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ON SECOND STAIR PARACOMPACT AND EXPANDABLE SPACES

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Abstract In this paper, a new type of paracompact and expandable space, “second stair paracompact spaces” and “second stair expandable spaces” are introduced respectively. In this study, several characterizations, properties and examples of these spaces are investigated. Also, we have investigated the relation between these spaces with the spaces that have previously been defined. Generalization of G_δ-set is a part of this paper.

1 Introduction

Levine [16] is the pioneer for generalizing the open set in literature. After that the authors Andrijević [3, 4], Mashhour et al. [2], Modak [17] and many others have studied the sequential generalizations in different angles. One of the angles is to study of paracompact and expandable space. The authors Al-Zoubi [12], Al-Zoubi and Al-Ghour [10], Demir and Ozbakir [5] and Krajewski [13] have studied these spaces. In this article, we shall study the nearly paracompact and extendable spaces via Andrijević’s b-open set [3] and also the b-locally finite. These concepts split the relation between p-locally finite [10] and β-locally finite [5], P_3-paracompact [10] and β-paracompact [5], and P_3-expandable [10] and β-expandable [5]. For further investigation, we have studied works related in [18, 19]. To do these, we shall consider and study the new concepts, b-normal space, b-G_δ set and b-extremally disconnected space.

Before studying the main section, we discuss a few words about the preliminaries of the article.

In a topological space (X, τ), a subset A of X is said to be b-open [3] if $A \subset Int(Cl(A)) \cup Cl(Int(A))$. Here “$Int$" and “$Cl$" stand for interior and closure operator respectively. Complement of a b-open set is called b-closed set. The class of all b-open sets in a topological space (X, τ) will be denoted by $BO(X)$. If A be a subset of a topological space (X, τ), the b-closure [3] of A, denoted by $bcl(A)$, is the smallest b-closed set containing A. The b-interior [3] of A, denoted by $bint(A)$, is the largest b-open set contained in A. For any $x \in X$ in a topological space (X, τ), the class of all b-open sets containing x will be denoted by $BO(X, x)$. If the collection of b-open subsets of a topological space (X, τ) is closed under finite intersection, then the space is called B^*-space [1]. A subset A of a topological space (X, τ) is called semi-open set [16] (resp., preopen set [2], β-open set [15]) if $A \subseteq Cl(Int(A))$ (resp., $A \subseteq Int(Cl(A))$, $A \subseteq Cl(Int(Cl(A)))$.

2 Family of second stair locally finiteness

Recall that a family $\mathscr{A} = \{A_\alpha : \alpha \in \Delta\}$ of subsets of a topological space (X, τ) is said to be locally finite [21] (resp., s-locally finite [11], p-locally finite [10], β-locally finite [5]) if for each $x \in X$, there exists an open (resp., a semi-open, a pre-open, a β-open) set Q_x containing x that intersects at most finitely many members of $\mathscr{A}$.

Definition 2.1. A collection $\mathscr{A} = \{A_\alpha : \alpha \in \Delta\}$ of subsets of a topological space (X, τ) is said to be second stair locally finite or simply $\Lsh\!\Rsh$-locally finite if for each $x \in X$, there exists a b-open set Q_x containing x such that $\{\alpha \in \Delta : Q_x \cap A_\alpha \neq \varnothing\}$ is at most finite.

Regarding to different types of locally finiteness mentioned just above we have following diagram where validation of reverse directions are considered in subsequent discussion.

$$\begin{tikzcd}
\text{locally finite} \arrow[r] \arrow[d] & s\text{-locally finite} \arrow[d] & \\
p\text{-locally finite} \arrow[r] & \Lsh\!\Rsh\text{-locally finite} \arrow[r] & \beta\text{-locally finite}
\end{tikzcd}$$

Figure 1. Diagram

In [11] (resp., [10]), it can be found that an s-locally finite (resp., p-locally finite) family may not be locally finite. Here, we demonstrate counter examples of $\Lsh\!\Rsh$-locally finite (resp., β-locally finite) family which is not s-locally and p-locally finite (resp., $\Lsh\!\Rsh$-locally finite).

Example 2.2. There exists a space and a family of subsets that is $\Lsh\!\Rsh$-locally finite but not s-locally finite. Suppose X is an infinite set endowed with the indiscrete topology, and consider the family $\mathscr{A} = \{\{x\} : x \in X\}$. Then $SO(X) = \{\varnothing, X\}$ and $BO(X) = 2^X$, the power set of X. These fulfill our pertinent requirements.

Example 2.3. There exists a space and a family of subsets that is $\Lsh\!\Rsh$-locally finite but not p-locally finite. Consider the set of reals $\mathbb{R}$ equipped with the usual topology τ_u, and pick the family $\mathscr{B} = \{[\frac{1}{n+1}, \frac{1}{n}] : n \in \mathbb{N}\}$, where $\mathbb{N}$ stands for the set of positive integers. Then $\Lsh\!\Rsh$-locally finiteness of $\mathscr{B}$ is pending only at 0, and one can tackle this issue by considering the b-open set $(-\frac{1}{2}, 0]$. Thus $\mathscr{B}$ is $\Lsh\!\Rsh$-locally finite. We claim that $\mathscr{B}$ is not p-locally finite. Indeed, for a pre-open set Q containing 0, we have $0 \in int(cl(Q))$, yielding that there exists $r > 0$ such that $(-r, r) \subseteq cl(Q)$. By Archimedean property, there exists $n_0 \in \mathbb{N}$ for which $\frac{1}{n_0} < r$ and so $(0, \frac{1}{n_0}) \subseteq cl(Q)$. Thus, $(0, \frac{1}{k}) \subseteq cl(Q)$ for all $k \geq n_0$. Since $(0, \frac{1}{k})$ is itself an open set containing $\frac{1}{k+1} \in cl(Q)$, we have $(0, \frac{1}{k}) \cap A \neq \varnothing$ for all $k \geq n_0$. This implies that Q intersects infinitely many members of $\mathscr{B}$. Hence, p-locally finiteness of $\mathscr{B}$ is not provided at 0.

