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BIPOLAR FUZZY ALGEBRAIC STRUCTURES ON IDEALS OF BE-ALGEBRAS

Beza Lamesgin Derseh

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Abstract. In this manuscript, the concept of bipolar fuzzy subsets is employed to analyze the ideals of BE-algebras within the context of bipolar fuzzification. The notion of bipolar fuzzy ideals in a BE-algebra is introduced, and some of their useful properties are investigated. Characterizations of bipolar fuzzy ideals are provided in terms of their level subsets. The homomorphism of bipolar fuzzy ideals in a BE-algebra is discussed, leading to intriguing results. Furthermore, the bipolar fuzzy ideals of a BE-algebra are studied in the context of Cartesian products, revealing several substantial results. Finally, the relationship between the strongest bipolar fuzzy relations in a BE-algebra and the bipolar fuzzy ideals in a BE-algebra is examined, yielding significant findings.

1 Introduction

L.A. Zadeh [18] first introduced the notion of fuzzy subsets of sets in 1965 as a way to demonstrate vagueness in real-life situations. Since then, the theory of fuzzy sets has evolved into a dynamic field of research, with applications across diverse disciplines, including engineering, computer science, economics and finance, natural language processing, image processing, control systems, decision-making, risk assessment, and so on. Various extensions of fuzzy sets have been developed by different authors to handle imprecision and uncertainty. Among these, bipolar-valued fuzzy sets stand out as one of the most notable extensions. W.R. Zhang [19, 20] initiated the idea of bipolar fuzzy sets as an extension of ordinary fuzzy sets by enlarging the range of their membership degree from the interval [0, 1] to [-1, 1]. A membership value of 0 indicates that the elements have no relevance to the corresponding property. Furthermore, membership values in the range (0, 1] imply that the elements partially fulfill the property, while membership values in the interval [-1, 0) suggest that the elements partially meet the opposite or contrary property.

K. Iseki and S. Tanaka [5] were the first to introduce the concept of BCK-algebras, while K. Iseki [6] initiated BCI-algebras as an extension of BCK-algebras and explored several results related to them. H. S. Kim et al. [7] introduced the notion of BE-algebras as a generalization of BCK-algebras and explored various related results. S. S. Ahn et al. [1] defined the theory of ideals in BE-algebras and subsequently provided detailed illustrations of these ideals. M. B. Prabhakar et al. [13] introduced the concept of generalized lower sets in transitive BE-algebras and examined their properties.

Many studies have been conducted on different algebraic structures based on fuzzy and bipolar fuzzy sets. S. Z. Song et al. [16] introduced the fuzzification of ideals in BE-algebras and discussed some of their properties. G. Muhiuddin et al. [11] applied linear Diophantine fuzzifications to BCK/BCI-algebras and introduced the concepts of linear Diophantine fuzzy subalgebras, linear Diophantine fuzzy ideals, and linear Diophantine fuzzy commutative ideals of a

BCK-algebra, exploring the relationships between them. M. Al-Tahan et al. [2] introduced the concept of a linear Diophantine fuzzy n-fold weak subalgebra of a BE-algebra, investigated its properties, and established its relationship with the notion of an n-fold weak subalgebra. A. Fayazi et al. [4] introduced fuzzy multi-hyperrings and investigated their fundamental properties. The authors also explored homomorphic properties and as well as the direct product of fuzzy multi-hyperrings. S. Dogra and M. Pal [3] introduced the concept of a picture fuzzy ideal in a multiplicative semigroup, initiating the study of various types of ideals within a picture fuzzy environment and establishing relationships between them. P. Uma Maheswari et al. [17] introduced the concept of a bipolar-valued fuzzy subrings within a ring, explored the relationships among such subrings, and established several characterization theorems related to bipolar-valued fuzzy subrings of a ring. S. Sabarinathan et al. [14] introduced the concept of a bipolar-valued fuzzy α-ideal in BF-algebras and investigated the relationship between bipolar-valued fuzzy ideals and bipolar-valued fuzzy α-ideals, deriving significant and insightful results. G. Muhiuddin et al. [10] introduced the notions of bipolar fuzzy closed ideals, bipolar fuzzy positive implicative ideals, and bipolar fuzzy implicative ideals in BCK-algebras and investigated their related properties. Furthermore, G. Muhiuddin et al. [12] explored the concepts of bipolar-valued fuzzy soft hyper BCK ideals and investigated their related properties and relationships using the framework of bipolar-valued fuzzy soft sets. J. G. Lee and K. Hur [8] introduced a bipolar fuzzy relation between two sets and the composition of two bipolar fuzzy relations. Furthermore, they defined the level set of a bipolar fuzzy relation and investigated some relationships between bipolar fuzzy relations and their level sets. They also indicated that bipolar fuzzy relations generalize fuzzy relations. However, to the best of our knowledge, no study has been conducted on the algebraic structure of bipolar fuzzy ideals in BE-algebras. Inspired by recent works on bipolar fuzzy algebraic structures and previous works related to fuzzy algebraic structures, we are motivated to study bipolar fuzzy ideals in BE-algebras.

In this manuscript, we employ the theory of bipolar fuzzy sets to analyze ideals in BE-algebras within the context of bipolar fuzzifications. We introduce bipolar fuzzy ideals in BE-algebras and explore their important properties in detail. We provide characterizations of bipolar fuzzy ideals in BE-algebras in terms of their level sets. Furthermore, we discuss the homomorphic image and inverse image of bipolar fuzzy ideals in a BE-algebra and examine the Cartesian product of these ideals, highlighting several key results. Finally, we investigate the relationship between the strongest bipolar fuzzy relations and bipolar fuzzy ideals in a BE-algebra.

2 Preliminaries

This section reviews key definitions and results essential for the development of this work.

Definition 2.1. [7] A nonempty set X, together with a binary operation '$\diamond$' and a constant element '1', is called a BE-algebra whenever it satisfies the following axioms:

(i). $x \diamond x = 1$

(ii). $x \diamond 1 = 1$

(iii). $1 \diamond x = x$

(iv). $x \diamond (y \diamond z) = y \diamond (y \diamond z), \forall x, y, z \in X$

The binary relation $\leq$ defined by $x \leq y \Leftrightarrow x \diamond y = 1, \forall x, y, z \in X$

Definition 2.2. [1] Let X be a BE-algebra. A nonempty subset I of X is said to be an ideal of X if it satisfies the following axioms:

(i). $x \diamond y \in$ I, $\forall x \in X, \forall y \in$ I.

(i). $(x \diamond (y \diamond z)) \diamond z \in$ I, $\forall z \in X, \forall x, y \in$ I.

Definition 2.3. [1] A BE-algebra X is called transitive if the inequality $y \diamond z \leq (x \diamond y) \diamond (x \diamond z)$ holds for all $x,\ y,\ z\ \in X$.

Definition 2.4. [15] A function $f : X \to Y$ is called a homomorphism of BE-algebras if $f\ (x \diamond y) = f\ (x) \diamond^{'} f(y)$, for all $x, y \in X$. f is called an epimorphism of BE-algebras if it is onto. Also, $f(1) = 1'$, where 1 and 1' are constant elements of X and Y respectively.

Definition 2.5. [18] In a nonempty set X, a fuzzy subset A is characterized by a function $\mu_A : X \to [0,1]$, where $\mu_A(x)$ is the membership degree of an element.

Definition 2.6. [15] A fuzzy set (FS) A in a BE-algebra X with a membership function $\mu_A : X \to [0,1]$ is considered to be a fuzzy ideal in X if the following two conditions hold:

(i). $\mu_A(x \diamond y) \geq \mu_A(y)$,

(ii). $\mu_A((x \diamond (y \diamond z)) \diamond z) \geq min\ \{\mu_A(x),\ \mu_A(y)\}, \forall\, x, y, z\ \in X.$

Definition 2.7. [9, 19] A Bipolar Fuzzy Set (BFS) A in a nonempty set X is defined as an expression of the type $A = \{\langle x, \mu_A^+(x), \mu_A^-(x)\rangle \mid x \in X\}$, where $\mu_A^+ : X \to [0,1]$ and $\mu_A^- : X \to [-1,0]$ are the positive and negative membership functions, respectively. The value $\mu_A^+(x)$ represents the degree to which an element x satisfies the property corresponding to A, while the negative membership degree $\mu_A^-(x)$ represents the degree to which an element x satisfies an implicit counter-property of A.
From now on, we will represent the BFS $A = \{\langle x, \mu_A^+(x), \mu_A^-(x)\rangle \mid x \in X\}$ as $A = \left(\mu_A^+,\ \mu_A^-\right)$.

Definition 2.8. [8] The complement of a BFS A in X, symbolized as $A^c = ((\mu_A^+)^c,\ (\mu_A^-)^c)$, is a bipolar fuzzy set in X defined as $(\mu_A^+)^c(x) = 1 - \mu_A^+(x)$ and $\left(\mu_A^-\right)^c(x) = -1 - \mu_A^-(x)$, for every $x \in X$.

Definition 2.9. [8, 20] Let $A = (\mu_A^+,\ \mu_A^-)$ and $B = (\mu_B^+,\ \mu_B^-)$ be any two BFSs in X, Therefore,
(i). $A \subseteq B \Leftrightarrow \mu_A^+(x) \leq \mu_B^+(x)$ and $\mu_A^-(x) \geq \mu_B^-(x)$
(ii). $A \cap B = \left\{\min\left\{\mu_A^+(x), \mu_B^+(x)\right\}, max\{\mu_A^-(x),\ \mu_B^-(x)\}\right\}$
(iii). $A \cup B = \left\{\max\left\{\mu_A^+(x), \mu_B^+(x)\right\}, min\{\mu_A^-(x),\ \mu_B^-(x)\}\right\}$, for each $x \in X$.

Definition 2.10. [19] Let $A = (\mu_A^+,\ \mu_A^-)$ be a BFS in X. Then for $s \in [0,1]$ and $t \in [-1,0]$, the sets $P(\mu_A^+, s) = \{x \in X | \mu_A^+(x) \geq s\}$ and $N(\mu_A^-, t) = \{x \in X | \mu_A^-(x) \leq t\}$ are called positive s-cut and negative t-cut respectively.

Definition 2.11. [9, 11] A BFS $A = (\mu_A^+,\ \mu_A^-)$ in a nonempty set X satisfies Sup-Inf property if there exists an element $x_0 \in$T such that $\mu_A^+(x_0) = \sup_{t \in T} \mu_A^+(t)$ and $\mu_A^-(x_0) = \inf_{t \in T} \mu_A^-(t)$, for every $T \subseteq X$.

Definition 2.12. [15, 14] Let $f : X \to Y$ be a homomorphism from with $f(X) = Y$. If $A = (\mu_A^+,\ \mu_A^-)$ and $B = (\mu_B^+,\ \mu_B^-)$ are any two bipolar fuzzy subsets of X and Y, respectively, then the image of A under f is defined as $f(A) = \{\langle y, f(\mu_A^+)(y),\ f(\mu_A^-)(y)\rangle \mid y\ \in Y\}$, where

$$f(\mu_A^+)(y) = \begin{cases} \sup\{\mu_A^+(x) : x \in f^{-1}(y)\}, \text{if } f^{-1}(y) \neq \emptyset \\ 0 \quad \text{otherwise} \end{cases}$$

and

$$f(\mu_A^-)(y) = \begin{cases} \inf\{\mu_A^+(x) : x \in f^{-1}(y)\}, \text{if } f^{-1}(y) \neq \emptyset \\ 0 \quad \text{otherwise} \end{cases}$$

The invers image of B under f is a bipolar fuzzy set defined as
$f^{-1}(B)(x) = \{\langle x, f^{-1}(\mu_A^+)(x), f^{-1}(\mu_A^-)(x)\rangle \mid x \in X\}$,
where $f^{-1}(\mu_A^+)(x) = \mu_A^+(f(x))$ and $f^{-1}(\mu_A^-)(x) = \mu_A^-(f(x))$, for each $x \in X$.

