomorphic to $C(X)$.

Since $C(X)$ is commutative and semisimple, by a result of Jonhson [3, Theorem 21.18], the only derivation on $C(X)$ is zero. Indeed, each derivation D on $C(X)$ is continuous and it follows from semisimplicty that $D(f) = 0$ for all $f \in C(X)$.

A theorem due to Milutin [10] (see also [12]) asserts that for any two uncountable compact metric spaces X and Y the spaces of continuous complex-valued functions $C(X)$ and $C(Y)$ are linearly isomorphic. Thus, $C([0,1])$ is isomorphic to $C(X)$, where X is uncountable compact metric space. Hence all properties which we present in this note for $C([0,1])$, also valid for $C(X)$. We mention that $C([0,1])$ is not isomorphic to the dual space of any normed space.

In this paper we study some interesting properties of $C(X)$, this properties are suitable for counter-examples in the general theory of Banach algebras.

For another different properties of $C(X)$, we refer the reader to [3, 4], and the chapter book [12] and its references.

Throughout the paper, $\mathfrak{M}$ denote the Banach algebra $C([0,1])$ equipped with the supremum norm, unless otherwise states.

2 Some properties of Banach algebra $\mathfrak{M}$

A Banach space X is said to have the *Donford-Pettis property* if for each Banach space Y every weakly compact linear operator $T : X \longrightarrow Y$ is completely continuous, i.e., T takes weakly compact sets in X onto norm compact sets in Y, or equivalently, T carries every weakly convergent sequence to a norm convergent sequence [6]. It is known that if X', the dual of X, has the Donford-Pettis property, then X has the property, too.

The following result is due to Donford-Pettis, concerning of completely continuous weakly compact linear operator.

Theorem 2.1. *[6] For any μ and any Banach space Y, if $T : L^1(\mu) \longrightarrow Y$ is a weakly compact linear operator, then T is completely continuous.*

From [6], we have the following fact. We include the proof for the sake of completeness.

Theorem 2.2. *$\mathfrak{M}$ has the Donford-Pettis property.*

Proof. Since the dual $\mathfrak{M}'$ of $\mathfrak{M}$ is the space $M([0, 1])$ of regular Borel measures on $[0, 1]$, and $M([0, 1])$ is itself L^1-space, it follows that $\mathfrak{M}'$ and hence $\mathfrak{M}$ has the Donford-Pettis property. □

It is known that, on the second dual space $\mathcal{A}''$ of a Banach algebra $\mathcal{A}$, there are two multiplications, called the first and second Arens products, which make $\mathcal{A}''$ into a Banach algebra. If these products coincide on $\mathcal{A}''$, then $\mathcal{A}$ is said to be Arens regular [4].

By [4, Corollary 3.2.37], every C^*-algebra $\mathcal{A}$ is Arens regular and $\mathcal{A}''$ is a C^*-algebra. Thus, we have the following result.

Theorem 2.3. *For all $n \in \mathbb{N}$, $\mathfrak{M}^{2n}$ is a C^*-algebra and Arens regular.*

Lemma 2.4. *Let X be a compact metric space. Then $C(X)$ is separable.*

Proof. Since every compact metric space is separable, X is separable. Fix a countable dense subset $\{x_1, x_2, ...\}$ of X and let $f_n : X \longrightarrow \mathbb{C}$ be a function defined by $f_n(t) = \|t - x_n\|$, for all $t \in X$.

Now, let $x, y \in X$ satisfy $x \neq y$, and put $\|x-y\| = 2\delta > 0$. Choose some n with $\|x-x_n\| < \delta$, then

$$f_n(y) = \|y - x_n\| \geq \|x - y\| - \|x - x_n\| \geq 2\delta - \delta = \delta > \|x - x_n\| = f_n(x).$$

So $f_n(x) \neq f_n(y)$, which implies that the algebra generated by $\{1, f_1, f_2, ...\}$ separates the points of X. By the Stone-Weierstrass Theorem this algebra is dense in $C(X)$.