Example 2.4. There exists a space and a family of subsets that is β-locally finite but not $\Lsh\!\Rsh$-locally finite. Let us consider the space in Example 2.3, and the family $\mathscr{C} = \{\{\frac{(-1)^n}{n}\} : n \in \mathbb{N}\}$. Then β-locally finiteness of $\mathscr{C}$ is very clear at all non-zero points of $\mathbb{R}$. For the point $0 \in \mathbb{R}$, the β-open set $\{0\} \cup ((0, 1) \cap (\mathbb{R} \setminus \mathbb{Q}))$, where $\mathbb{Q}$ is the set of all rational numbers, fulfills our requirements. We now show that $\mathscr{C}$ is not $\Lsh\!\Rsh$-locally finite at 0. In fact, for a b-open set Q containing 0, we have $0 \in int(cl(Q))$ or $0 \in cl(int(Q))$. For the first case, a parallel argument as in Example 2.3 concludes that $\mathscr{C}$ is not $\Lsh\!\Rsh$-locally finite. For the second case, $(-\frac{1}{n}, \frac{1}{n}) \cap int(Q) \neq \varnothing$ for every $n \in \mathbb{N}$. Pick $x \in (-\frac{1}{n}, \frac{1}{n}) \cap int(Q)$. Then there exists $r > 0$ such that $(x - r, x + r) \subseteq Q$ and $-\frac{1}{n} < x < \frac{1}{n}$ for every n. By Archimedean property, there exists $n_0 \in \mathbb{N}$ for which $0 < \frac{1}{n_0} < r$, implying that $x - r < 0$ and $x + r > 0$. So $(x - r, x + r)$ intersects infinitely many elements of $\mathscr{C}$ and hence Q also, as required.

Remark 2.5. In a b-space [1], the notion of $\Lsh\!\Rsh$-locally finiteness coincides with that of locally, s-locally and p-locally finiteness. Note that converse is not true in general.

Lemma 2.6. *Let (X, τ) be a topological space, and $\mathscr{A} = \{A_\alpha : \alpha \in \Delta\}$ its a collection of subsets. Then*

(i) $\mathscr{A}$ is $\Lsh\!\Rsh$-locally finite and X is B^*-space imply $\bigcup\limits_{\alpha \in \Delta} bcl(A_\alpha) = bcl(\bigcup\limits_{\alpha \in \Delta} A_\alpha)$;

(ii) $\mathscr{A}$ is $\Lsh\!\Rsh$-locally finite and $B_\alpha \subseteq A_\alpha$ for each $\alpha \in \Delta$ implies $\{B_\alpha : \alpha \in \Delta\}$ is $\Lsh\!\Rsh$-locally finite;

(iii) $\mathscr{A}$ is $\Lsh\!\Rsh$-locally (resp., locally, s-locally, p-locally) finite if and only if $\{bcl(A_\alpha) : \alpha \in \Delta\}$ is $\Lsh\!\Rsh$-locally (resp., locally, s-locally, p-locally) finite.

Proof. We provide the proof of 1 only. The inclusion $\bigcup\limits_{\alpha\in\Delta} bcl(A_\alpha) \subseteq bcl(\bigcup\limits_{\alpha\in\Delta} A_\alpha)$ is obvious. For reverse inclusion, let $x \notin \bigcup\limits_{\alpha\in\Delta} bcl(A_\alpha)$. Since $\mathscr{A}$ is $\Lsh\Rsh$-locally finite, there exists $Q \in BO(X,x)$ and a finite subset $\Delta_0 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of Δ such that $Q \cap A_\alpha = \varnothing$ for all $\alpha \in \Delta \setminus \Delta_0$. Now, for each $i = 1, 2, \ldots, n$, $x \notin bcl(A_{\alpha_i})$ and hence there exists $G_{\alpha_i(x)} \in BO(X,x)$ such that $G_{\alpha_i(x)} \cap A_{\alpha_i} = \varnothing$. Take $G = Q \cap (G_{\alpha_1(x)} \cap G_{\alpha_2(x)} \cap \cdots \cap G_{\alpha_n(x)})$. Then $G \in BO(X,x)$ because X is B^*-space, and $G \cap A_\alpha = \varnothing$ for all $\alpha \in \Delta$. So $G \cap (\bigcup\limits_{\alpha\in\Delta} A_\alpha) = \varnothing$, establishing that $x \notin bcl(\bigcup\limits_{\alpha\in\Delta} A_\alpha)$. Hence, $bcl(\bigcup\limits_{\alpha\in\Delta} A_\alpha) \subseteq \bigcup\limits_{\alpha\in\Delta} bcl(A_\alpha)$, and this completes the proof. □

Recall that a function $f : (X,\tau) \to (Y,\sigma)$ is called b-closed [14] (resp., b-open) if for every b-closed (resp., b-open) set U in X, $f(U)$ is b-closed (resp., b-open) in Y. A characterization of b-closed maps is exhibited below without its proof.