Definition 2.13. [17] The Cartesian product of the BFSs $A = \left(\mu_A^+, \mu_A^-\right)$ and $B = \left(\mu_B^+, \mu_B^-\right)$ in a nonempty set X is defined as $A \times B$=$\{\langle (x,y), \mu_{A\times B}^+(x,y),\ \mu_{A\times B}^-(x,y)\rangle \mid (x,y) \in X \times X\}$, where $\mu_{A\times B}^+ : X \times X \to [0,\ 1]$ and $\mu_{A\times B}^- : X \times X \to [-1,\ 0]$, for each $(x,\ y) \in X \times X$. with $\mu_{A\times B}^+(x,\ y) = min\{\mu_A^+(x),\ \mu_B^+(y)\}$ and $\mu_{A\times B}^-(x,\ y) = \max\left\{\mu_{A,}^-(x),\ \mu_{B,}^-(y)\right\}$,

Definition 2.14. [19] A bipolar fuzzy relation (BFR) R on a nonempty set X is an expression of the form $R = \{\langle (x,y), \mu_R^+(x,\ y),\ \mu_R^-(x,\ y)\rangle \mid x,\ y \in X\}$, where $\mu_R^+ : X \times X \to [0,\ 1]$ and $\mu_R^- : X \times X \to [-1,\ 0]$ are positive and negative membership functions, respectively. For simplicity, we denote the relation $R = \left\{\langle (x,y), \mu_R^+(x,y), \mu_R^-(x,y)\rangle \mid x,y \in X\right\}$ as $R = (\mu_R^+,\ \mu_R^-)$.

Definition 2.15. [17] Let $A = \left(\mu_A^+,\ \mu_A^-\right)$ be a BFS and $R = (\mu_R^+, \mu_R^-)$ be a BFR on a nonempty set X. The strongest BFR on X, associated with A , denoted by R_A, is defined as

$$R_A = \{\langle(x,y), \mu_{R_A}^+(x,y), \mu_{R_A}^-(x,y)\rangle \,|x, y \in X\},$$

where $\mu_{R_A}^+(x,y) \ = min\{\mu_A^+(x), \mu_A^+(y)\}$ and $\mu_{R_A}^-(x,y) \ = min\{\mu_A^-(x), \mu_A^-(y)\}, \forall\, x, y \in X.$

3 Bipolar Fuzzy Ideals in BE-algebra

This part of the paper presents the idea of bipolar fuzzy ideals (BFIs) in the context of a BE-algebra and explores several important properties associated with them. Unless stated otherwise, let X and Y represent BE-algebras. In this and subsequent sections, "BFS" refers to a bipolar fuzzy set, while "BFI" refers to a bipolar fuzzy ideal.

Definition 3.1. . A BFS $A = (\mu_A^+, \mu_A^-)$ in X is a BFI of X if the following conditions hold:

(i). $\mu_A^+(x \diamond y) \geq \mu_A^+(y)$ and $\mu_A^-(x \diamond y) \leq \mu_A^-(y)$

(ii). $\mu_A^+\left((x \diamond (y \diamond z)) \diamond z\right) \geq min\{\mu_A^+(x).\mu_A^+(y)\}$

(iii). $\mu_A^-\left((x \diamond (y \diamond z)) \diamond z\right) \leq max\{\mu_A^+(x).\mu_A^+(y)\}$

Example 3.2. Let $X = \{1,\ a,\ b,\ c, d, 0\}$ be a set with the following table.

$*$	1	α	β	γ	δ	σ
1	1	α	β	γ	δ	σ
α	1	1	a	γ	γ	δ
β	1	1	1	γ	γ	γ
γ	1	α	β	1	α	β
δ	1	1	α	1	1	α
σ	1	1	1	1	1	1

Table 1.

Clearly, $(X;\ *,\ 1)$ is a BE-algebra and consequently I $= \{1, \alpha, \beta\}$ is an ideal of X. Suppose $A = \left(\mu_A^+,\ \mu_A^-\right)$ be a BFS in X defined by

$$\mu_A^+(x) = \begin{cases} 0.8 \ if \ x \in \{1, \alpha, \beta\} \\ 0.5 \ if \ x \in \{\gamma, \delta, \sigma\} \end{cases} \text{ and } \mu_A^-(x) = \begin{cases} -0.7 \ if \ x \in \{1, \alpha, \beta\} \\ -0.2 \ if \ x \in \{\gamma, \delta, \sigma\} \end{cases}$$

Then by simple calculation we can verify that $A = \left(\mu_A^+,\ \mu_A^-\right)$ is a BFI of X.

Lemma 3.3. *If $A = \left(\mu_A^+,\ \mu_A^-\right)$ is a BFI of X, then*
(i). $\mu_A^+(1) \geq \mu_A^+(x)$ and $\mu_A^-(1) \leq \mu_A^-(x), \forall x \in X$.
(ii). $\mu_A^+\left((x \diamond y) \diamond y\right) \geq \mu_A^+\ (x)$ and $\mu_A^-\left((x \diamond y) \diamond y\right) \leq \mu_A^-\ (x)$, for all $x, y \in X$.

Proof. (i) By using definitions 2.1 (i) and 3.1 (i), we have
$\mu_A^+(1) = \mu_A^+(x \diamond x) \geq \mu_A^+(x)$ and $\mu_A^-(1) = \mu_A^-(x \diamond x) \leq \mu_A^-(x)$
Hence, $\mu_A^+(1) \geq \mu_A^+(x)$ and $\mu_A^-(1) \leq \mu_A^-(x), \forall x \in X$
(ii) By Definitions 2.1 (iii), 3.1 (ii, iii) and using the result in (i), we get
$\mu_A^+((x \diamond y) \diamond y) = \ \mu_A^+((x \diamond (1 \diamond y)) \diamond y) \ \geq \ \min\{\mu_A^+(x),\ \mu_A^+(1)\} \ = \ \mu_A^+((x)$
$\mu_A^-((x \diamond y) \diamond y) = \ \mu_A^-((x \diamond (1 \diamond y)) \diamond y) \ \leq \ \max\{\mu_A^-(x),\ \mu_A^-(1)\} \ = \ \mu_A^-((x)$
Therefore, $\mu_A^+\left((x \diamond y) \diamond y\right) \geq \mu_A^+\ (x)$ and $\mu_A^-\left((x \diamond y) \diamond y\right) \leq \mu_A^-\ (x), \forall x,\ y \in X$. □

Theorem 3.4. *Suppose $A = \left(\mu_A^+,\ \mu_A^-\right)$ is a BFI of X. If $x \leq y$ for $x, y \in X$, then μ_A^+ order preserving and $\mu_A^-(x)$ is order reversing*

Proof. Let $x \leq y$, for $x, y \in X$. Then by the definition of the binary relation '$\leq$' it follows that $x \diamond y = 1$. Thus by Definition 2.1 (iii) and Lemma 3.3 (ii), $\mu_A^+(y) = \mu_A^+(1 \diamond y) = \mu_A^+((x \diamond y) \diamond y) \geq \mu_A^+(x)$ and $\mu_A^-(y) = \mu_A^-(1 \diamond y) = \mu_A^-\left((x \diamond y) \diamond y\right) \leq \mu_A^-(x)$. Therefore, $\mu_A^+(x) \leq \mu_A^+(y)$ and $\mu_A^-(x) \geq \mu_A^-(y)$. □

Theorem 3.5. *Let $A = (\mu_A^+,\ \mu_A^-)$ be a BFI of X. If $x \diamond (y \diamond z) = 1$, for all $x,\ y,\ z\ \in X$ then $\mu_A^+(z) \geq \min\{\mu_A^+(x),\ \mu_A^+(y)\}$ and $\mu_A^-(z) \leq \max\{\mu_A^-(x),\ \mu_A^-(y)\}$*

Proof. Since $\text{x} \diamond (y \diamond z) = 1,\ we\ have\ \ z = 1 \diamond z = (x \diamond (y \diamond z)) \diamond z$, for all elements $x,\ y,\ z \in X$. Then, $\mu_A^+(z) = \mu_A^+((x \diamond (y \diamond z)) \diamond z) \geq \min\{\mu_A^+(x),\ \mu_A^+(y)\}$ and

$\mu_A^-(z) = \mu_A^-((x \diamond (y \diamond z)) \diamond z) \leq \max\{\mu_A^-(x),\ \mu_A^-(y)\}$

Therefore, $\mu_A^+(z) \geq \min\{\mu_A^+(x),\ \mu_A^+(y)\}$ and $\mu_A^-(z) \leq \max\{\mu_A^-(x),\ \mu_A^-(y)\}$. □

Theorem 3.6. *Suppose $A = (\mu_A^+,\ \mu_A^-)$ is a BFS in X such that*

1. $\mu_A^+(1) \geq \mu_A^+(x)$ *and* $\mu_A^-(1) \leq\ \mu_A^-(x)$
2. $\mu_A^+(x \diamond z) \geq\ \min\{\mu_A^+(x \diamond (y \diamond z)),\ \mu_A^+(y)\}$
3. $\mu_A^+(x \diamond z) \leq\ \max\{\mu_A^+(x \diamond (y \diamond z)),\ \mu_A^+(y)\}$*, for all* $x,\ y, z \in X$.

Then μ_A^+ is order preserving and μ_A^- is order reversing.

Proof. Assume that $x \leq y$ for $x, y \in X$. Then $x \diamond y = 1$. By Definition 2.1 (i, iii) and conditions (1), (2) and (3), we have
$\mu_A^+(y) = \mu_A^+(1 \diamond y) \geq \min\{\mu_A^+(1 \diamond (x \diamond y)),\ \mu_A^+(x)\} = min\{\mu_A^+(1 \diamond 1),\ \mu_A^+(x)\} = \mu_A^+(x)$ and
$\mu_A^-(y) = \mu_A^-(1 \diamond y) \leq \max\{\mu_A^-(1 \diamond (x \diamond y)),\ \mu_A^-(x)\} = max\{\mu_A^-(1 \diamond 1),\ \mu_A^-(x)\} = \mu_A^-(x)$.
Hence $\mu_A^+(y) \geq \mu_A^+(x)$ and $\mu_A^-(y) \leq \mu_A^-(x)$ □

Theorem 3.7. *If X is transitive, then a BFS $A = (\mu_A^+, \mu_A^-)$ in X is a BFI of X if and only if it holds the conditions (1)-(3) given in Theorem 3.6 for all $x, y, z \in X$.*

Proof. Suppose $A = (\mu_A^+,\ \mu_A^-)$ is a BFI of X. By Lemma 3.3 (i), $A = (\mu_A^+,\ \mu_A^-)$ satisfies (1). Since X is transitive, we have $(y \diamond z) \diamond z \leq (x \diamond (y \diamond z)) \diamond (x \diamond z)$, for all $x,\ y,\ z\ \in X$.

$\Longrightarrow ((y \diamond z) \diamond z) \diamond ((x \diamond y \diamond z)) \diamond x \diamond z))\ =\ 1.$

By Definition 2.1 (iii), Definition 3.1 and Lemma 3.3 (ii), we have

$$\begin{aligned}\mu_A^+(x \diamond z) = \mu_A^+(1 \diamond (x \diamond z)) &= \mu_A^+((((y \diamond z) \diamond z) \diamond ((x \diamond (y \diamond z)) \diamond (x \diamond z))) \diamond (x \diamond z))\\ &\geq \min\{\mu_A^+((y \diamond z) \diamond z), \mu_A^+(x \diamond (y \diamond z))\}\\ &\geq \min\{\mu_A^+(x \diamond (y \diamond z)), \mu_A^+(y)\}.\text{and}\end{aligned}$$

$$\begin{aligned}\mu_A^-(x \diamond z) = \mu_A^-(1 \diamond (x \diamond z)) &= \mu_A^-((((y \diamond z) \diamond z) \diamond ((x \diamond (y \diamond z)) \diamond (x \diamond z))) \diamond (x \diamond z))\\ &\leq \max\{\mu_A^-((y \diamond z) \diamond z),\ \mu_A^-(x \diamond (y \diamond z))\}\\ &\leq \max\{\mu_A^-(x \diamond (y \diamond z)),\ \mu_A^-(y)\}.\end{aligned}$$

Hence $A = (\mu_A^+,\ \mu_A^-)$ satisfies (2)and (3).