Let $\mathcal{E}$ denote the collection of all finite products of the countable collection $\{1, f_1, f_2, ...\}$ and note that $\mathcal{E}$ is a countable set, say, $\mathcal{E} = \{\varphi_1, \varphi_2, ...\}$. Then the finite linear combinations of $\{1, \varphi_1, \varphi_2, ...\}$ with rational coefficients form a countable dense subset of $C(X)$. This completes the proof. □

Theorem 2.5. *$\mathfrak{M}$ is separable, but $\mathfrak{M}'$ is not.*

Proof. By Lemma 2.4, $\mathfrak{M}$ is separable and since $[0, 1]$ is uncountable set, $\mathfrak{M}'$ is not separable by [11, Theorem 1]. □

Theorem 2.6. $Card\, \mathfrak{M} \leq \mathcal{P}(\mathbb{N})$.

Proof. First note that $\mathfrak{M}$ is separable by Theorem 2.5. Suppose that $\{f_1, f_2, ...\}$ is a countable dense subset of $\mathfrak{M}$, and take

$$\mathcal{E} = \{B(f_i, \frac{1}{k}) : \; i, k \in \mathbb{N}\}.$$

Then $\mathcal{E}$ is a countable set. Let $\{B_1, B_2, ...\}$ be one of its enumerations. Now for each $f \in \mathfrak{M}$ define the set $S_f = \{n \in \mathbb{N} : \; f \in B_n\}$. Then $\varphi : \mathfrak{M} \longrightarrow \mathcal{P}(\mathbb{N})$ defined by $\varphi(f) = S_f$ is one to one. Thus, $Card\, \mathfrak{M} \leq Card\, \mathcal{P}(\mathbb{N})$. □

Let $\mathcal{A}$ be a Banach algebra and $n \in \mathbb{N}$. Then $\mathcal{A}$ is *amenable* if every derivation of $\mathcal{A}$ into dual Banach $\mathcal{A}$-bimodule X' is inner, and it is n-*weakly amenable* if every derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^n$ is inner. Moreover, $\mathcal{A}$ is *permanently weakly amenable* if it is n-weakly amenable for all $n \in \mathbb{N}$.

A C^*-algebra $\mathcal{A}$ is called *nuclear* if there is only one C^*-norm on $\mathcal{A} \otimes \mathcal{B}$, for every C^*-algebra $\mathcal{B}$. It is known that a C^*-algebra $\mathcal{A}$ is amenable if and only if it is nuclear [4].

Note that not all C^*-algebras are amenable, in particular, the C^*-algebra $B(H)$ is not nuclear and hence not amenable, in the case where H is an infinite dimensional Hilbert space. However, every C^*-algebra is permanently weakly amenable [5, Theorem 2.1].

Theorem 2.7. *$\mathfrak{M}$ is amenable, but not biprojective.*

Proof. By [4, Theorem 5.6.2], $\mathfrak{M}$ is amenable and hence it is nuclear. The second statement follows from [4, Corollary 2.8.42]. □

A sequence $\{x_n\}$ in a Banach space X is weak Cauchy if $\lim_{n\to\infty} f(x_n)$ exists for all $f \in X'$, and a Banach space X is called weakly sequentially complete(=WSC) if every weak Cauchy sequence is weakly convergent in X.