Theorem 2.7. *A function $f : (X,\tau) \to (Y,\sigma)$ is b-closed if and only if for every $Q \subseteq Y$ and every b-open set $U \supseteq f^{-1}(Q)$, there exists a b-open set V in Y such that $f^{-1}(V) \subseteq U$ and $Q \subseteq V$.*

Corollary 2.8. *A function $f : (X,\tau) \to (Y,\sigma)$ is b-closed if and only if for every $y \in Y$ and every b-open set $U \supseteq f^{-1}(y)$, there exists a b-open set V in Y containing y such that $f^{-1}(V) \subseteq U$.*

Recall that a function $f : (X,\tau) \to (Y,\sigma)$ is called b-irresolute if for every b-open set V in Y, $f^{-1}(V)$ is b-open in X. In next result, we have tried to find functions under which direct and inverse images of $\Lsh\Rsh$-locally finite family are $\Lsh\Rsh$-locally finite. We rule out the proofs for easiness.

Theorem 2.9. *Suppose $f : (X,\tau) \to (Y,\sigma)$ is a function. Then*

(i) f is a bijective b-open map and $\mathscr{A} = \{A_\alpha : \alpha \in \Delta\}$ is a $\Lsh\Rsh$-locally finite family of subsets of X imply $f(\mathbf{A}) = \{f(A_\alpha) : \alpha \in \Delta\}$ is $\Lsh\Rsh$-locally finite.

(ii) f is a b-irresolute map and $\mathscr{B} = \{B_\alpha : \alpha \in \Delta\}$ is a $\Lsh\Rsh$-locally finite family of subsets of Y implies $f^{-1}(\mathbf{B}) = \{f^{-1}(B_\alpha) : \alpha \in \Delta\}$ is $\Lsh\Rsh$-locally finite.

Theorem 2.10. *A surjective b-closed function $f : (X,\tau) \to (Y,\sigma)$ with the property that for every $y \in Y$, $f^{-1}(y)$ is b-compact relative to X preserves $\Lsh\Rsh$-locally finite families.*

Proof. Let $\mathscr{A} = \{A_\alpha : \alpha \in \Delta\}$ be a $\Lsh\Rsh$-locally finite family of subsets of X, and $f(\mathscr{A}) = \{f(A_\alpha) : \alpha \in \Delta\}$. For $y \in Y$, let $x \in f^{-1}(y)$. Since $\mathscr{A}$ is $\Lsh\Rsh$-locally finite, there exists a $Q_x \in BO(X,x)$ such that $Q_x \cap A_\alpha = \varnothing$ except for finitely many α's. Clearly then $\{Q_x : x \in f^{-1}(y)\}$ covers $f^{-1}(y)$. So, we can pick $x_1, x_2, \ldots, x_n \in f^{-1}(y)$ such that $f^{-1}(y) \subseteq \bigcup\limits_{i=1}^{n} Q_{x_i}$. Take $Q = \bigcup\limits_{i=1}^{n} Q_{x_i}$. Then Q is a b-open set and $f^{-1}(y) \subseteq Q$. By Corollary 2.8, there exists a $V \in BO(Y,y)$ such that $f^{-1}(V) \subseteq Q$. Since Q intersects finitely many A_α's, we have $V \cap f(A_\alpha) = \varnothing$ except for finitely many α's. This completes the proof. □

3 Second stair paracompactness

Recall that a topological space (X,τ) is termed S-paracompact [12] (resp., P_3-paracompact [10], β-paracompact [5]) if for every open covering $\mathscr{A}$ of X has a locally (resp., p-locally, β-locally) finite semi-open (resp., preopen, β-open) refinement.

Definition 3.1. A topological space (X,τ) is said to be second stair paracompact or simply $\Lsh\Rsh$-paracompact if every open covering $\mathscr{A}$ of X has a $\Lsh\Rsh$-locally finite b-open refinement.

In view of the above context of paracompact spaces, following diagram is very transparent.

$$\begin{tikzcd}
\text{paracompact} \arrow[r] \arrow[d] & S\text{-paracompact} \arrow[d] & \\
P_3\text{-paracompact} \arrow[r] & \Lsh\Rsh\text{-paracompact} \arrow[r] & \beta\text{-paracompact}
\end{tikzcd}$$

Figure 2. Diagram

If the space (X, τ) becomes b-space [1], then ↰↱-paracompactness coincides with paracompactness.

Definition 3.2. A subset A of a topological space (X, τ) is said to be b-regular [9] if it is b-open as well as b-closed.

The collection of all b-regular sets in a topological space (X, τ) is denoted by $BR(X)$.

Proposition 3.3 ([1]). *(i) If P is b-open, then $bcl(P)$ is b-open.*

(ii) If Q is b-closed, then $bint(Q)$ is b-closed.

Theorem 3.4. *If (X, τ) is ↰↱-paracompact, Hausdorff and B^*-space, then for any closed subset A of X and $x \notin A$, there exists a b-regular set U and a b-open set V such that $x \in U$, $A \subseteq V$ and $U \cap V = \varnothing$.*

Proof. The Hausdorff condition enables us to choose for each $p \in A$, an open set O_p such that $p \in O_p$ and $x \notin Cl(O_p)$. Therefore the family $\mathscr{O} = \{O_p : p \in A\} \cup \{X \setminus A\}$ is an open cover of X and so it has a ↰↱-locally finite b-open refinement $\mathscr{H}$. Put $V = \cup\{H \in \mathscr{H} : H \cap A \neq \varnothing\}$. Then V is a b-open set containing A. Also since (X, τ) is a B^*-space, $bcl(V) = \cup\{bcl(H) : H \in \mathscr{H}$ and $H \cap A \neq \varnothing\}$. Since V is a b-open set, then $bcl(V)$ is also b-open set and by definition $bcl(V)$ is the smallest b-closed set containing V. Hence $bcl(V)$ is a b-regular subset of X. Therefore $U = X \setminus bcl(V)$ is a b-regular set such that $U \cap V = \varnothing$. Now we have to show that $x \in U$. Since $H \in \mathscr{H}$, then $Cl(H)$ is disjoint from p and hence $bcl(H)$ is disjoint from p as $bcl(H) \subseteq Cl(H)$ for all H. For each $H \in V$, there exists $O_p \in \mathscr{O}$ such that $H \subseteq O_p$ and $x \notin Cl(O_p)$. This implies $x \notin Cl(H)$ and hence $x \notin bcl(H)$. This implies $x \in X \setminus bcl(H)$ and hence $x \in U$. Hence the required result follows. □