Conversely suppose that $A = (\mu_A^+,\ \mu_A^-)$ satisfies the conditions (1), (2) and (3). By definitions 2.1(i, ii) and condition (1), (2) and (3), we have

$$\begin{aligned}\mu_A^+(x \diamond y) \geq \min\{\mu(x \diamond (y \diamond y)),\ \mu(y)\} &= min\{\mu_A^+(x \diamond 1), \mu_A^+(y)\}\\ &=\ min\{\mu_A^+(1), \mu_A^+(y)\} = \mu_A^+(y) \text{ and}\end{aligned}$$

$$\begin{aligned}\mu_A^-(x \diamond y) \leq \max\{\mu(x \diamond (y \diamond y)),\ \mu(y)\} &= max\{\mu_A^-(x \diamond 1),\ \mu_A^-(y)\}\\ &= min\{\mu_A^-(1), \mu_A^-(y)\} = \mu_A^-(y)\end{aligned}$$

Since X is a transitive BE-algebra, by the above theorem it follows that
$\mu_A^+((y \diamond z) \diamond z) \leq \mu_A^+((x \diamond (y \diamond z)) \diamond (x \diamond z))$ and $\mu_A^-((y \diamond z) \diamond z) \geq \mu_A^-((x \diamond (y \diamond z)) \diamond (x \diamond z))$.
So that from conditions (2) and (3) given in Theorem 3.6, we have

$$\begin{aligned}\mu_A^+((x \diamond (y \diamond z)) \diamond z) &\geq \min\{\mu_A^+((((x \diamond (y \diamond z)) \diamond (x \diamond z)),\ \mu_A^+((x)\}\\ &\geq min\{\ \mu_A^+((y \diamond z) \diamond z), \mu_A^+((x)\}\\ &\geq\ min\{\mu_A^+((x), \mu_A^+((y)\}\text{, for all } x,\ y,\ z \in X.\text{and}\end{aligned}$$

$$\begin{aligned}\mu_A^-((x \diamond (y \diamond z)) \diamond z) &\leq \max\{\mu_A^-((x \diamond (y \diamond z)) \diamond (x \diamond z)), \mu_A^-(x)\}\\ &\leq \max\{\mu_A^-((y \diamond z) \diamond z), \mu_A^-(x)\}\\ &\leq max\{\mu_A^-(x), \mu_A^-(y)\}\text{, for all } x, y,\ z \in X.\end{aligned}$$

Hence, $A = (\mu_A^+,\ \mu_A^-)$ is a bipolar fuzzy ideal of X. □

Theorem 3.8. *The intersection of any two BFIs of X is also a BFI.*

Proof. Let $A = \left(\mu_A^+,\ \mu_A^-\right)$ and $B = \left(\mu_B^+, \mu_B^-\right)$ be any two BFI of X. Let $x, y, z \in X$. Then $\mu_{A\cap B}^+(x\diamond y) = \min\left\{\mu_A^+(x\diamond y),\ \mu_B^+(x\diamond y)\right\} \geq \min\left\{\mu_A^+(y),\ \mu_B^+(y)\right\} = \mu_{A\cap B}^+(y)$ and $\mu_{A\cap B}^-(x\diamond y) = \max\left\{\mu_A^-(x\diamond y),\ \mu_B^-(x\diamond y)\right\} \leq \max\left\{\mu_A^-(y),\ \mu_B^-(y)\right\} = \mu_{A\cap B}^-(y)$.

$$\begin{aligned} Also, \mu_{A\cap B}^+\left((x\diamond(y\diamond z))\diamond z\right) &= \min\left\{\mu_A^+\left((x\diamond(y\diamond z))\diamond z\right), \mu_B^+\left((x\diamond(y\diamond z))\diamond z\right)\right\} \\ &\geq min\{min\left\{\mu_A^+(x).\mu_A^+(y)\right\}, \min\left\{\mu_B^+(x), \mu_B^+(y)\right\}\} \\ &= min\{min\left\{\mu_A^+(x), \mu_B^+(x)\right\}, \min\left\{\mu_A^+(y), \mu_B^+(y)\right\}\} \\ &= min\{\mu_{A\cap B}^+(x),\ \mu_{A\cap B}^+(y)\} \text{ and} \\ \mu_{A\cap B}^-\left((x\diamond(y\diamond z))\diamond z\right) &= \max\left\{\mu_A^-\left((x\diamond(y\diamond z))\diamond z\right), \mu_B^-\left((x\diamond(y\diamond z))\diamond z\right)\right\} \\ &\leq max\{max\left\{\mu_A^-(x).\mu_A^-(y)\right\}, \max\left\{\mu_B^-(x), \mu_B^-(y)\right\}\} \\ &= max\{max\left\{\mu_A^-(x), \mu_B^-(x)\right\}, \max\left\{\mu_A^-(y), \mu_B^-(y)\right\}\} \\ &= max\{\mu_{A\cap B}^-(x),\ \mu_{A\cap B}^-(y)\}. \end{aligned}$$

Hence, $A \cap B$ is a BFI of X. □

The above theorem can be generalized to any set of BFI X as follows.

Theorem 3.9. *The intersection of any arbitrary set of BFIs of a BE-algebra X is also a BFI.*

Remark 3.10. The union of any two BFI in a BE-algebra X is not necessarily a BFI of X. This fact is illustrated by the next Example.

Example 3.11. Let $\mathbb{N}$ be the set of positive integers and $'\diamond'$ be the binary operation defined on $\mathbb{N}$ by

$$x\diamond y = \begin{cases} y \text{ if x } = 1 \\ 1\ \ if\ \ x \neq 1 \end{cases}$$

So, it is straightforward to verify that $(\mathbb{N};\ \diamond, 1)$ is a BE-algebra.
Define BFSs $A = \left(\mu_A^+,\ \mu_A^-\right)$ and $B = \left(\mu_B^+,\ \mu_B^-\right)$ in N by

$$\mu_A^+(x) = \begin{cases} 0.7\ if\ x \in \{2n-1 : n \in \mathbb{N}\} \\ 0.5\ if\ x\ \in \{2n : n \in \mathbb{N}\} \end{cases} \text{ and } \mu_A^-(x) = \begin{cases} -0.6\ if\ x\ \in \{2n-1 : n \in \mathbb{N}\} \\ -0.4\ if\ x\ \in \{2n : n \in \mathbb{N}\} \end{cases}$$

and

$$\mu_B^+(x) = \begin{cases} 0.8\ if\ x\ is\ 1 \\ 0.4\ otherwise \end{cases} \text{ and } \mu_B^-(x) = \begin{cases} -0.7\ if\ x\ is\ 1 \\ -0.3\ \ otherwise \end{cases}$$

Clearly, $A = \left(\mu_A^+,\ \mu_A^-\right)$ and $B = \left(\mu_B^+,\ \mu_B^-\right)$ are BFIs of $\mathbb{N}$. Now,

$$\mu_{A\cup B}^+(x) = \begin{cases} 0.8\ if\ x = \ 1 \\ 0.7\ if\ x \in \{2n+1 : n \in \mathbb{N}\} \\ 0.5\ if\ x\ \in \{2n : n \in \mathbb{N}\} \end{cases} \text{ and } \mu_{A\cup B}^-(x) = \begin{cases} -0.7\ if\ x = \ 1 \\ -0.6\ \ if\ x \in \{2n+1 : n \in \mathbb{N}\} \\ -0.4\ if\ x\ \in \{2n : n \in \mathbb{N}\} \end{cases}$$

Let $x = 1,\ y = 3$ and $z = 2$. Then $\mu_{A\cup B}^+\left((x\diamond(y\diamond z))\diamond z\right) = \mu_{A\cup B}^+\left((1\diamond(3\diamond 2))\diamond 2\right) = \mu_{A\cup B}^+\left((1\diamond 1)\diamond 2\right) = \mu_{A\cup B}^+(1\diamond 2) = \mu_{A\cup B}^+(2) = 0.5$ and $\min\{\mu_{A\cup B}^+(x), \mu_{A\cup B}^+(y)\} = \min\{\mu_{A\cup B}^+(1), \mu_{A\cup B}^+(3)\} = \min\{0.8, 0.7\} = 0.7$. $\Longrightarrow \mu_{A\cup B}^+\left((1\diamond(3\diamond 2))\diamond 2\right) = 0.5 < 0.7 = \min\{\ \mu_{A\cup B}^+(1), \mu_{A\cup B}^+(3)\}$. This contradicts Definition 3.1. Hence, $A \cup B$ is not necessarily a BFI of X.

Theorem 3.12. *A BFS* $A = \left(\mu_A^+,\ \mu_A^-\right)$ *of* X *is a BFI of* X *if and only if the fuzzy subsets* μ_A^+ *and* $\left(\mu_A^-\right)^c$ *are both fuzzy ideals of* X.

Proof. Assume that $A = \left(\mu_A^+, \mu_A^-\right)$ be a BFI of X. By the definition of the BFI $A = \left(\mu_A^+,\ \mu_A^-\right)$ in X, μ_A^+ is a fuzzy ideal of X. So, it remains to show that $\left(\mu_A^-\right)^c$ is also a fuzzy ideal in X. Then for each $x, y, z \in X$, $\left(\mu_A^-\right)^c(x\diamond y) = -1 - \mu_A^-(x\diamond y) \geq -1 - \mu_A^-(y) = \left(\mu_A^-\right)^c(y)$

$$\Longrightarrow \left(\mu_A^-\right)^c(x\diamond y) \geq \left(\mu_A^-\right)^c(y)$$

Also, $(\mu_A^-)^c((x \diamond (y \diamond z)) \diamond z) = -1 - \mu_A^-((x \diamond (y \diamond z)) \diamond z)$

$$\geq -1 - \max\{\mu_A^-(x), \mu_A^-(y)\}$$

$$= \min\{-1 - \mu_A^-(x), -1 - \mu_A^-(y)\}$$

$$= \min\left\{(\mu_A^-)^c(x),\ (\mu_A^-)^c(y)\right\}$$

$$\Longrightarrow (\mu_A^-)^c((x \diamond (y \diamond z)) \diamond z) \geq \min\{(\mu_A^-)^c(x),\ (\mu_A^-)^c(y)\}$$

Therefore, μ_A^+ and $(\mu_A^-)^c$ are fuzzy ideals of X.

On the other hand, let both μ_A^+ and $(\mu_A^-)^c$ be fuzzy ideals of X. By definition 2.6, $\mu_A^+(x \diamond y) \geq \mu_A^+(y)$ and $\mu_A^+((x \diamond (y \diamond z)) \diamond z) \geq \min\{\mu_A^+(x),\ \mu_A^+(y)\}$, for all $x, y, z \in X$, Now we prove that $\mu_A^-(x \diamond y) \leq \mu_A^-(y)$ and $\mu_A^-((x \diamond (y \diamond z)) \diamond z) \leq \max\{\mu_A^-(x),\ \mu_A^-(y)\}$. Then, $-1 - \mu_A^-(x \diamond y) = (\mu_A^-)^c(x \diamond y) \geq (\mu_A^-)^c(y) = -1 - \mu_A^-(y)$. $\Longrightarrow \mu_A^-(x \diamond y) \leq \mu_A^-(y)$.

Also, $-1 - \mu_A^-((x \diamond (y \diamond z)) \diamond z) = (\mu_A^-)^c((x \diamond (y \diamond z)) \diamond z)$

$$\geq \min\left\{(\mu_A^-)^c(x),\ (\mu_A^-)^c(y)\right\} (\because (\mu_A^-)^c \text{ is a fuzzy ideal of} X)$$

$$= \min\{-1 - \mu_A^-(x), -1 - \mu_A^-(y)\}$$

$$= -1 - \max\{\mu_A^-(x), \mu_A^-(y)\}$$

$$\Longrightarrow \mu_A^-((x \diamond (y \diamond z)) \diamond z) \leq \max\{\mu_A^-(x), \mu_A^-(y)\}$$

Therefore**,** $A = (\mu_A^+,\ \mu_A^-)$ is a BFI of X. □

Theorem 3.13. *A BFS $A = (\mu_A^+,\ \mu_A^-)$ of X is a BFI of X if and only if $\oplus A = (\ \mu_A^+, (\mu_A^+)^c)$ and $\ominus A = ((\mu_A^-)^c,\ \mu_A^-)$ are BFIs in X.*

Proof. Suppose $A = (\mu_A^+,\ \mu_A^-)$ be a BFI of X. Then by Theorem 3.12, μ_A^+ and $(\mu_A^-)^c$ are both fuzzy ideals of X. Next we have to show that $(\mu_A^+)^c$ satisfies the condition $(\mu_A^+)^c(x \diamond y) \leq (\mu_A^+)^c(y)$ and $(\mu_A^+)^c((x \diamond (y \diamond z)) \diamond z) \leq \max(\mu_A^+)^c(x), (\mu_A^+)^c(y)\}$, for all $x, y, z \in X$. Now for all $x, y, z \in X$, $(\mu_A^+)^c(x \diamond y) = 1 - \mu_A^+(x \diamond y) \leq 1 - \mu_A^+(y) = (\mu_A^+)^c(y)$

$$\Longrightarrow (\mu_A^+)^c(x \diamond y) \leq (\mu_A^+)^c(y).$$

Also, $(\mu_A^+)^c((x \diamond (y \diamond z)) \diamond z) = 1 - \mu_A^+((x \diamond (y \diamond z)) \diamond z)$

$$\leq 1 - \min\{\mu_A^+(x), \mu_A^+(y)\}$$

$$= max\{1 - \mu_A^+(x), 1 - \mu_A^+(y)\}$$

$$= max\{(\mu_A^+)^c(x), (\mu_A^+)^c(y)\}$$

$$\Longrightarrow (\mu_A^+)^c((x \diamond (y \diamond z)) \diamond z) \leq\ max\{(\mu_A^+)^c(x), (\mu_A^+)^c(y)\}$$

Therefore, $\oplus A = (\ \mu_A^+,\ (\mu_A^+)^c)$ and $\ominus A = ((\mu_A^-)^c,\ \mu_A^-)$ are BFIs of X.