Theorem 2.8. *$\mathfrak{M}$ is not WSC, but $\mathfrak{M}'$ is WSC.*

Proof. This fact that $\mathfrak{M}$ is not WSC follows from [16, Theorem 3.4], (see also [2, Example 1]), and the second statement is [4, Theorem 3.2.42]. □

Theorem 2.9. *$\mathfrak{M}$ is not complete with respect to the norm $\|f\|_2$ which is obtained by the inner product*

$$< f, g >= \int_0^1 f(x)\overline{g(x)}dx, \quad f, g \in \mathfrak{M}.$$

Proof. The sequence $\{f_n\}$ defined by

$$f_n(x) = \begin{cases} 0 & 0 \le x < \frac{1}{2}, \\ (n+3)(x - \frac{1}{2}) & \frac{1}{2} \le x \le \frac{1}{2} + \frac{1}{n+3}, \\ 1 & \frac{1}{2} + \frac{1}{n+3} < x \le 1. \end{cases}$$

is a Cauchy sequence but it is not convergent. Thus, $\mathfrak{M}$ is not complete with $\|f\|_2$. Note that by [4, Corollary 2.3.4], $\mathfrak{M}$ has a unique complete norm which is the supremum norm. □

Theorem 2.10. *$C(X)$ has no nontrivial idempotent if and only if X is connected.*

Proof. Suppose that $f \in C(X)$ is a nontrivial idempotent, i.e., $f \neq 0, 1$. Then

$$X = f^{-1}(\{0\}) \cup f^{-1}(\{1\})$$

implies that X is not connected, which is a contradiction. Consequently, f is a trivial idempotent.

For the converse, let $C(X)$ have no nontrivial idempotent. Assume, on the contrary, that X is disconnected and let $X = U \cup V$ with open disjoint sets U and V, then the map

$$f(x) = \begin{cases} 1 & x \in U, \\ 0 & x \in V, \end{cases}$$

is a trivial idempotent of $C(X)$. On the other hand, since $sp(f)$ is not connected, where $sp(f)$ is the spectral radius of $f \in C(X)$, by Remarks of Proposition 7.9 of [3], $C(X)$ has a nontrivial idempotent. This leads to a contradiction, hence X is connected. □

Since $X = [0, 1]$ is connected, we obtain the following fact.

Theorem 2.11. *$\mathfrak{M}$ has no nontrivial idempotent.*

A topological space X is called *totally disconnected* if for every distinct $x, y \in X$, there exist disjoint open sets U and V such that $x \in U$, $y \in V$ and $X = U \cup V$.

Theorem 2.12. $\mathfrak{M}$ *cannot be generated by its idempotents.*

Proof. Let $\mathfrak{M}$ be generated by idempotents and let $x, y \in [0,1]$. Then by Urysohn's Lemma there exists $f \in \mathfrak{M}$ such that $f(x) = 1$ and $f(y) = 0$. Since every element of the self-adjoint subalgebra generated by idempotents is the form

$$F = \sum_{i=1}^{k} \alpha_i f_i, \tag{2.1}$$

for some $f_i \in \mathfrak{M}$ and $\alpha_i \in \mathbb{C}$, there is a sequence $\{F_n\}$ of elements of the form (2.1) such that $F_n \longrightarrow f$ uniformly on $[0,1]$. Hence,

$$\lim_n F_n(x) = 1, \quad and \qquad \lim_n F_n(y) = 0.$$

So there exists a number N such that $|F_N(x)| > 1/2$ and $|F_N(y)| < 1/2$. Take

$$U = F_N^{-1}(\{z \in \mathbb{C} : \ |z| > 1/2\}) \ \ and \ \ \ V = F_N^{-1}(\{z \in \mathbb{C} : \ |z| < 1/2\}).$$

Then $x \in U$, $y \in V$, $U \cap V = \varnothing$ and $[0,1] = U \cup V$. Thus, $[0,1]$ is totally disconnected which is a contradiction. □

Note that the Banach algebra $C(X)$ is generated by its idempotents if and only if X is totally disconnected. Indeed, let X be totally disconnected, $x \neq y$, $x \in U$, $y \in V$, where U and V are disjoint open sets and $X = U \cup V$. Then the continuous function

$$f(x) = \begin{cases} 1 & x \in U, \\ \\ 0 & x \in V, \end{cases}$$

separates x and y. Consequently, by the Stone-Weierstrass theorem the closed self-adjoint subalgebra generated by idempotents is $C(X)$. The converse was proved in the preceding result.