Corollary 3.5. *If (X, τ) is Hausdorff ↰↱-paracompact space, then for any closed subset A of X and $x \notin A$, there exists an open set U and a b-open set V such that $x \in U$, $A \subseteq V$ and $U \cap V = \varnothing$.*

Proposition 3.6 ([3]). *If S is a b-open set such that $Int(S) = \varnothing$, then S is preopen.*

Lemma 3.7 ([6]). *A space (X, τ) is submaximal if and only if every preopen set is open.*

Theorem 3.8. *If (X, τ) is a submaximal space and S be a b-open set with $Int(S) = \varnothing$, then S is open.*

Proof. Let S be a b-open set with $Int(S) = \varnothing$. Then S is a preopen set by Theorem 3.6. Also since (X, τ) is submaximal, then S is open by Lemma 3.7. □

Definition 3.9. A topological space (X, τ) is said to be b-regular [9] if for each closed set F and each point $x \notin F$, there exists disjoint b-open sets U and V such that $x \in U$ and $F \subseteq V$.

Definition 3.10. A topological space (X, τ) is said to be b-normal if for any two disjoint closed subsets C and D, there exists disjoint b-open sets U and V such that $C \subseteq U$ and $D \subseteq V$.

Clearly every normal spaces is b-normal. But the reverse may not be true.

Lemma 3.11. *A topological space (X, τ) is b-normal if and only if given a closed set A and an open set U containing A, there exists a b-regular set V' containing A such that $V' \subseteq U$.*

Proof. Suppose that X is b-normal and suppose that a closed set A and an open set U containing A are given. Put $B = X \setminus U$. Then B is a closed set. Thus by hypothesis, there exist two disjoint b-open sets V and W containing A and B respectively. If $y \in B$, then W is a b-open set containing y such that $W \cap V = \varnothing$ and hence $y \notin bcl(V)$. Since y is an arbitrary member of B, then $bcl(V) \cap B = \varnothing$ and hence $bcl(V) \subseteq U$. Since $V \in BO(X)$, then $bcl(V) \in BR(X)$. Put $V' = bcl(V)$. Then $V' \subseteq U$, where V' a b-regular set containing A.

To prove the converse, consider two disjoint closed sets A and B. Put $U = X \setminus B$. Then U is an open set containing A. So by hypothesis, there exists a b-regular set V' containing A such that $V' \subseteq U$. This implies V' and $X \setminus V'$ are two disjoint b-regular sets containing A and B respectively. Since b-regular set is also b-open, then the space (X, τ) is b-normal. □

Theorem 3.12. *Every $\Lsh\!\Rsh$-paracompact, Hausdorff and B^*-space is b-normal.*

Proof. Let (X, τ) be a $\Lsh\!\Rsh$-paracompact, Hausdorff space and A and B be two disjoint closed subsets of X. Then by Corollary 3.5, for each $b \in B$, there exist an open set U_b containing b and a b-open set V containing A such that $U_b \cap V = \varnothing$. Thus $b \notin bcl(V)$. Therefore the family $\mathscr{W} = \{U_b : b \in B\} \cup \{X \setminus B\}$ is an open cover of X. So it has a $\Lsh\!\Rsh$-locally finite b-open refinement $\mathscr{H}$. Put $V' = \bigcup\{H \in \mathscr{H} : H \cap B \neq \varnothing\}$. Then V' is a b-open set containing B and hence $bcl(V') = \cup\{bcl(H) : H \in \mathscr{H}$ and $H \cap B \neq \varnothing\}$ is disjoint from A as X is a B^*-space. Therefore $U = X \setminus bcl(V')$ is a b-regular set containing A such that $U \cap V' = \varnothing$. Since b-regular set is also a b-open, thus any two disjoint closed sets can be separated by two disjoint b-open sets. Hence the required result. □

Definition 3.13. A topological space (X, τ) is said to be b-compact [9] if every cover of X by b-open sets has a finite subcover.

Theorem 3.14. *Every b-compact space is $\Lsh\!\Rsh$-paracompact.*

Proof. The proof is obvious and hence omitted. □

Definition 3.15. A subset **A** of a topological space (X, τ) is said to be b-G_δ set in X if it equals the intersection of a countable collection of b-open subsets of X.

Every G_δ set in X is a b-G_δ set in X but the reverse may not be true.

Definition 3.16. A collection $\mathscr{B}$ of subsets of X is said to be countably b-locally finite if $\mathscr{B}$ can be written as the countable union of collection $\mathscr{B}_n$, each of which is b-locally finite.