Conversely, suppose $\oplus A = (\mu_A^+, (\mu_A^+)^c)$ and $\ominus A = ((\mu_A^-)^c, \mu_A^-)$ are BFIs of X. Then by Definition 3.1, it follows that A $= (\ \mu_A^+, \mu_A^-)$ is a BFI of X. □

Theorem 3.14. *If $A = (\mu_A^+, \mu_A^-)$ is a BFI of X, then the set $X_{\mu_A^+} = \{x \in X : \mu_A^+(x) = \mu_A^+(1)\}$ and $X_{\mu_A^-} = \{x \in X :\ \mu_A^-(x) = \mu_A^-(1)\}$ are ideals of X.*

Proof. Let $A = (\mu_A^+,\ \mu_A^-)$ be a BFI of X. Let $x, y \in X$ with $y \in X_{\mu_A^+}$ and $y \in X_{\mu_A^-}$. Then $\mu_A^+(x \diamond y) \geq \mu_A^+(y) = \mu_A^+(1)$ and $\mu_A^-(x \diamond y) \leq \mu_A^-(y) = \mu_A^-(1)$ but by Lemma 3.3 (i), we have $\mu_A^+(1) \geq \mu_A^+(x \diamond y)$ and $\mu_A^-(1) \leq \mu_A^-(x \diamond y)$. This implies $\mu_A^+(x \diamond y) = \mu_A^+(1)$ and $\mu_A^-(x \diamond y) = \mu_A^-(1)$. Hence $x \diamond y \in X_{\mu_A^+}$ and $x \diamond y \in X_{\mu_A^-}$.

Also, Let $x, y, z \in X$ such that $x, y \in X_{\mu_A^+}$. Then

$$\mu_A^+(x) = \mu_A^+(1) = \mu_A^+(y) \text{ and } \mu_A^-(x) = \mu_A^-(1) = \mu_A^-(y).$$

So, $\mu_A^+((x \diamond (y \diamond z)) \diamond z) \geq min\{\mu_A^+(x), \mu_A^+(y)\} = \min\{\mu_A^+(1), \mu_A^+(1)\} = \mu_A^+(1)$ and

$\mu_A^-((x \diamond (y \diamond z)) \diamond z) \leq max\{\mu_A^+(x), \mu_A^+(y)\} = \max\{\mu_A^+(1), \mu_A^+(1)\} = \mu_A^+(1)$.

As $\mu_A^+(1) \geq \mu_A^+((x \diamond (y \diamond z)) \diamond z)$ and $\mu_A^-(1) \leq \mu_A^-((x \diamond (y \diamond z)) \diamond z)$ by Lemma 3.3 (i) it follows that $\mu_A^+((x \diamond (y \diamond z)) \diamond z) = \mu_A^+(1)$ and $\mu_A^-((x \diamond (y \diamond z)) \diamond z) = \mu_A^-(1)$.

Thus, $(x \diamond (y \diamond z)) \diamond z \in X_{\mu_A^+}$ and $(x \diamond (y \diamond z)) \diamond z \in X_{\mu_A^-}$.

Therefore, $X_{\mu_A^+}$ and $X_{\mu_A^-}$ are ideals of X. □

Theorem 3.15. *If $A = (\mu_A^+,\ \mu_A^-)$ is a BFI of X with $\mu_A^+(1) \geq s$ and $\mu_A^-(1) \leq t$, then $P(\mu_A^+, s)$ and $N(\mu_A^-, t)$ are ideals of X.*

Proof. Suppose that $A = (\mu_A^+,\ \mu_A^-)$ is a BFI in X with $\mu_A^+(1) \geq s$ and $\mu_A^-(1) \leq t$. Clearly, P$(\mu_A^+, s) \neq \varnothing$ and $N(\mu_A^-, t) \neq \varnothing$ as both contain 1. Let $x, y \in$X such that $y \in P\,(\mu_A^+, s)$ and $y \in N(\mu_A^-, t)$. Then $\mu_A^+(x \diamond y) \geq \mu_A^+(y) \geq s$ and $\mu_A^-(x \diamond y) \leq \mu_A^-(y) \leq t$.
Hence $x \diamond y \in P(\mu_A^+, s)$ and $x \diamond y \in N(\mu_A^-, t)$.
Also, Let $x, y, z \in X$ such that $x, y \in P(\mu_A^+, s)$ and $x, y \in N(\mu_A^-, t)$. Then
$\mu_A^+(x \diamond (y \diamond z)) \diamond z) \geq min\{\mu_A^+(x),\ \mu_A^+(y)\} \geq s$ and
$\mu_A^-((x \diamond (y \diamond z)) \diamond z) \leq max\{\mu_A^+(x), \mu_A^+(y)\} \leq t$.
Hence, $(x \diamond (y \diamond z)) \diamond z \in P(\mu_A^+, s)$ and $(x \diamond (y \diamond z)) \diamond z \in$ N(μ_A^-, t).
Therefore, P(μ_A^+, s) and $N(\mu_A^-, t)$ are ideals of X. □

Theorem 3.16. *A BFS $A = (\ \mu_A^+,\ \mu_A^-)$ in X is a BFI in X if and only if the nonempty level cuts $P(\mu_A^+, s)$ and $N(\mu_A^-, t)$ are ideals in X, $\forall s \in [0.1]$ and $t \in [-1.0]$.*

Proof. Suppose that $A = (\ \mu_A^+,\ \mu_A^-)$ is a BFI in X. Let P(μ_A^+, s) and $N(\mu_A^-, t)$ are nonempty level cuts for all $s \in [0.1]$ and $t \in [-1.0]$. We claim that $P(\mu_A^+, s)$ and $N(\mu_A^-, t)$ are ideals of X. Let $x, y \in X$such that $y \in P(\mu_A^+, s)$ Then by Definition 2.10, $\mu_A^+(y) \geq s$, and so by Definition 3.1 (i), $\mu_A^+(x \diamond y) \geq \mu_A^+(y) \geq s$. This implies $x \diamond y \in P(\mu_A^+, s)$.
Also, let $x, y, z \in X$ such that $x,\ y \in P\,(\mu_A^+, s)$. Then, $\mu_A^+(x) \geq s$ and $\mu_A^+(y) \geq s$. By Definition 3.1 (ii), it follows that $\mu_A^+\,((x \diamond (y \diamond z)) \diamond z) \geq \min\left\{\mu_A^+(x),\ \mu_A^+(y)\right\} \geq s$. This implies that $(x \diamond (y \diamond z)) \diamond z \in P(\mu_A^+, s)$. Therefore, $P(\mu_A^+, s)$ is an ideal in X.
Similarly, if we assume $x, y \in X$ such that $y \in N(\mu_A^-, t)$, we have $\mu_A^-(y) \leq t$ by Definition 2.9. Also, by Definition 3.1 (i), $\mu_A^-(x \diamond y) \leq \mu_A^-(y) \leq t$. This implies $x \diamond y \in N(\mu_A^-, t)$.
Let $x, y, z \in X$ such that $x, y \in N(\mu_A^-, t)$. Then $\mu_A^-(x) \leq t$ and $\mu_A^-(y) \leq t$. By Definition 3.1 (iii), we have that $\mu_A^-((x \diamond (y \diamond z)) \diamond z) \leq \max\{\mu_A^-(x),\ \mu_A^-(y)\} \leq t$. This implies that $(x \diamond (y \diamond z)) \diamond z \in N(\mu_A^-, t)$. Hence $N(\mu_A^-, t)$ is an ideal of X. Therefore, P(μ_A^+, s) and $N(\mu_A^-, t)$ are ideals of X for all $s \in [0.1]$ and $t \in [-1.0]$.
Conversely, assume that the nonempty level cuts $P(\mu_A^+, s)$ and $N(\mu_A^-, t)$ are ideals in X, $\forall\ s \in [0.1]$ and $t \in [-1.0]$. Then we need to show that$A = (\mu_A^+, \mu_A^-)$ is a BFI of X. Let $x, y \in X$ such that $y \in P(\mu_A^+, s)$. Assume that $\mu_A^+(x \diamond y) < \mu_A^+(y)$ for some $x, y \in X$.
Take s$_0 = \frac{1}{2}(\mu_A^+(x \diamond y) + \mu_A^+(y))$, where s$_0 \in [0, 1]$. This implies $\mu_A^+(x \diamond y) < s_0 < \mu_A^+(y)$.
Hence y $\in P(\mu_A^+, s_0)$ but x $\diamond\, y \notin P(\mu_A^+, s_0)$, which is a contradiction.
Assume that $\mu_A^+((x \diamond (y \diamond z)) \diamond z) < min\{\mu_A^+(x), \mu_A^+(y)\}$ for some $x, y, z \in X$ such that $x, y \in P\,(\mu_A^+, s)$. By taking s$_0 = \frac{1}{2}(\mu_A^+((x \diamond (y \diamond z)) \diamond z) + min\{\mu_A^+(x),\ \mu_A^+(y)\}),\ where$ $s_0 \in [0,\ 1]$, we have $\mu_A^+((x \diamond (y \diamond z)) \diamond z) < s_0 < min\{\mu_A^+(x),\ \mu_A^+(y)\}$. Then it follows that $x, y \in P(\mu_A^+, s_0)$ and $(x \diamond (y \diamond z)) \diamond z \notin P(\mu_A^+, s_0)$.This is a contradiction, since $P(\mu_A^+, s_0)$ is an ideal of X.
Therefore, $\mu_A^+(x{\diamond}y) \geq \mu_A^+(y)$ and $\mu_A^+((x{\diamond}(y{\diamond}z)){\diamond}z) \geq min\{\mu_A^+(x),\ \mu_A^+(y)\}$, for all $x, y, z \in X$. Similarly, we can also prove that $\mu_A^-(x{\diamond}y) \leq \mu_A^-(y)$ and $\mu_A^-((x{\diamond}(y{\diamond}z)){\diamond}z) \leq max\{\mu_A^-(x),\ \mu_A^-(y)\}$, for all $x, y, z \in X$. Therefore, $A = (\mu_A^+,\ \mu_A^-)$ is a BFI of X. □

Theorem 3.17. *Each ideal in a BE-algebra X is a level ideal of some BFI of X.*

Proof. Consider an ideal I of X and a BFS $A = \left(\mu_A^+, \mu_A^-\right)$ of X defined by

$$\mu_A^+(x) = \begin{cases} s\ if\ x \in \text{I} \\ 0\ \ \text{otherwise} \end{cases} \text{ and } \mu_A^-(x) = \begin{cases} t\ if\ x \in \text{I} \\ 0\ \text{otherwise} \end{cases}, \text{ where } s \in [0, 1] \text{ and } t \in [-1, 0].$$

Clearly, P(μ_A^+, s) = N $\left(\mu_A^-, t\right)$. Now, let's examine the following cases:
Case (i) If $x \in X, y \in$ I , then $x \diamond y \in$ I as I is an ideal X. Therefore $\mu_A^+(x \diamond y) = s = \mu_A^+(y)$ and $\mu_A^-(x \diamond y) = t = \mu_A^-(y)$. Also, if $x, y, z \in X$ such that$x, y \in$ I then $(x \diamond (y \diamond z)) \diamond z \in$ I . So, $\mu_A^+\,((x \diamond (y \diamond z)) \diamond\ z) = s = \min\{s, s\} = \min\left\{\mu_A^+(x), \mu_A^+(y)\right\}$ and $\mu_A^-\,((x \diamond (y \diamond z)) \diamond z) = t = \max\{t, t\} = max\left\{\mu_A^-(x), \mu_A^-(y)\right\}$.
Case (ii) If$x, y \in X$ such that $y \notin$ I , then $\mu_A^+(x \diamond y) \geq 0 = \mu_A^+(y)$ and $\mu_A^-(x \diamond y) \leq 0 = \mu_A^-(y)$. Also, if $x, y,\ z \in$X such that $x \notin$ I or $y \notin$ I , then $\mu_A^+(x) = 0$ or $\mu_A^+(y) = 0$.
Therefore, $\mu_A^+\,((x \diamond (y \diamond z)) \diamond z) \geq 0 = \min\{0, 0\} = \min\left\{\mu_A^+(x), \mu_A^+(y)\right\}$ and
$\mu_A^-\,((x \diamond (y \diamond z)) \diamond z) \leq 0 = \max\{0, 0\} = \max\left\{\mu_A^-(x), \mu_A^-(y)\right\}$.