A linear functional φ on Banach algebra $\mathcal{A}$ is called (ε, n)-multiplicative if there exists $\varepsilon \geq 0$ such that

$$|\varphi(x_1 x_2 ... x_n) - \varphi(x_1)...\varphi(x_n)| \leq \varepsilon \|x_1\|...\|x_n\|, \quad x_1, x_2, ..., x_n \in \mathcal{A}.$$

If $n = 2$, then it is called ε-multiplicative, in the usual sense. If $\varepsilon = 0$, then (ε, n)-multiplicative turns out to be n-multiplicative.

We mention that there is (ε, n)-multiplicative functional φ on $\mathfrak{M}$ which is not n-multiplicative. To see this, let $\varphi : \mathfrak{M} \longrightarrow \mathbb{C}$ be defined by $\varphi(f) = \lambda f(a)$, where $0 < \lambda < 1$ and $a \in [0,1]$ be a fixed. Then for all $f_1, f_2, ..., f_n \in \mathfrak{M}$,

$$\begin{aligned} |\varphi(f_1...f_n) - \varphi(f_1)...\varphi(f_n)| &= |\lambda f_1(a)...f_n(a) - \lambda^n f_1(a)...f_n(a)| \\ &= |\lambda - \lambda^n||f_1(a)...f_n(a)| \\ &\leq |\lambda - \lambda^n|\|f_1\|...\|f_n\|. \end{aligned}$$

Therefore φ is (ε, n)-multiplicative for $\varepsilon = |\lambda - \lambda^n|$, but it is not n-multiplicative.

Following Johnson [8] we say that the Banach algebra $\mathcal{A}$ is an algebra in which approximately multiplicative functionals are near multiplicative functionals, or $\mathcal{A}$ is AMNM for short, if for each $\varepsilon > 0$ there is $\delta > 0$ such that for every linear δ-multiplicative functional φ on $\mathcal{A}$ there exists a linear multiplicative functional θ on $\mathcal{A}$ such that

$$|\theta(x) - \varphi(x)| \leq \varepsilon, \quad x \in \mathcal{A}.$$

Moreover, $\mathcal{A}$ is AHNH (almost homomorphisms are near homomorphisms) [15], if for each $\varepsilon > 0$ there is $\delta > 0$ such that for every δ-homomorphism $\varphi : \mathcal{A} \longrightarrow \mathbb{R}$ there exists a linear multiplicative functional $\theta : \mathcal{A} \longrightarrow \mathbb{R}$ such that $|\theta(x) - \varphi(x)| \leq \varepsilon$ for every $x \in \mathcal{A}$.

Theorem 2.13. *$\mathfrak{M}$ is both AMNM and AHNH.*

Proof. This is [8, Theorem 4.1], and [14, Theorem 3.1], respectively. □

Theorem 2.14. *The supremum norm on $\mathfrak{M}$ cannot be obtained from an inner product.*

Proof. Lat $f, g \in \mathfrak{M}$ with $f(x) = 1$ and $g(x) = x$. Then $\|f\| = \|g\| = 1$, and

$$\|f - g\| = \sup\{|1 - x| : \ x \in [0, 1]\} = 1, \quad \|f + g\| = \sup\{|1 + x| : \ x \in [0, 1]\} = 2.$$

Thus, the parallelogram equality

$$\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2|g\|^2,$$

which is satisfied in every inner product spaces does not hold. Hence the supremum norm on $\mathfrak{M}$ cannot be obtained from an inner product. □

The Banach space X is called *primary* if X is isomorphic to $Y \oplus Z$, then X is isomorphic to Y or to Z. By [12, Corollary 5.4], we have the following.