Lemma 3.17. *Let (X, τ) be a B^* and b-regular space with a basis $\mathscr{B}$ that is countably $\Lsh\!\Rsh$-locally finite. Then X is b-normal and every closed set in X is a b-G_δ set in X.*

Proof. We consider the following three steps:

Step 1: Let Ω be an open set in X. We are to show that there is a countable collection $\{U_n\}$ of b-open subsets of X such that $\Omega = \bigcup U_n = \bigcup bcl(U_n)$. Since the basis $\mathscr{B}$ for X is countably $\Lsh\!\Rsh$-locally finite, then $\mathscr{B} = \bigcup \mathscr{B}_n$, where each collection $\mathscr{B}_n$ is $\Lsh\!\Rsh$-locally finite. Let $\mathscr{C}_n$ be the collection of those basis elements B such that $B \in \mathscr{B}_n$ and $bcl(B) \subseteq \Omega$. Then $\mathscr{C}_n$ is $\Lsh\!\Rsh$-locally finite by Lemma 2.6(2). Define $U_n = \bigcup\limits_{B \in \mathscr{C}_n} B$. Then U_n is a b-open set and by Lemma 2.6(1), $bcl(U_n) = \bigcup\limits_{B \in \mathscr{C}_n} bcl(B)$ as X is B^*-space. Since for each $B \in \mathscr{C}_n$, $bcl(B) \subseteq \Omega$, then $\bigcup\limits_{B \in \mathscr{C}_n} bcl(B) \subseteq \Omega$ and hence $bcl(U_n) \subseteq \Omega$, so that $\bigcup U_n \subseteq \bigcup bcl(U_n) \subseteq \Omega$. We assert that equality holds. Let $x \in \Omega$, then by b-regularity, there exists a b-regular set B_1 such that $x \in B_1$ and $B_1 \subseteq \Omega$ [7]. Since B_1 is b-regular, then $bcl(B_1) = B_1$ and hence $bcl(B_1) \subseteq \Omega$. Since $B_1 \in \mathscr{B}_n$ for some n, then $B_1 \in \mathscr{C}_n$ by definition and hence $x \in U_n$. Thus $\Omega \subseteq \bigcup U_n$ and hence $\Omega = \bigcup U_n = \bigcup bcl(U_n)$.

Step 2: We are to show that every closed set C in X is a b-G_δ set in X. Let $\Omega = X \setminus C$. Then by Step 1, there are b-open sets U_n in X such that $\Omega = \bigcup bcl(U_n)$. Since each U_n is a b-open sets, then each $bcl(U_n)$ is b-regular set. Hence Ω is a b-open set. Also since $bcl(U_n)$

is a b-regular set, then $bcl(U_n)$ must be a b-closed set and $X \setminus bcl(U_n)$ is a b-open set. Now from $\Omega = X \setminus C$, $C = \Omega^c = (\bigcup(bcl(U_n)))^c = \bigcap(X \setminus bcl(U_n))$, where Ω^c denotes the complement of Ω. Thus C equals countable intersection of b-open sets of X. Hence C is a b-G_δ set in X.

Step 3: Now we are to show that X is b-normal. Let C and D be any two disjoint closed subsets of X. Applying Step 1 to the open set $X \setminus D$, we construct a collection $\{U_n\}$ of b-open sets such that $C \subseteq X \setminus D = \bigcup U_n = bcl(U_n)$. Then $\{U_n\}$ covers C and each set $bcl(U_n)$ is disjoint from D. Similarly, there is a countable covering $\{V_n\}$ of D by b-open sets whose b-closures are disjoint from C. Define ${U_n}' = U_n \setminus \bigcup\limits_{i=1}^{n} bcl(V_i)$ and ${V_n}' = V_n \setminus \bigcup\limits_{i=1}^{n} bcl(U_i)$. Then ${U_n}'$ and ${V_n}'$ both are b-open sets for each $n \in \mathbb{N}$, since complement of b-closed set is b-open set and in a B^*-space, intersection of two b-open sets is b-open. The collection $\{{U_n}'\}$ covers C as each $x \in C$ belongs to U_n for some $n \in \mathbb{N}$ and x belongs to none of the set $bcl(V_i)$. Similarly, the collection $\{{V_n}'\}$ covers D. Put $U' = \bigcup\limits_{n \in \mathbb{Z}_+} {U_n}'$ and $V' = \bigcup\limits_{n \in \mathbb{Z}_+} {V_n}'$. Then U' and V' both are b-open sets. Now we claim that U' and V' are disjoint. If not, let $x \in U' \cap V'$, then $x \in U_j' \cap V_k'$ for some j and k. Suppose $j \leq k$. It follows from definition of U_j that $x \in U_j$ and since $j \leq k$, it follows from definition of V_k', that $x \notin bcl(U_j)$, which is a contradiction. A similar contradiction arise if $j \geq k$. Hence U' and V' are disjoint. Thus the sets U' and V' are disjoint b-open sets containing C and D respectively. Hence X is b-normal. □

Corollary 3.18. *If (X, τ) is B^* and regular space with a basis $\mathscr{B}$ that is countably $\Lsh\!\Rsh$-locally finite. Then X is b-normal and every closed set in X is a b-G_δ set in X.*

Definition 3.19. A space (X, τ) is called b-extremally disconnected if the b-closure of every open subset is open.

Theorem 3.20. *Let (X, τ) be a b-extremally disconnected regular space. If (X, τ) is a $\Lsh\!\Rsh$-paracompact space, then (X, τ) is a paracompact space.*