Thus, $\mu_A^+(x \diamond y) \geq \mu_A^+(y)$ and $\mu_A^-(x \diamond y) \leq \mu_A^-(y)$, $\mu_A^+((x \diamond (y \diamond z)) \diamond z) \geq \min\{\mu_A^+(x), \mu_A^+(y)\}$ and $\mu_A^-((x \diamond (y \diamond z)) \diamond z) \leq \max\{\mu_A^-(x), \mu_A^-(y)\}$. Therefore $A = (\mu_A^+,\ \mu_A^-)$ is a BFI of X and thus I is a level ideal of X corresponding to BFI A of X. □

Theorem 3.18. *If $A = (\mu_A^+,\ \mu_A^-)$ is a BFS of X, then the two level cuts $P(\mu_A^+, s_1) = P(\mu_A^+, s_2)$ and $N(\mu_A^-, t_1) = N(\mu_A^-, t_2)$ if and only if there is no $x \in$X such that $s_1 \leq \mu_A^+(x) < s_2$ and $t_1 \geq \mu_A^-(x) > t_2$.*

Proof. Let $P(\mu_A^+, s_1) = P(\mu_A^+, s_2)$ and $N(\mu_A^-, t_1) = N(\mu_A^-, t_2)$. Assume that there exist $x \in$ X such that $s_1 \leq \mu_A^+(x) < s_2$ and $t_1 \geq \mu_A^-(x) > t_2$. If $\mu_A^+(x) \geq s_1$ and $\mu_A^-(x) \leq t_1$, then we have $x \in P(\mu_A^+, s_1) = P(\mu_A^+, s_2)$ and $x \in N(\mu_A^-, t_1) = N(\mu_A^-, t_2)$ if $\mu_A^-(x) \leq t_1$. Thus, $\mu_A^+(x) \geq s_2$ and $\mu_A^-(x) \leq t_2$. This is a contradiction to the assumption that $\mu_A^+(x) < s_2$ and $\mu_A^-(x) > t_2$. Thus there is no $x \in X$ such that $s_1 \leq \mu_A^+(x) < s_2$ and $t_1 \geq \mu_A^-(x) > t_2$.
Conversely, suppose that there is no $x \in X$ such that $s_1 \leq \mu_A^+(x) < s_2$ and $t_1 \geq \mu_A^-(x) > t_2$. Assume that $P(\mu_A^+, s_1) \neq P(\mu_A^+, s_2)$ and $N(\mu_A^-, t_1) \neq N(\mu_A^-, t_2)$. Then there exist $x \in P(\mu_A^+, s_1)$ but $x \notin P(\mu_A^+, s_2)$ and also there exist $x \in N(\mu_A^-, t_1)$ but $x \notin N(\mu_A^-, t_2)$. This implies $\mu_A^+(x) \geq s_1$, $\mu_A^+(x) < s_2$, $\mu_A^-(x) \leq t_1$ and $\mu_A^-(x) > t_2$. So that $s_1 \leq \mu_A^+(x) < s_2$ and $t_1 \geq \mu_A^-(x) > t_2$. Thus, $P(\mu_A^+, s_1) \neq P(\mu_A^+, s_2)$ and $N(\mu_A^-, t_1) \neq N(\mu_A^-, t_2)$ implies there exists $x \in X$ such that $s_1 \leq \mu_A^+(x) < s_2$ and $t_1 \geq \mu_A^-(x) > t_2$. Therefore, by contrapositive it follows that $P(\mu_A^+, s_1) = P(\mu_A^+, s_2)$ and $N(\mu_A^-, t_1) = N(\mu_A^-, t_2)$. □

Theorem 3.19. *Let $A = (\mu_A^+,\ \mu_A^-)$ be a BFI of X. Then $P(\mu_A^+, s) \subseteq P(\mu_A^+, u)$ and $N(\mu_A^-, t) \subseteq N(\mu_A^-, v)$ if $s \geq u$ and $t \leq v$, where $s, u \in [0, 1]$ and $t, v \in [-1,\ 0]$.*

Proof. Let $x \in P(\mu_A^+, s)$. $Then, \mu_A^+(x) \geq s \geq u \Longrightarrow x \in P(\mu_A^+, u)$.
Hence, $P(\mu_A^+, s) \subseteq P(\mu_A^+, u)$.
Similarly, let $x \in N(\mu_A^-, t)$. $Then,\ \mu_A^-(x) \leq t \leq v \Longrightarrow x \in N(\mu_A^-, v)$.
$So,$ $\mathrm{N}(\mu_A^-, t) \subseteq N(\mu_A^-, v)$. □

Theorem 3.20. *Let $A = (\mu_A^+,\ \mu_A^-)$ and $B = (\mu_B^+,\ \mu_B^-)$ be BFIs of X. $P(\mu_A^+, s) \subseteq P(\mu_B^+, s)$ and $N(\mu_A^-, t) \subseteq N(\mu_B^-, t)$, for $s \in [0, 1]$ and $t \in [-1,\ 0]$ only if $A \subseteq B$*

Proof. Let $x \in P(\mu_A^+, s)$. Then $\mu_A^+(x) \geq s$. Since $A \subseteq B$, we have $\mu_B^+(x) \geq \mu_A^+(x) \geq s$.
This implies $x \in P(\mu_B^+, s)$. Thus, $P(\mu_A^+, s) \subseteq P(\mu_B^+, s)$. Similarly, if we let $x \in N(\mu_A^-, t)$, we have, $\mu_A^-(x) \leq t$. As $A \subseteq B$, we also have $\mu_B^-(x) \leq \mu_A^-(x) \leq t$. This implies $x \in N(\mu_B^-, t)$. Hence $\mathrm{N}(\mu_A^-, t) \subseteq N(\mu_B^-, t)$. □

Theorem 3.21. *Let $A = (\mu_A^+,\ \mu_A^-)$ and $B = (\mu_B^+,\ \mu_B^-)$ be any two BFIs of X. Then, $P(\mu_{A\cap B}^+, s) = P(\mu_A^+, s) \cap P(\mu_B^+, s)$ and $N(\mu_{A\cap B}^-, t) = N(\mu_A^-, t) \cap N(\mu_B^-, t)$, for $s \in [0, 1]$ and $t \in [-1, 0]$.*

Proof. By Definition 2.10, we have $P(\mu_{A\cap B}^+, s) = \{x \in X : \mu_{A\cap B}^+(x) \geq s\}$ and $N(\mu_{A\cap B}^-, t) = \{x \in X : \mu_{A\cap B}^-(x) \leq t\}$. Now,

$$\begin{aligned} x \in P(\mu_{A\cap B}^+, s) &\Longleftrightarrow \mu_{A\cap B}^+(x) \geq s \\ &\Longleftrightarrow \min\{\mu_A^+(x), \mu_B^+(x)\} \geq s \\ &\Longleftrightarrow \mu_A^+(x) \geq s \text{ and } \mu_B^+(x) \geq s \\ &\Longleftrightarrow x \in P(\mu_A^+, s) \text{ and } x \in P(\mu_B^+, s) \\ &\Longleftrightarrow x \in P(\mu_A^+, s) \cap P(\mu_B^+, s) \end{aligned}$$

Therefore, $P(\mu_{A\cap B}^+, s) = P(\mu_A^+, s) \cap P(\mu_B^+, s)$
Similarly,

$$\begin{aligned} x \in N(\mu_{A\cap B}^-, t) &\Longleftrightarrow \mu_{A\cap B}^-(x) \leq t \\ &\Longleftrightarrow \max\{\mu_A^-(x), \mu_B^-(x)\} \leq t \\ &\Longleftrightarrow \mu_A^-(x) \leq t \text{ and } \mu_B^-(x) \leq t \\ &\Longleftrightarrow x \in N(\mu_A^-, t) \text{ and } x \in N(\mu_B^-, t) \\ &\Longleftrightarrow x \in N(\mu_A^-, t) \cap N(\mu_B^-, t) \end{aligned}$$

Therefore, $P(\mu_{A\cap B}^+, s) = P(\mu_A^+, s) \cap P(\mu_B^+, s)$. □

Theorem 3.22. *Suppose* $A = (\mu_A^+,\ \mu_A^-)$ *and* $B = (\mu_B^+,\ \mu_B^-)$ *be any two BFIs of X. Then,* $P\left(\mu_{A\cup B}^+, s\right) = P\left(\mu_A^+, s\right) \cup P\left(\mu_B^+, s\right)$ *and* $N\left(\mu_{A\cup B}^-, t\right) = N\left(\mu_A^-, t\right) \cup N\left(\mu_B^-, t\right)$, *for* $s \in [0,\ 1]$ *and* $t \in [-1,\ 0]$.

Proof. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by the theorem 3.20 $P\left(\mu_A^+, s\right) \subseteq P\left(\mu_{A\cup B}^+, s\right)$ and P$\left(\mu_B^+, s\right) \subseteq P\left(\mu_{A\cup B}^+, s\right)$. Therefore, $P\left(\mu_A^+, s\right) \cup P\left(\mu_B^+, s\right) \subseteq P\left(\mu_{A\cup B}^+, s\right)$.
Let $x \in P\left(\mu_{A\cup B}^+, s\right)$. Then, $\mu_{A\cup B}^+(x) \geq s$. This implies that max $\left\{\mu_A^+(x), \mu_B^+(x)\right\} \geq s$. Thus, $\mu_A^+(x) \geq s$ or $\mu_B^+(x) \geq s \Longrightarrow x \in P\left(\mu_A^+, s\right)$ or $x \in P\left(\mu_B^+, s\right)$

$$\Longrightarrow x \in P\left(\mu_A^+, s\right) \cup P\left(\mu_B^+, s\right).$$

Therefore, $P\left(\mu_{A\cup B}^+, s\right) \subseteq P\left(\mu_A^+, s\right) \cup P\left(\mu_B^+, s\right)$.
Consequently, we have P$\left(\mu_{A\cup B}^+, s\right) = P\left(\mu_A^+, s\right) \cup P\left(\mu_B^+, s\right)$.
And also by the theorem 3.20, we have, N$\left(\mu_A^-, t\right) \subseteq N\left(\mu_{A\cup B}^-, t\right)$ and $N\left(\mu_B^-, t\right) \subseteq N\left(\mu_{A\cup B}^-, t\right)$. Thus, N$\left(\mu_A^-, t\right) \cup N\left(\mu_B^-, t\right) \subseteq N\left(\mu_{A\cup B}^-, t\right)$.
Let $x \in N\left(\mu_{A\cup B}^-, t\right)$. Then, $\mu_{A\cup B}^-(x) \leq t$. This implies that min $\left\{\mu_A^-(x), \mu_B^-(x)\right\} \leq t$. Thus $\mu_A^-(x) \leq t$ or $\mu_B^-(x) \leq t \Longrightarrow x \in N\left(\mu_A^-, t\right)$ or $x \in N\left(\mu_B^-, t\right)$

$$\Longrightarrow x \in N\left(\mu_A^-, t\right) \cup N\left(\mu_B^-, t\right).$$

Therefore, $N\left(\mu_{A\cup B}^-, t\right) \subseteq N\left(\mu_A^-, t\right) \cup N\left(\mu_B^-, t\right)$.
Consequently, we have N$\left(\mu_{A\cup B}^-, t\right) = N\left(\mu_A^-, t\right) \cup N\left(\mu_B^-, t\right)$.
Hence, $P\left(\mu_{A\cup B}^+, s\right) = P\left(\mu_A^+, s\right) \cup P\left(\mu_B^+, s\right)$ and $N\left(\mu_{A\cup B}^-, t\right) = N\left(\mu_A^-, t\right) \cup N\left(\mu_B^-, t\right)$, for $s \in [0,\ 1]$ and $t \in [-1,\ 0]$. □

4 Homomorphism of Bipolar Fuzzy ideals in BE-Algebras

In this section, we explore the effect of homomorphism on the bipolar fuzzy ideals of a BE-algebra. We demonstrate that both the homomorphic image and the homomorphic pre-image of bipolar fuzzy ideals in a BE-algebra are themselves bipolar fuzzy ideals.