Theorem 2.15. *$\mathfrak{M}$ is primary.*

Theorem 2.16. *Every maximal ideal in $\mathfrak{M}$ is closed.*

Proof. This is [3, Corollary 5.9]. □

Let $\mathcal{A}$ be an Banach algebra and $\mathcal{X}$ be an arbitrary Banach space. Then the continuous bilinear mapping $\varphi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{X}$ preserves zero products if

$$a, b \in \mathcal{A}, \quad ab = 0 \quad \Longrightarrow \quad \varphi(a, b) = 0, \tag{2.2}$$

and $\mathcal{A}$ *has the property* $(\mathbb{B})$ if for every continuous bilinear mapping $\varphi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{X}$, where $\mathcal{X}$ is an arbitrary Banach space, the condition (2.2) implies that

$$\varphi(ab, c) = \varphi(a, bc), \quad a, b, c \in \mathcal{A}.$$

Moreover, $\mathcal{A}$ is said to be *zero product determined* if for every continuous bilinear mapping $\varphi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{X}$ preserving zero products, there is a continuous linear mapping $T : \mathcal{A} \longrightarrow \mathcal{X}$ such that $\varphi(a, b) = T(ab)$, for all $a, b \in \mathcal{A}$.

Theorem 2.17. *$\mathfrak{M}$ has the property $(\mathbb{B})$ and it is zero product determined.*

Proof. The first statement follows from Theorem 2.11 of [1]. Let $\varphi : \mathfrak{M} \times \mathfrak{M} \longrightarrow \mathcal{X}$ be a continuous bilinear map preserving zero products. Then by the property $(\mathbb{B})$,

$$\varphi(fg, h) = \varphi(f, gh), \quad f, g, h \in \mathfrak{M}.$$

Let I be the unit element of $\mathfrak{M}$. Now take $h = I$ and note that T can be defined according to $T(f) = \varphi(f, I)$. □

Corollary 2.18. *Every continuous bilinear mapping $\varphi : \mathfrak{M} \times \mathfrak{M} \longrightarrow \mathcal{X}$ preserving zero products is symmetric, that is, $\varphi(f, g) = \varphi(g, f)$ for all $f, g \in \mathfrak{M}$.*

Recall that the normed space X is *strictly convex* if $\|tx + (1 - t)y\| < 1$ whenever x and y are different points of S_X and $0 < t < 1$, where S_X is the unit sphere of X, or equivalently, if $x, y \in S_X$ and $x \neq y$, then $\|x + y\| < 2$, [9].

Theorem 2.19. *$\mathfrak{M}$ is not strictly convex.*

Proof. This is [9, Example 5.1.7]. □

Although, $\mathfrak{M}$ is not strictly convex with supremum norm, but there are equivalent norms on $\mathfrak{M}$ which make $\mathfrak{M}$ to be strictly convex. To see this, define

$$\|f\|' := \|f\| + \|f\|_2,$$

where $\|f\|$ is the supremum norm. Then for all $f \in \mathfrak{M}$,

$$\|f\| \le \|f\|' \le 2\|f\|.$$

Therefore $\|f\|$ and $\|f\|'$ are equivalent on $\mathfrak{M}$. On the other hand, since $\mathfrak{M}$ is a subspace of l^2, hence $\mathfrak{M}$ equipped with $\|f\|_2$ is strictly convex, because each subspace of strictly convex space is strictly convex [9]. Now let $f, g \in \mathfrak{M}$ and $\|f\|', \|g\|' \le 1$. Then

$$\begin{aligned}\|\frac{1}{2}(f+g)\|' &= \frac{1}{2}\big(\|(f+g\| + \|f+g\|_2\big)\\ &< \frac{1}{2}\big(\|f\| + \|f\|_2 + \|g\| + \|g\|_2\big)\\ &= \frac{1}{2}\big(\|f\|' + \|g\|'\big)\\ &\le 1.\end{aligned}$$

Consequently, $\mathfrak{M}$ is strictly convex with norm $\|f\|'$.