Proof. Let $\mathscr{U}$ be an open cover of X. For each $x \in X$, choose a member $U_x \in \mathscr{U}$ and by regularity, there exists $V_x \in \tau_x$ where $\tau_x = \{V \in \tau : x \in V\}$ such that $x \in V_x \subseteq Cl(V_x) \subseteq U_x$. Therefore $\mathscr{W} = \{V_x : x \in X\}$ is an open cover of X and so by assumption it has a $\Lsh\!\Rsh$-locally finite b-open refinement, say $\mathscr{W} = \{W_\alpha : \alpha \in \Delta\}$. For every $\alpha \in \Delta$, choose an open set H_α such that $H_\alpha \subseteq W_\alpha \subseteq bcl(H_\alpha)$. But for every $\alpha \in \Delta$, $bcl(H_\alpha) = bcl(W_\alpha) \subseteq Cl(W_\alpha) \subseteq Cl(V_x)$ for some $V_x \in \tau(x)$ and so $bcl(H_\alpha) \subseteq U$ for some $U \in \mathscr{U}$. On the other hand, since (X, τ) is b-extremally disconnected, then $bcl(H_\alpha) \in \tau$ for every $\alpha \in \Delta$. Now we have to show that the collection $\mathscr{H} = \{bcl(H_\alpha) : \alpha \in \Delta\}$ is a locally finite. Since $\{bcl(W_\alpha) : \alpha \in \Delta\}$ is $\Lsh\!\Rsh$-locally finite by Lemma 2.6(3) and $bcl(H_\alpha) \subseteq bcl(W_\alpha)$ for all $\alpha \in \Delta$, then the collection $\mathscr{H} = \{bcl(H_\alpha) : \alpha \in \Delta\}$ is $\Lsh\!\Rsh$-locally finite by Lemma 2.6(2). Let $x \in X$, then there exists a b-open set B_x containing x that intersects at most finitely many members of $\mathscr{H}$. Finally, choose $A_x \in \tau(x)$ such that $A_x \subseteq B_x \subseteq bcl(A_x)$. Since (X, τ) is b-extremally disconnected, $bcl(A_x)$ is an open set containing x with the property $bcl(A_x) \cap bcl(H_\alpha) \neq \varnothing$ if and only if $B_x \cap bcl(H_\alpha) \neq \varnothing$. Then $\mathscr{H}$ is a locally finite open refinement of $\mathscr{U}$. Hence (X, τ) is a paracompact space. □

Theorem 3.21. *Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous, b-open, b-closed and surjective function such that $f^{-1}(y)$ is b-compact for each $y \in Y$. If (X, τ) be $\Lsh\!\Rsh$-paracompact, then (Y, σ) is also $\Lsh\!\Rsh$-paracompact.*

Proof. Let $\mathscr{U} = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of (Y, σ). Then $f^{-1}(\mathscr{U}) = \{f^{-1}(U_\alpha) : \alpha \in \Delta\}$ is an open cover of (X, τ). So by assumption, $f^{-1}(\mathscr{U})$ has a $\Lsh\!\Rsh$-locally finite b-open refinement say, $\mathscr{V} = \{V_\alpha : \alpha \in \Delta\}$. Since f is b-open, then the collection $f(\mathscr{V}) = \{f(V_\alpha) : \alpha \in \Delta\}$ is a b-open refinement of $\mathscr{U}$. Now we are to show that the collection $f(\mathscr{V})$ is b-locally finite in (Y, σ). Let $y \in Y$. Then for each $x \in f^{-1}(y)$, there exists a b-open set U_x containing x such that U_x intersects at most finitely many members of $\mathscr{V}$. The collection $\{U_x : x \in f^{-1}(y)\}$ is a b-open cover of $f^{-1}(y)$ and therefore, there exists a finite subset K of $f^{-1}(y)$ such that $f^{-1}(y) \subseteq \bigcup\limits_{x \in K} U_x$ as $f^{-1}(y)$ is b-compact for each $y \in Y$. Since f is a b-closed, then by Corollary 2.8, there exists a b-open set O_y containing y in (Y, σ) such that $f^{-1}(O_y) \subseteq \bigcup\limits_{x \in K} U_x$. Then $f^{-1}(O_y)$ intersects at most finitely many members of $\mathscr{V}$. Then O_y intersects at most finitely many members of $f(\mathscr{V})$. Thus $f(\mathscr{V})$ is $\Lsh\Rsh$-locally finite in (Y, σ). Hence the result. □

Proposition 3.22 ([3]). *In a topological space, intersection of an open set and a b-open set is a b-open set.*

Theorem 3.23. *Let $f : (X, \tau) \to (Y, \sigma)$ is a b-irresolute, closed and surjective function such that $f^{-1}(y)$ is compact for each $y \in Y$. If (Y, σ) be $\Lsh\Rsh$-paracompact, then (X, τ) is also $\Lsh\Rsh$-paracompact.*

Proof. Let $\mathscr{U} = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of (X, τ). Since $f^{-1}(y)$ is compact for each $y \in Y$, then there exists a finite subset $\Delta(y)$ of Δ such that $f^{-1}(y) \subseteq \bigcup\limits_{\alpha \in \Delta(y)} U_\alpha$. Also since f is closed, then there exists an open set V_y in Y containing y such that $f^{-1}(V_y) \subseteq \bigcup\limits_{\alpha \in \Delta(y)} U_\alpha$. Therefore $\{V_y : y \in V\}$ is an open cover of (Y, σ). So by assumption it has a $\Lsh\Rsh$-locally finite b-open refinement $\mathscr{W} = \{W_\alpha : \alpha \in \Delta\}$. Since f is b-irresolute, $\{f^{-1}(W_\alpha) : \alpha \in \Delta\}$ is a $\Lsh\Rsh$-locally finite by Theorem 2.9(2). Again, since for all $\alpha \in \Delta$, W_α is b-open in (Y, σ) and f is b-irresolute, then each $f^{-1}(W_\alpha)$ is b-open in (X, τ). Then the collection $\{f^{-1}(W_\alpha) : \alpha \in \Delta\}$ becomes a $\Lsh\Rsh$-locally finite collection of b-open subsets in (X, τ). As $\{W_\alpha : \alpha \in \Delta\}$ is a refinement of $\{V_y : y \in V\}$, then for all $\alpha \in \Delta$, there exists $y(\alpha) \in Y$ such that $W_\alpha \subseteq V_{y(\alpha)}$. Therefore $f^{-1}(W_\alpha) \subseteq f^{-1}(V_{y(\alpha)}) \subseteq \bigcap\limits_{\delta \in \Delta(y(\alpha))} U_\delta = F_{y(\alpha)}$. Put $\mathscr{F} = \{f^{-1}(W_\alpha) \bigwedge F_{y(\alpha)} : \alpha \in \Delta \text{ and } y(\alpha) \in Y\}$ where $f^{-1}(W_\alpha) \bigwedge F_{y(\alpha)} = \{f^{-1}(W_\alpha) \cap U_\delta : \alpha \in \Delta \text{ and } \delta \in \Delta(y(\alpha))\}$. Hence, by Proposition 3.22, each set of $\mathscr{F}$ is a b-open subset of X. Then the family $\mathscr{F}$ is a $\Lsh\Rsh$-locally finite b-open refinement of $\mathscr{U}$. Thus, (X, τ) is $\Lsh\Rsh$-paracompact. □