Theorem 4.1. *Let f be an epimorphism of BE-algebras. The homomorphic image of A under f is a BFI in Y only if* $A = (\mu_A^+,\ \mu_A^-)$ *is a BFI in X that satisfies* sup-inf *property.*

Proof. Let $A = (\mu_A^+,\ \mu_A^-)$ be a BFI in X that satisfies the condition of sup-inf property. Let $a, b, c \in Y$ with $x_0 \in f^{-1}(a)$, $y_0 \in f^{-1}(b)$ and $z_0 \in f^{-1}(c)$ such that

$$\mu_A^+(x_0) = \sup_{t\in f^{-1}(a)} \mu_A^+(t),\ \mu_A^+(y_0) = \sup_{t\in f^{-1}(b)} \mu_A^+(t),\ \mu_A^+(z_0) = \sup_{t\in f^{-1}(c)} \mu_A^+(t),\ \text{and}$$

$$\mu_A^-(x_0) = \inf_{t\in f^{-1}(a)} \mu_A^-(t),\ \mu_A^-(y_0) = \inf_{t\in f^{-1}(b)} \mu_A^-(t),\ \mu_A^-(z_0) = \inf_{t\in f^{-1}(c)} \mu_A^-(t),$$

Then, by definitions 2.11 and 2.12, we have

$$f(\mu_A^+)(a \diamond b) = \sup_{t\in f^{-1}(a\diamond b)} \mu_A^+(t) = \mu_A^+(x_0 \diamond y_0) \geq \mu_A^+(y_0) = \sup_{t\in f^{-1}(b)} \mu_A^+(t) = f(\mu_A^+)(b) \text{ and}$$

$$f(\mu_A^-)(a \diamond b) = \inf_{t\in f^{-1}(a\diamond b)} \mu_A^-(t) = \mu_A^+(x_0 \diamond y_0) \leq \mu_A^-(y_0) = \inf_{t\in f^{-1}(b)} \mu_A^-(t) = f(\mu_A^-)(b)$$

$$\Longrightarrow f(\mu_A^+)(a \diamond b) \geq f(\mu_A^+)(b) \text{ and } f(\mu_A^-)(a \diamond b) \leq f(\mu_A^-)(b).$$

$$\begin{aligned}
f(\mu_A^+)\left((a \diamond (b \diamond c)) \diamond c\right) &= \sup_{t\in f^{-1}((a\diamond(b\diamond c))\diamond c)} \mu_A^+(t) \\
&= \mu_A^+\left((x_0 \diamond (y_0 \diamond z_0)) \diamond z_0\right) \\
&\geq min\left\{\mu_A^+(x_0), \mu_A^+(y_0)\right\} \\
&= min\{\sup_{t\in f^{-1}(a)} \mu_A^+(t), \sup_{t\in f^{-1}(b)} \mu_A^+(t)\} \\
&= min\left\{f(\mu_A^+)(a), f(\mu_A^+)(b)\right\}
\end{aligned}$$

$$\Longrightarrow f(\mu_A^+)((a \diamond (b \diamond c)) \diamond c) \geq min\left\{f(\mu_A^+)(a), f(\mu_A^+)(b)\right\} \text{ and}$$

$$\begin{aligned}
f(\mu_A^-)\left((a \diamond (b \diamond c)) \diamond c\right) &= \inf_{t\in f^{-1}((a\diamond(b\diamond c))\diamond c)} \mu_A^+(t) \\
&= \mu_A^-\left((x_0 \diamond (y_0 \diamond z_0)) \diamond z_0\right) \\
&\leq max\left\{\mu_A^-(x_0), \mu_A^-(y_0)\right\}
\end{aligned}$$

$$= max\{\inf_{t\in f^{-1}(a)} \mu_A^-(t), \inf_{t\in f^{-1}(a)} \mu_A^-(t)\}$$
$$= max\left\{f(\mu_A^+)(a), f(\mu_A^+)(b)\right\}$$
$$\Longrightarrow f(\mu_A^-)((a\diamond(b\diamond c))\diamond c) \leq max\left\{f(\mu_A^-)(a), f(\mu_A^-)(b)\right\}$$

Therefore, $f(A)$ is a BFI of Y. □

Theorem 4.2. *Let* $f : X \to Y$ *be an epimorphism and* $A = (\mu_A^+,\ \mu_A^-)$ *is a BFS in Y. Then,* $A = (\mu_A^+,\ \mu_A^-)$ *is a BFI in Y if and only if* $f^{-1}(A)$ *is a BFI in* X*.*

Proof. Suppose that $f : X \to Y$ is an epimorphism of BE-algebras and $A = (\mu_A^+,\ \mu_A^-)$ is a BFI of Y. Then for $x, y, z \in Y$, we have

$f^{-1}(\mu_A^+)(x\diamond y) = \mu_A^+(f(x\diamond y)) = \mu_A^+(f(x)\diamond f(y)) \geq \mu_A^+(f(y)) = f^{-1}(\mu_A^+)(y)$ and
$f^{-1}\left(\mu_A^-\right)(x\diamond y) = \mu_A^-(f(x\diamond y)) = \mu_A^-(f(x)\diamond f(y)) \leq \mu_A^-(f(y)) = f^{-1}\left(\mu_A^-\right)(y)$.

$\Longrightarrow f^{-1}(\mu_A^+)(x\diamond y) \geq f^{-1}(\mu_A^+)(y)$ and $f^{-1}\left(\mu_A^-\right)(x\diamond y) \leq f^{-1}\left(\mu_A^-\right)(y)$.

Also,
$$\begin{aligned}
f^{-1}\left(\mu_A^+\right)((x\diamond(y\diamond z))\diamond z) &= \mu_A^+(f((x\diamond(y\diamond z))\diamond z))\\
&= \mu_A^+(f((x\diamond(y\diamond z))\diamond f(z))\\
&= \mu_A^+((f(x)\diamond f(y\diamond z))\diamond f(z))\\
&= \mu_A^+((f(x)\diamond(f(y)\diamond f(z)))\diamond f(z))\\
&\geq min\left\{\mu_A^+(f(x)), \mu_A^+(f(y))\right\}\\
&= min\left\{f^{-1}\left(\mu_A^+\right)(x), f^{-1}\left(\mu_A^+\right)(y)\right\}
\end{aligned}$$
$$\Longrightarrow f^{-1}\left(\mu_A^+\right)((x\diamond(y\diamond z))\diamond z) \geq min\left\{f^{-1}\left(\mu_A^+\right)(x), f^{-1}\left(\mu_A^+\right)(y)\right\} \text{ and}$$
$$\begin{aligned}
f^{-1}\left(\mu_A^-\right)((x\diamond(y\diamond z))\diamond z) &= \mu_A^-(f((x\diamond(y\diamond z))\diamond z))\\
&= \mu_A^-(f((x\diamond(y\diamond z))\diamond f(z))\\
&= \mu_A^-((f(x)\diamond f(y\diamond z))\diamond f(z))\\
&= \mu_A^-((f(x)\diamond(f(y)\diamond f(z)))\diamond f(z))\\
&\leq max\left\{\mu_A^-(f(x)), \mu_A^-(f(y))\right\}\\
&= max\left\{f^{-1}\left(\mu_A^-\right)(x), f^{-1}\left(\mu_A^-\right)(y)\right\}
\end{aligned}$$
$$\Longrightarrow f^{-1}\left(\mu_A^-\right)((x\diamond(y\diamond z))\diamond z) \leq \max\left\{f^{-1}\left(\mu_A^-\right)(x), f^{-1}\left(\mu_A^-\right)(y)\right\}$$

Therefore, $f^{-1}(A)$ is a bipolar fuzzy ideal of X.

Conversely, suppose that $f^{-1}(A)$ is a BFI of X. Now we need to show that $A = (\mu_A^+,\ \mu_A^-)$ is a BFI of Y. Since f is an epimorphism of BE-algebras for any $x, y, z \in Y$, there exist $a, b, c \in X$ such that $f(a) = x, f(b) = y$ and $f(c) = z$. Then,

$$\begin{aligned}
\mu_A^+(x\diamond y) = \mu_A^+(f(a)\diamond f(b)) &= \mu_A^+(f(a\diamond b))\\
&= f^{-1}\left(\mu_A^+\right)(a\diamond b)\\
&\geq f^{-1}\left(\mu_A^+\right)(a) = \mu_A^+(f(a)) = \mu_A^+(y) \text{ and}
\end{aligned}$$
$$\begin{aligned}
\mu_A^-(x\diamond y) = \mu_A^-(f(a)\diamond f(b)) &= \mu_A^-(f(a\diamond b))\\
&= f^{-1}\left(\mu_A^-\right)(a\diamond b)\\
&\leq f^{-1}\left(\mu_A^-\right)(a) = \mu_A^-(f(a)) = \mu_A^-(y)
\end{aligned}$$

$\Longrightarrow \mu_A^+(x\diamond y) \geq \mu_A^+(y)$ and $\mu_A^-(x\diamond y) \leq \mu_A^-(y)$.

Also,
$$\begin{aligned}
\mu_A^+((x\diamond(y\diamond z))\diamond z) &= \mu_A^+((f(a)\diamond(f(b)\diamond f(c)))\diamond f(c))\\
&= \mu_A^+((f(a)\diamond(f(b\diamond c)))\diamond f(c))\\
&= \mu_A^+((f(a\diamond(b\diamond c))\diamond f(c))\\
&= \mu_A^+((f(a\diamond(b\diamond c))\diamond c))\\
&= f^{-1}\left(\mu_A^+\right)((a\diamond(b\diamond c))\diamond c)\\
&\geq min\left\{f^{-1}\left(\mu_A^+\right)(a), f^{-1}\left(\mu_A^+\right)(b)\right\}
\end{aligned}$$

$$= min\left\{\mu_A^+\left(f(a)\right), \mu_A^+\left(f(b)\right)\right\}$$
$$= min\left\{\mu_A^+(x), \mu_A^+(y)\right\}$$
$$\Longrightarrow \mu_A^+\left((x\diamond(y\diamond z))\diamond z\right) \geq \ min\left\{\mu_A^+(x), \mu_A^+(y)\right\} \text{ and}$$
$$\mu_A^-\left((x\diamond(y\diamond z))\diamond z\right) = \mu_A^-\left((f(a)\diamond(f(b)\diamond f(c)))\diamond f(c)\right)$$
$$= \mu_A^-\left((f(a)\diamond(f(b\diamond c)))\diamond f(c)\right)$$
$$= \mu_A^-\left((f(a\diamond(b\diamond c))\diamond f(c)\right)$$
$$= \mu_A^-\left((f\left(a\diamond(b\diamond c)\right)\diamond c)\right)$$
$$= f^{-1}\left(\mu_A^-\right)\left((a\diamond(b\diamond c))\diamond c\right)$$
$$\leq max\left\{f^{-1}\left(\mu_A^-\right)(a), f^{-1}\left(\mu_A^-\right)(b)\right\}$$
$$= max\left\{\mu_A^-\left(f(a)\right), \mu_A^-\left(f(b)\right)\right\}$$
$$= max\left\{\mu_A^-(x), \mu_A^-(y)\right\}$$
$$\Longrightarrow \mu_A^-\left((x\diamond(y\diamond z))\diamond z\right) \leq \ max\left\{\mu_A^-(x), \mu_A^-(y)\right\}.$$

Therefore,$A = (\mu_A^+,\ \mu_A^-)$ is a BFI of Y. □

Theorem 4.3. *Let $A = (\mu_A^+,\ \mu_A^-)$ be a BFI of Y and $f : X \to Y$ is an isomorphism in BE-algebras. Then $Aof = (\mu_A^+ \circ f, \mu_A^- \circ f)$ is a BFI of X, where $'\circ'$ is composition of functions.*

Proof. Let $x, y, z \in X$. Then,

$(\mu_A^+ \circ f)(x\diamond y) = \mu_A^+\left(f(x\diamond y)\right) = \mu_A^+\left(f\left(x\right)\diamond f(y)\right) \geq \mu_A^+\left(f(y)\right) = (\mu_A^+ \circ f)(y)$ and

$(\mu_A^- \circ f)(x\diamond y) = \mu_A^-\left(f(x\diamond y)\right) = \mu_A^-\left(f\left(x\right)\diamond f(y)\right) \leq \mu_A^-\left(f(y)\right) = (\mu_A^- \circ f)(y)$

$\Longrightarrow (\mu_A^+ \circ f)(x\diamond y) \geq (\mu_A^+ \circ f)(y)$ and $(\mu_A^- \circ f)(x\diamond y) \leq (\mu_A^- \circ f)(y)$.