The normed space X is *uniformly convex* if for every $0 < \varepsilon \le 2$, there is a $\delta > 0$ depending on ε such that

$$\|\frac{1}{2}(x+y)\| \le 1 - \delta,$$

whenever x, y are different points of S_X and $\|x - y\| \ge \varepsilon$. Now by [9, Proposition 5.2.6], every uniformly convex normed space is strictly convex.

Theorem 2.20. $\mathfrak{M}$ *is not uniformly convex.*

Proof. Suppose that $\mathfrak{M}$ is uniformly convex. Then by [9, Theorem 5.2.15], $\mathfrak{M}$ is reflexive which is not possible. Thus, $\mathfrak{M}$ fails to be uniformly convex. □

Let E be a nonempty convex subset of Banach space X. We say that E has a normal structure if for every bounded and convex subset $C \subseteq E$, there exists $x_0 \in C$ such that

$$\sup\{\|x_0 - x\| : \ x \in C\} < diam(C).$$

A Banach space X has a normal structure if every closed, convex and bounded subset E of X with $diam(E) > 0$, has a normal structure. For example, every compact convex subset E of Banach space X has a normal structure.

If we consider $\mathfrak{M}$ just for real-valued functions on $[0, 1]$, then we get the following result.

Theorem 2.21. $\mathfrak{M}$ *has not the normal structure.*

Proof. Let

$$E = \{f \in \mathfrak{M} : \ 0 = f(0) \le f(t) \le f(1) = 1, \ t \in [0, 1]\}.$$

Suppose that $f_1, f_2 \in E$, $\lambda \in [0, 1]$ and

$$f = \lambda f_1 + (1 - \lambda) f_2.$$

Since $f(0) = 0$, $f(1) = 1$ and $0 \le f(t) \le 1$ for all $t \in [0, 1]$, E is convex. On the other hand, E is a closed and bounded subset of $\mathfrak{M}$. As

$$diam(E) = \sup\{\|f - g\| : \ f, g \in E\} = 1,$$

thus for every $g \in E$,

$$\sup\{\|g - f\| : \ f \in E\} = 1 = diam(E),$$

and hence $\mathfrak{M}$ has not the normal structure. □

Definition 2.22. A topological algebra $\mathcal{A}$ is called

(i) *n-functionally continuous*, if every n-multiplicative linear functional on $\mathcal{A}$ is continuous,

(ii) *n-Jordan functionally continuous*, if every linear functional φ defined by $\varphi(a^n) = \varphi(a)^n$, for all $a \in \mathcal{A}$, is continuous,

(iii) *sequentially complete* if every Cauchy sequence in $\mathcal{A}$ converges.

It is known that every Banach algebra $\mathcal{A}$ is 2-Jordan functionally continuous [17], and hence it is functionally continuous.

Theorem 2.23. $\mathfrak{M}$ *is n-Jordan functionally continuous.*

Proof. From [18, Theorem 3.1], $\mathfrak{M}$ is n-Jordan functionally continuous. In particular, $\mathfrak{M}$ is n-functionally continuous. □

A unital topological algebra $\mathcal{A}$ is called a *Q-algebra* if $Inv(\mathcal{A})$, the set of all invertible elements of $\mathcal{A}$, is an open set. Banach algebras are important examples of Q-algebras.

A locally convex topological algebra $\mathcal{A}$ is called locally multiplicatively convex algebra , if it has a neighborhood basis of (absolutely) convex sets V_n of zero such that V_n is multiplicative idempotent, i.e., $V_nV_n \subseteq V_n$ for all $n \in \mathbb{N}$.

In the following result we assume that $\mathfrak{M}$ is endowed with the uniform convergence topology on countable compact subsets of $[0, 1]$.