Theorem 3.24. *Let (X, τ) be a compact space and (Y, σ) be $\Lsh\Rsh$-paracompact space. Then $(X \times Y, \tau \times \sigma)$ is $\Lsh\Rsh$-paracompact.*

Proof. Let $f : (X, \tau) \times (Y, \sigma) \to (Y, \sigma)$ be a projection map. Then f is open, surjection and continuous. Thus f is b-irresolute. Also, f is closed because X is compact. For each $y \in Y$, $f^{-1}(y) = X \times \{y\}$ is a compact subset of $X \times Y$ because $X \times \{y\}$ is homeomorphism to X. Since (Y, σ) is $\Lsh\Rsh$-paracompact, then by Theorem 3.23, $(X \times Y, \tau \times \sigma)$ is $\Lsh\Rsh$-paracompact. □

Theorem 3.25. *Let (X, τ) be a regular space. Then (X, τ) is $\Lsh\Rsh$-paracompact if and only if every open cover of X has a $\Lsh\Rsh$-locally finite b-closed refinement.*

Proof. Let $\mathscr{U}$ be an open cover of X. For each $x \in X$, we choose a member $U_x \in \mathscr{U}$ and by the regularity of (X, τ), an open subset V_x such that $x \in V_x \subseteq Cl(V_x) \subseteq U_x$. Therefore $\mathscr{V} = \{V_x : x \in X\}$ is an open cover of X and so by assumption, it has a $\Lsh\Rsh$-locally finite b-open refinement, say $\mathscr{W} = \{\omega_\alpha : \alpha \in \Delta\}$. Now, consider the collection $bcl(\mathscr{W}) = \{bcl(\omega_\alpha) : \alpha \in \Delta\}$. Then by Lemma 2.6 (3), $bcl(\mathscr{W})$ is a $\Lsh\Rsh$-locally finite collection of b-regular subsets of (X, τ) such that for every $\alpha \in \Delta$, $bcl(\omega_\alpha) \subseteq bcl(V_x) \subseteq Cl(V_x) \subseteq U_x$ for some $U_x \in \mathscr{U}$. Since b-regular set is b-open as well as b-closed, then $bcl(\mathscr{W})$ is a $\Lsh\Rsh$-locally finite b-closed refinement of $\mathscr{U}$.

Conversely, let $\mathscr{U}$ be an open cover of X and let $\mathscr{V}$ be a $\Lsh\Rsh$-locally finite b-closed refinement of $\mathscr{U}$. For $x \in X$, choose $\omega_x \in BO(X, \tau)$ such that $x \in \omega_x$ and ω_x intersects at most finitely many members of $\mathscr{V}$. Let $\mathscr{H}$ be a b-closed $\Lsh\Rsh$-locally finite refinement of $\mathscr{W} = \{W_x : x \in X\}$. For each $V \in \mathscr{V}$, let $V' = X \setminus \bigcap\{H \in \mathscr{H} : H \cap V = \varnothing\}$. Then V' is b-open and the collection $\{V' : V' \in \mathscr{V}\}$ becomes a b-open cover of X. Finally, for each $V \in \mathscr{V}$, choose $U_V \in \mathscr{U}$ such that $V \subseteq U_V$. Therefore the collection $\{U_V \cap V : V \in \mathscr{V}\}$ is $\Lsh\Rsh$-locally finite b-open refinement of $\mathscr{U}$ as intersection of a b-open set and an open set is b-open by Proposition 3.22. Hence (X, τ) is $\Lsh\Rsh$-paracompact. □

4 Second stair expandable spaces

Recall that a space (X, τ) is termed expandable [13] (resp., S-expandable [11], P_3-expandable [10], β-expandable [5]) if for every locally finite collection $\mathscr{F} = \{F_\alpha : \alpha \in \Lambda\}$ of subsets of X, there exists a locally finite (resp., s-locally, p-locally, β-locally) collection $\mathscr{G} = \{G_\alpha : \alpha \in \Lambda\}$ of open (resp., s-open, p-open, β-open) subsets of X such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Lambda$.

Definition 4.1. A space (X, τ) is said to be second stair expandable or simply ↰↱-expandable if for every locally finite collection $\mathscr{F} = \{F_\alpha : \alpha \in \Lambda\}$ of subsets of X, there exists a ↰↱-locally finite collection $\mathscr{G} = \{G_\alpha : \alpha \in \Lambda\}$ of b-open subsets of X such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Lambda$.