$$(\mu_A^+ \circ f)\left((x\diamond(y\diamond z))\diamond z\right) = \mu_A^+\left(f\left((x\diamond(y\diamond z))\diamond z\ \right)\right.$$
$$= \mu_A^+\left(f\left((x\diamond(y\diamond z))\diamond f(z)\right)\right.$$
$$= \mu_A^+\left((f(x)\diamond f(y\diamond z))\diamond f(z)\right)$$
$$= \mu_A^+\left((f(x)\diamond(f(y)\diamond f(z)))\diamond f(z)\right)$$
$$\geq min\left\{\mu_A^+\left(f(x)\right), \mu_A^+\left(f(y)\right)\right\}$$
$$= min\left\{(\mu_A^+ \circ f)(x), (\mu_A^+ \circ f)(y)\right\}$$
$$\Longrightarrow (\mu_A^+ \circ f)((x\diamond(y\diamond z))\diamond z) \geq \min\left\{(\mu_A^+ \circ f)(x), (\mu_A^+ \circ f)(y)\right\} \text{ and}$$
$$(\mu_A^- \circ f)\left((x\diamond(y\diamond z))\diamond z\right) = \mu_A^-\left(f\left((x\diamond(y\diamond z))\diamond z\right)\right.$$
$$= \mu_A^-\left(f\left((x\diamond(y\diamond z))\diamond f(z)\right)\right.$$
$$= \mu_A^-\left((f(x)\diamond f(y\diamond z))\diamond f(z)\right)$$
$$= \mu_A^-\left((f(x)\diamond(f(y)\diamond f(z)))\diamond f(z)\right)$$
$$\leq max\left\{\mu_A^-\left(f(x)\right), \mu_A^-\left(f(y)\right)\right\}$$
$$= max\left\{(\mu_A^- \circ f)(x), (\mu_A^- \circ f)(y)\right\}$$
$$\Longrightarrow (\mu_A^+ \circ f)((x\diamond(y\diamond z))\diamond z) \leq \max\left\{(\mu_A^+ \circ f)(x), (\mu_A^+ \circ f)(y)\right\}.$$

Therefore, $Aof = (\mu_A^+ \circ f, \mu_A^- \circ f)$ is a BFI of X. □

5 Cartesian Products of Bipolar Fuzzy BE-ideals in BE-Algebra

In this part of the manuscript, we address the Cartesian product of BFIs in BE-algebras and demonstrate that the Cartesian product of BFIs is a BFI, along with some other interesting results.

Lemma 5.1. *If $A = (\mu_A^+,\ \mu_A^-)$ and $B = (\mu_B^+,\ \mu_B^-)$ are any two BFIs of X, then $\mu_{A\times B}^+(1,1) \geq \mu_{A\times B}^+(x_1, x_2)$ and $\mu_{A\times B}^-(1,1) \leq \mu_{A\times B}^-(x_1, x_2)$, for all $x_1, x_2 \in X$.*

Proof. Let $(x_1, x_2) \in X \times X$. Then by using Definition 2.13 and Lemma 3.3, we have
$\mu^+_{A\times B}(1,1) = min\,\{\mu^+_A(1), \mu^+_B(1)\} \geq min\,\{\mu^+_A(x_1), \mu^+_B(x_2)\} = \mu^+_{A\times B}(x_1, x_2)$ and
$\mu^-_{A\times B}(1,1) = max\,\{\mu^-_A(1), \mu^-_B(1)\} \leq max\,\{\mu^-_A(x_1), \mu^-_B(x_2)\} = \mu^-_{A\times B}(x_1, x_2)$
Therefore, $\mu^+_{A\times B}(1,1) \geq \mu^+_{A\times B}(x_1, x_2)$ and $\mu^-_{A\times B}(1,1) \leq \mu^-_{A\times B}(x_1, x_2), \forall\, x_1, x_2 \in X$. □

Theorem 5.2. *If $A = (\mu^+_A,\ \mu^-_A)$ and $B = (\mu^+_B,\ \mu^-_B)$ are any two BFIs of X, then $A \times B$ is a BFI of $X \times X$.*

Proof. Let $(x_1, x_2), (y_1,\ y_2), (z_1, z_2) \in X \times X$. Then

$$\begin{aligned}
\mu^+_{A\times B}((x_1, x_2) \diamond (y_1,\ y_2)) &= \mu^+_{A\times B}(x_1 \diamond y_1, (x_2 \diamond y_2) \\
&= min\{\mu^+_A\,(x_1 \diamond y_1),\ \mu^+_B\,(x_2 * y_2)\} \\
&\geq \min\{\mu^+_A\,(y_1), \mu^+_B\,(y_2)\} \\
&= \mu^+_{A\times B}(y_1, y_2)\ and \\
\mu^-_{A\times B}((x_1, x_2) \diamond (y_1,\ y_2)) &= \mu^-_{A\times B}(x_1 \diamond y_1, (x_2 \diamond y_2) \\
&= max\{\mu^-_A\,(x_1 \diamond y_1), \mu^-_B\,(x_2 \diamond y_2)\} \\
&\leq \max\{\mu^-_A(y_1), \mu^-_B(y_2)\} \\
&= \mu^-_{A\times B}(y_1, y_2)
\end{aligned}$$

$\Longrightarrow \mu^+_{A\times B}((x_1, x_2) \diamond (y_1,\ y_2)) \geq \mu^+_{A\times B}(y_1,\ y_2)$ and $\mu^-_{A\times B}((x_1, x_2) \diamond (y_1,\ y_2)) \leq \mu^-_{A\times B}(y_1,\ y_2)$.

$$\begin{aligned}
\mu^+_{A\times B}(((x_1, x_2) \diamond ((y_1,\ y_2) \diamond (z_1, z_2))) \diamond (z_1, z_2)) &= \mu^+_{A\times B}(((x_1, x_2) \diamond (y_1 \diamond z_1, y_2 \diamond z_2)) \diamond (z_1, z_2)) \\
&= \mu^+_{A\times B}(((x_1 \diamond (y_1 \diamond z_1), (x_2 \diamond (y_2 \diamond z_2)) \diamond (z_1, z_2)) \\
&= \mu^+_{A\times B}((x_1 \diamond (y_1 \diamond z_1)) \diamond z_1, (x_2 \diamond (y_2 \diamond z_2)) \diamond z_2) \\
&= min\{\mu^+_A((x_1 \diamond (y_1 \diamond z_1)) \diamond z_1), \mu^+_B((x_2 \diamond (y_2 \diamond z_2)) \diamond z_2)\} \\
&\geq min\left\{\min\left\{\mu^+_A\,(x_1)\,, \mu^+_A\,(y_1)\right\}, min\left\{\mu^+_B\,(x_2)\,, \mu^+_B\,(y_2)\right\}\right\} \\
&= min\left\{\min\left\{\mu^+_A\,(x_1)\,, \mu^+_B\,(x_2)\right\}, min\left\{\mu^+_A\,(y_1)\,, \mu^+_B\,(y_2)\right\}\right\} \\
&= min\left\{\mu^+_{A\times B}\,(x_1, x_2)\,, \mu^+_{A\times B}\,(y_1, y_2)\right\}
\end{aligned}$$

$\Longrightarrow \mu^+_{A\times B}(((x_1, x_2) \diamond ((y_1, y_2) \diamond (z_1, z_2))) \diamond (z_1, z_2)) \geq min\{\mu^+_{A\times B}(x_1, x_2), \mu^+_{A\times B}(y_1, y_2)\}$ and

$$\begin{aligned}
\mu^-_{A\times B}\,(((x_1, x_2) \diamond ((y_1,\ y_2) \diamond (z_1, z_2))) \diamond (z_1, z_2)) &= \mu^-_{A\times B}\,(((x_1, x_2) \diamond (y_1 \diamond z_1, y_2 \diamond z_2)) \diamond (z_1, z_2)) \\
&= \mu^-_{A\times B}\,(((x_1 \diamond (y_1 \diamond z_1), (x_2 \diamond (y_2 \diamond z_2)) \diamond (z_1, z_2)) \\
&= \mu^-_{A\times B}\,((x_1 \diamond (y_1 \diamond z_1)) \diamond z_1, (x_2 \diamond (y_2 \diamond z_2)) \diamond z_2) \\
&= max\{\mu^-_A((x_1 \diamond (y_1 \diamond z_1)) \diamond z_1), \mu^-_B((x_2 \diamond (y_2 \diamond z_2)) \diamond z_2)\} \\
&\leq max\left\{\max\left\{\mu^-_A\,(x_1)\,, \mu^-_A\,(y_1)\right\},\ max\left\{\mu^-_B\,(x_2)\,, \mu^-_B\,(y_2)\right\}\right\} \\
&= max\left\{\max\left\{\mu^-_A\,(x_1)\,, \mu^+_B\,(x_2)\right\}, max\left\{\mu^-_A\,(y_1)\,, \mu^-_B\,(y_2)\right\}\right\} \\
&= max\left\{\mu^-_{A\times B}\,(x_1, x_2)\,, \mu^-_{A\times B}\,(y_1, y_2)\right\}
\end{aligned}$$

$\Longrightarrow \mu^-_{A\times B}\,(((x_1, x_2) \diamond ((y_1,\ y_2) \diamond (z_1, z_2))) \diamond (z_1, z_2)) \leq\ max\left\{\mu^-_{A\times B}\,(x_1, x_2)\,, \mu^-_{A\times B}\,(y_1, y_2)\right\}$
Therefore, $A \times B$ is a BFI of $X \times X$. □

Lemma 5.3. *Let $A = \left(\mu^+_A,\ \mu^-_A\right)$ and $B = \left(\mu^+_B,\ \mu^-_B\right)$ be any two bipolar fuzzy subsets in X. If $A \times B$ is a BFI of $X \times X$, then*
(i). $\mu^+_A(1) \geq \mu^+_B(x_2)$ and $\mu^+_B(1) \geq \mu^+_A(x_1)$
(ii). $\mu^-_A(1) \leq \mu^-_B(x_2)$ and $\mu^-_B(1) \leq \mu^-_A(x_1)$, $\forall x_1, x_2 \in X$.

Proof. Suppose $A \times B$ be a BFI of $X \times X$ and $x_1, x_2 \in X$.
(i). Assume that $\mu^+_A(1) < \mu^+_B(x_2)$ and $\mu^+_B(1) < \mu^+_A(x_1)$. Then,
$\mu^+_{A\times B}\,(x_1, x_2) = min\left\{\mu^+_A(x_1), \mu^+_B\,(x_2)\right\} > min\left\{\mu^+_A(1), \mu^+_B(1)\right\} = \mu^+_{A\times B}(1,1)$
$\Longrightarrow \mu^+_{A\times B}\,(x_1, x_2) > \mu^+_{A\times B}(1,1)$, which contradicts Lemma 5.1.
(ii). Assume that $\mu^-_A(1) > \mu^-_B(x_2)$ and $\mu^-_B(1) > \mu^-_A(x_1)$. Then
$\mu^-_{A\times B}\,(x_1, x_2) = max\left\{\mu^-_A(x_1), \mu^-_B\,(x_2)\right\} < min\left\{\mu^+_A(1), \mu^+_B(1)\right\} = \mu^+_{A\times B}(1,1)$

$\Longrightarrow \mu^+_{A\times B}(x_1, x_2) < \mu^+_{A\times B}(1, 1)$, which contradicts Lemma 5.1.
Hence, conditions (i) and (ii) hold true for all $x_1, x_2 \in X$. □

Theorem 5.4. *Let A and B be any two BFSs in X. If $A \times B$ is a BFI in $X \times X$, then either A or B is a BFI of X.*

Proof. Let $A \times B$ be a BFI of $X \times X$. Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$. Then
$\mu^+_{A\times B}((x_1, x_2) \diamond (y_1, y_2)) \geq \mu^+_{A\times B}(y_1, y_2)$ and $\mu^-_{A\times B}((x_1, x_2) \diamond (y_1, y_2)) \leq \mu^-_{A\times B}(y_1, y_2)$
$\Longrightarrow \mu^+_{A\times B}(x_1 \diamond y_1, x_2 \diamond y_2) \geq \mu^+_{A\times B}(y_1, y_2)$ and $\mu^-_{A\times B}(x_1 \diamond y_1, x_2 \diamond y_2) \leq \mu^-_{A\times B}(y_1, y_2)$
$\Longrightarrow$ min $\{\mu^+_A(x_1 \diamond y_1), \mu^+_B(x_2 \diamond y_2)\} \geq \min\{\mu^+_A(y_1), \mu^+_B(y_2)\}$ and

$$\max\{\mu^-_A(x_1 \diamond y_1), \mu^-_B(x_2 \diamond y_2)\} \leq \max\{\mu^-_A(y_1), \mu^-_B(y_2)\} \quad (i)$$