Theorem 2.24. $\mathfrak{M}$ *is a sequentially complete LMC algebra which is neither metrizable nor it is a Q-algebra, but it is n-functionally continuous.*

Proof. The first statement is [7, Example 3.22]. Since each n-multiplicative linear functional on $\mathfrak{M}$ is continuous with the supremum norm, it is easy to see that it is continuous on $\mathfrak{M}$ with respect to its topology. Hence $\mathfrak{M}$ is n-functionally continuous. □

The next characterization of multiplicative linear functionals on $\mathfrak{M}$ was appeared in [7].

Theorem 2.25. *For each nonzero multiplicative linear functional φ on $\mathfrak{M}$, there exists $x \in [0, 1]$ such that $\varphi(f) = f(x)$ for all $f \in \mathfrak{M}$.*

The Jacobson radical $\mathfrak{J}(\mathcal{A})$ of $\mathcal{A}$ is the intersection of maximal modular left ideals of $\mathcal{A}$. An algebra $\mathcal{A}$ is called *semisimple* whenever $\mathfrak{J}(\mathcal{A}) = \{0\}$. We know that every C^*-algebra is semisimple [4], and hence it is semiprime by [3, Poroposition 5.30], but we adapt the following proof which is contained in [3].

Let $\Phi_{\mathfrak{M}}$ denote the maximal ideal speace of $\mathfrak{M}$, i.e., the space of all multiplicative linear functional on $\mathfrak{M}$.

Theorem 2.26. $\mathfrak{M}$ *is semisimple.*

Proof. For all $t \in [0, 1]$, define $h : [0, 1] \longrightarrow \Phi_{\mathfrak{M}}$ by $h(t) = \varphi_t$, where $\varphi_t(f) = f(t)$, for each $f \in \mathfrak{M}$. This follows from Theorem 2.5. Clearly, h is a continuous homomorphism. Suppose that $t_1, t_2 \in [0, 1]$ with $t_1 \neq t_2$. Then by Urysohn's Lemma there exists $f \in \mathfrak{M}$ such that $f(t_1) = 1$ and $f(t_2) = 0$. Thus, $\varphi_{t_1}(f) \neq \varphi_{t_1}(f)$, and hence h is one to one. Let $\varphi \in \Phi_{\mathfrak{M}}$ and $\{f_1, f_2, ..., f_n\}$ be a finite subset of $ker\varphi$. We claim that

$$\bigcap_{f_i \in ker\varphi} ker f_i \neq \varnothing. \tag{2.3}$$

Since each f_i is continuous, $ker f_i$ is closed for every $i = 1, ..., n$. Assume, on the contrary, that

$$ker f_1 \cap ker f_2 \cap ... \cap ker f_n = \varnothing.$$

Let us define $p : [0, 1] \longrightarrow \mathbb{C}$ by

$$p(t) = \sum_{i=1}^{n} f_i(t)\overline{f_i}(t).$$

Then

$$0 \neq \varphi(p) = \sum_{i=1}^{n} \varphi(f_i(t))\varphi(\overline{f_i}(t)) = 0,$$

which is a contradiction. Hence (2.3) holds. Now by the finite intersection property,

$$\bigcap_{f \in ker\varphi} kerf \neq \varnothing.$$

Let t be a common zero of all functions in $ker\varphi$ and let $f \in ker\varphi$. Then $f(t) = 0$ and hence $\varphi_t(f) = 0$. Therefore $ker\varphi \subseteq ker\varphi_t$, but $\varphi(1) = \varphi_t(1) = 1$, it follows that $\varphi = \varphi_t$. Consequently, h is surjective. Now let $f, g \in \mathfrak{M}$ and $f \neq g$. Then there exists $t \in [0, 1]$ such that $f(t) \neq g(t)$. Thus, $\varphi_t(f) \neq \varphi_t(g)$, and hence $\Phi_{\mathfrak{M}}$ separates the points of $\mathfrak{M}$. This finishes the proof. □

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