In view of the above context of ↰↱-expandable spaces, following diagram is very transparent.

$$\begin{tikzcd}
\text{Expandable} \arrow[r] \arrow[d] & S\text{-expandable} \arrow[d] & \\
P_3\text{-expandable} \arrow[r] & \Lsh\Rsh\text{-expandable} \arrow[r] & \beta\text{-expandable}
\end{tikzcd}$$

Figure 3. Diagram

Theorem 4.2. *Let (X, τ) be a topological space. Then the followings are equivalent:*
(1) *X is ↰↱-expandable.*
(2) *For every locally finite collection $\mathscr{F} = \{F_\alpha : \alpha \in \Lambda\}$ of closed subsets of X there exists a ↰↱-locally finite collection $\mathscr{G} = \{G_\alpha : \alpha \in \Lambda\}$ of b-open subsets of X such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Lambda$.*

Proof. (1) $\implies$ (2): Let X is ↰↱-expandable and assume that $\mathscr{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a locally finite collection of closed subsets of X. Then there exists a ↰↱-locally finite collection $\mathscr{G} = \{G_\alpha : \alpha \in \Lambda\}$ of b-open subsets of X such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Lambda$.
(2) $\implies$ (1): Let $\mathscr{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a locally finite collection of subsets of X. Then $\{Cl(F_\alpha) : \alpha \in \Lambda\}$ is a locally finite collection of closed subsets of X. Then, by hypothesis, there exists a ↰↱-locally finite collection $\mathscr{G} = \{G_\alpha : \alpha \in \Lambda\}$ of b-open subsets of X such that $Cl(F_\alpha) \subseteq G_\alpha$ for all $\alpha \in \Lambda$. Since $F_\alpha \subseteq Cl(F_\alpha)$ for all $\alpha \in \Lambda$, then for all $\alpha \in \Lambda$, $F_\alpha \subseteq G_\alpha$ and hence X is ↰↱-expandable. □

Theorem 4.3. *Every ↰↱-paracompact space is ↰↱-expandable.*

Proof. Let X is a ↰↱-paracompact space and $\mathscr{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a locally finite collection of closed subsets of X. Let Γ be the collection of all finite subsets of Λ. For each $\gamma \in \Lambda$, let $V_\gamma = X \setminus \bigcup\{F_\alpha : \alpha \notin \gamma\}$. Since $\mathscr{F}$ is the locally finite collection of closed subsets of X, V_γ is open. Also V_γ meets only finitely many elements of $\mathscr{F}$. Let $\mathscr{V} = \{V_\gamma : \gamma \in \Gamma\}$. Then $\mathscr{V}$ is an open cover of X. Then $\mathscr{V}$ has a ↰↱-locally finite b-open refinement, say $\mathscr{W} = \{W_\delta : \delta \in \Delta\}$, because (X, τ) is ↰↱-paracompact. Set $U_\alpha = \{W_\delta \in \mathscr{W} : W_\delta \cap F_\alpha \neq \varnothing \text{ for all } \alpha \in \Lambda\}$. Since arbitrary union of b-open sets is b-open, then U_α is b-open and $F_\alpha \subseteq U_\alpha$ for all $\alpha \in \Lambda$. Now we are to prove that $\{U_\alpha : \alpha \in \Lambda\}$ is ↰↱-locally finite. Since $\mathscr{W}$ is ↰↱-locally finite, for each $x \in X$, there exists a b-open set U_x containing x in (X, τ) such that $\Delta_1 = \{\delta \in \Delta : U_x \cap W_\delta \neq \varnothing\}$ is at most finite. Also by the definition of U_α, we say that $U_x \cap U_\delta \neq \varnothing$ if and only if $U_x \cap W_\delta \neq \varnothing$ and $W_\delta \cap F_\alpha \neq \varnothing$ for some $\delta \in \Delta$. Since $\mathscr{W}$ is refinement of $\mathscr{V}$, then there is an element V_γ of $\mathscr{V}$ containing W_δ for each member W_δ of $\mathscr{W}$. Then W_δ meets only finitely many F_α for each $\delta \in \Delta$. Thus $\{U_\alpha : \alpha \in \Lambda\}$ is ↰↱-locally finite. Hence the required result. □

Corollary 4.4. *Let (X, τ) be a space. Then (X, τ) is ↰↱-expandable if every open cover X has a ↰↱-locally finite b-open refinement.*

Let (X, τ) be a space and U be a clopen subset of X. Then every locally finite family of U is a locally finite in X. Moreover, the intersection of a b-open subset and an open subset is a b-open subset by Proposition 3.22. Thus we have the following theorem:

Theorem 4.5. *Let (X, τ) be a ↰↱-expandable space. Then every clopen subset of (X, τ) is ↰↱-expandable.*

Proof. Let U be a clopen subset of X and let $\mathscr{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a locally finite family of U. Since U is a clopen, then the family $\mathscr{F} = \{F_\alpha : \alpha \in \Lambda\}$ becomes a locally finite family of X. Also since X is ↰↱-expandable, then it has a ↰↱-locally finite b-open refinement $\mathscr{G} = \{G_\alpha : \alpha \in \Lambda\}$ such that $G_\alpha \subseteq F_\alpha$ for all $\alpha \in \Lambda$. Thus U is ↰↱-expandable. □

5 Conclusion remarks

(i) One can modify the '**Nagata-Smirnov metrization theorem**' by the following:

"A space X is $\Lsh\Rsh$-metrizable if X is regular and has a basis that is countably $\Lsh\Rsh$-locally finite".

or

"A space X is $\Lsh\Rsh$-metrizable if X is b-regular and has a basis that is countably $\Lsh\Rsh$-locally finite".

(ii) One can redefine the Uryshon's Lemma by the following modified version:

"Let X be a b-normal space, A and B be disjoint closed subsets of X. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous (resp. b-irresolute) map $f : X \to [a, b]$ such that $f(x) = a$ for every $x \in A$, and $f(x) = b$ for every $x \in B$".

(iii) Further one can define a modified version of completely regular space:

"A space X is b-completely regular if one-point sets are closed in X and for each point x_0 and each b-regular closed set A not containing x_0, there is a continuous (resp. b-irresolute) function such that $f(x_0) = 1$ and $f(A) = \{0\}$".

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