If we put $x_2 = 1 = y_2$ (or resp. $x_1 = 1 = y_1$), in (i) we have
min $\{\mu^+_A(x_1 \diamond y_1), \mu^+_B(1 \diamond 1)\} \geq \min\{\mu^+_A(y_1), \mu^+_B(1)\}$ and

$$\max\{\mu^-_A(x_1 \diamond y_1), \mu^-_B(1 \diamond 1)\} \leq \max\{\mu^-_A(y_1), \mu^-_B(1)\}.$$

$\Longrightarrow \mu^+_A(x_1 \diamond y_1) \geq \mu^+_A(y_1)$ and $\mu^-_A(x_1 \diamond y_1) \leq \mu^-_A(y_1)$. (By Lemma 5.3) (ii)

Or

min $\{\mu^+_A(1 \diamond 1), \mu^+_B(x_2 \diamond y_2)\} \geq \min\{\mu^+_A(1), \mu^+_B(y_2)\}$ and

$$\max\{\mu^-_A(1 \diamond 1), \mu^-_B(x_2 \diamond y_2)\} \leq \max\{\mu^-_A(1), \mu^-_B(y_2)\}.$$

$\Longrightarrow \mu^+_B(x_2 \diamond y_2) \geq \mu^+_B(y_2)$ and $\mu^-_B(x_2 \diamond y_2) \leq \mu^-_B(y_2)$. (By Lemma 5.3) (iii)

So, from (ii) and (iii) we have $\mu^+_A(x_1 \diamond y_1) \geq \mu^+_A(y_1)$ and $\mu^-_A(x_1 \diamond y_1) \leq \mu^-_A(y_1)$ or $\mu^+_B(x_2 \diamond y_2) \geq \mu^+_B(y_2)$ and $\mu^-_B(x_2 \diamond y_2) \leq \mu^-_B(y_2)$.
Also, as $A \times B$ be a BFI of $X \times X$, we have
$\mu^+_{A\times B}(((x_1, x_2) \diamond ((y_1, y_2) \diamond (z_1, z_2))) \diamond (z_1, z_2)) \geq min\{\mu^+_{A\times B}(x_1, x_2), \mu^+_{A\times B}(x_1, x_2)\}$ and
$\mu^-_{A\times B}(((x_1, x_2) \diamond ((y_1, y_2) \diamond (z_1, z_2))) \diamond (z_1, z_2)) \leq max\{\mu^-_{A\times B}(x_1, x_2), \mu^-_{A\times B}(y_1, y_2)\}$
$\Longrightarrow \mu^+_{A\times B}((x_1 \diamond (y_1 \diamond z_1)) \diamond z_1, (x_2 \diamond (y_2 \diamond z_2)) \diamond z_2) \geq min\{\mu^+_{A\times B}(x_1, x_2), \mu^+_{A\times B}(x_1, x_2)\}$
and $\mu^-_{A\times B}((x_1 \diamond (y_1 \diamond z_1)) \diamond z_1, (x_2 \diamond (y_2 \diamond z_2)) \diamond z_2) \leq max\{\mu^-_{A\times B}(x_1, x_2), \mu^-_{A\times B}(x_1, x_2)\}$
$\Longrightarrow min\{\mu^+_A((x_1 \diamond (y_1 \diamond z_1)) \diamond z_1), \mu^+_B((x_2 \diamond (y_2 \diamond z_2)) \diamond z_2)\}$

$$\geq \min\{\min\{\mu^+_A(x_1), \mu^+_B(x_2)\}, min\{\mu^+_A(y_1), \mu^+_B(y_2)\}\}$$ and

max $\{\mu^-_A(x)_1 \diamond (y_1 \diamond z_1)) \diamond z_1), \mu^-_B((x_2 \diamond (y_2 \diamond z_2)) \diamond z_2)\}$

$$\leq \max\{\max\{\mu^-_A(x_1), \mu^-_B(x_2)\}, max\{\mu^-_A(y_1), \mu^-_B(y_2)\}\} \quad (iv)$$

If we put $x_2 = y_2 = z_2 = 1$ (or resp. $x_1 = y_1 = z_1 = 1$) in (iv), by Lemma 5.3 we get

$$\mu^+_A((x_1 \diamond (y_1 \diamond z_1)) \diamond z_1) \geq \min\{\mu^+_A(x_1), \mu^+_A(y_1)\} \; and$$

$$\mu^-_A((x_1 \diamond (y_1 \diamond z_1)) \diamond z_1) \leq \max\{\mu^-_A(x_1), \mu^-_A(y_1)\} \quad (v)$$

or

$$\mu^+_B((x_2 \diamond (y_2 \diamond z_2)) \diamond z_2) \geq \min\{\mu^+_B(x_2), \mu^+_B(y_2)\} \; and$$

$$\mu^-_B((x_2 \diamond (y_2 \diamond z_2)) \diamond z_2) \leq \max\{\mu^-_B(x_2), \mu^-_B(y_2)\} \quad (vi)$$

Hence From $(ii), (iii), (v)$ and (vi) we conclude that either A or B is a BFI of X. □

Theorem 5.5. *Let $A = (\mu^+_A, \mu^-_A)$ be a BFS of X and, let R_A be the strongest BFR on X. A is a BFI in X if and only if R_A is a BFI in $X \times X$.*

Proof. Assume that A is a BFI of X. Let $(x_1, x_2), (y_1, y_2) \in X \times X$. Then,

$$\begin{aligned}
\mu^+_{R_A}((x_1, x_2) \diamond (y_1, y_2)) &= \mu^+_{R_A}(x_1 \diamond y_1, x_2 \diamond y_2) \\
&= \min\{\mu^+_A(x_1 \diamond y_1), \mu^+_A(x_2 \diamond y_2)\} \\
&\geq \min\{\mu^+_A(y_1), \mu^+_A(y_2)\} = \mu^+_{R_A}(y_1, y_2) \; and \\
\mu^-_{R_A}((x_1, x_2) \diamond (y_1, y_2)) &= \mu^-_{R_A}(x_1 \diamond y_1, x_2 \diamond y_2) \\
&= \max\{\mu^-_A(x_1 \diamond y_1), \mu^-_A(x_2 \diamond y_2)\} \\
&\leq \min\{\mu^-_A(y_1), \mu^-_A(y_2)\} = \mu^-_{R_A}(y_1, y_2)
\end{aligned}$$

$\Longrightarrow \mu^+_{R_A}((x_1, x_2) \diamond (y_1, y_2)) \geq \mu^+_{R_A}(y_1, y_2)$ and $\mu^-_{R_A}((x_1, x_2) \diamond (y_1, y_2)) \leq \mu^-_{R_A}(y_1, y_2)$.
Also, let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$. Then

$$\mu^+_{R_A}(((x_1,x_2)\diamond((y_1,\ y_2)\diamond(z_1,z_2)))\diamond(z_1,z_2)) = \mu^+_{R_A}((x_1\diamond(y_1\diamond z_1))\diamond z_1,(x_2\diamond(y_2\diamond z_2))\diamond z_2)$$
$$= min\{\mu^+_A((x_1\diamond(y_1\diamond z_1))\diamond z_1\ ),\mu^+_A((x_2\diamond(y_2\diamond z_2))\diamond z_2)\}$$
$$\geq\ min\{min\{\mu^+_A(x_1),\mu^+_A(y_1)\},min\{\mu^+_A(x_2),\mu^+_A(y_2\ )\}\}$$
$$= min\{min\{\mu^+_A(x_1),\mu^+_A(x_2)\},min\{\mu^+_A(y_1),\mu^+_A(y_2\ )\}\}$$
$$=\ min\ \{\mu^+_{R_A}(x_1,x_2),\ \mu^+_{R_A}(y_1,\ y_2)\}$$
$$\Longrightarrow \mu^+_{R_A}(((x_1,x_2)\diamond((y_1,\ y_2)\diamond(z_1,z_2)))\diamond(z_1,z_2)) \geq min\ \{\mu^+_{R_A}(x_1,x_2),\ \mu^+_{R_A}(y_1,\ y_2)\}$$
and
$$\mu^-_{R_A}(((x_1,x_2)\diamond((y_1,\ y_2)\diamond(z_1,z_2)))\diamond(z_1,z_2)) = \mu^-_{R_A}((x_1\diamond(y_1\diamond z_1))\diamond z_1,(x_2\diamond(y_2\diamond z_2))\diamond z_2)$$
$$= max\{\mu^-_A((x_1\diamond(y_1\diamond z_1))\diamond z_1\ ),\mu^-_A((x_2\diamond(y_2\diamond z_2))\diamond z_2)\}$$
$$\leq\ max\{max\{\mu^-_A(x_1),\mu^-_A(y_1)\},max\{\mu^-_A(x_2),\mu^-_A(y_2\ )\}\}$$
$$= max\{max\{\mu^-_A(x_1),\mu^-_A(x_2)\},max\{\mu^-_A(y_1),\mu^-_A(y_2\ )\}\}$$
$$=\ max\ \{\mu^-_{R_A}(x_1,x_2),\ \mu^-_{R_A}(y_1,\ y_2)\}$$
$$\Longrightarrow \mu^-_{R_A}(((x_1,x_2)\diamond((y_1,\ y_2)\diamond(z_1,z_2)))\diamond(z_1,z_2)) \leq max\{\mu^-_{R_A}(x_1,x_2),\ \mu^-_{R_A}(y_1,\ y_2)\}$$

Therefore R_A is a bipolar fuzzy ideal of $X\times X$.

Conversely, Let R_A is a BFI of $X\times X$. Then for all $(x_1,x_2),(y_1,y_2)\in X\times X$,

$$\mu^+_{R_A}((x_1,x_2)\diamond(y_1,\ y_2)) \geq \mu^+_{R_A}(y_1,y_2) \Longrightarrow \mu^+_{R_A}(x_1\diamond y_1,x_2\diamond y_2)\geq \min\{\mu^+_A(y_1),\mu^+_A(y_2)\}$$
$$\Longrightarrow \min\left\{\mu^+_A(x_1\diamond y_1),\mu^+_A(x_2\diamond y_2)\right\}\geq min\{\mu^+_A(y_1),\mu^+_A(y_2)\}$$

and $\mu^-_{R_A}((x_1,x_2)\diamond(y_1,\ y_2)) \leq\ \mu^-_{R_A}(y_1,y_2) \Longrightarrow \mu^-_{R_A}(x_1\diamond y_1,x_2\diamond y_2)\leq \max\{\mu^-_A(y_1),\mu^-_A(y_2)\}$

$$\Longrightarrow \max\left\{\mu^-_A(x_1\diamond y_1),\mu^-_A(x_2\diamond y_2)\right\}\leq max\{\mu^-_A(y_1),\mu^-_A(y_2)\}$$

If we put $x_2=y_2=1\ in\ (i)$ and (ii), we get

$$\mu^+_A(x_1\diamond y_1)\geq \mu^+_A(y_1) \text{ and } \mu^-_A(x_1\diamond y_1)\leq \mu^-_A(y_1).$$

Also, for all $(x_1,x_2),(y_1,\ y_2)),(z_1,z_2)\in X\times X$, we have

$$\mu^+_{R_A}(((x_1,x_2)\diamond((y_1,\ y_2)\diamond(z_1,z_2)))\diamond(z_1,z_2)) \geq min\ \{\mu^+_{R_A}(x_1,x_2),\ \mu^+_{R_A}(y_1,\ y_2)\}$$
$$\Longrightarrow \mu^+_{R_A}((x_1\diamond(y_1\diamond z_1))\diamond z_1,(x_2\diamond(y_2\diamond z_2))\diamond z_2)$$
$$\geq min\{min\{\mu^+_A(x_1),\mu^+_A(x_2)\},min\{\mu^+_A(y_1),\mu^+_A(y_2\ )\}\}$$
$$\Longrightarrow \min\{\mu^+_A((x_1\diamond(y_1\diamond z_1))\diamond z_1\ ),\mu^+_A((x_2\diamond(y_2\diamond z_2))\diamond z_2)\}$$
$$\geq min\{min\{\mu^+_A(x_1),\mu^+_A(y_1)\},min\{\mu^+_A(x_2),\mu^+_A(y_2\ )\}\}$$

If we put $x_2=y_2=1$,we get $\mu^+_A((x_1\diamond(y_1\diamond z_1))\diamond z_1)\geq min\{\mu^+_A(x_1),\mu^+_A(y_1)\}$

Similarly, $\mu^-_{R_A}(((x_1,x_2)\diamond((y_1,y_2)\diamond(z_1,z_2)))\diamond(z_1,z_2)) \leq max\{\mu^-_{R_A}(x_1,x_2),\ \mu^-_{R_A}(y_1,\ y_2)\}$

$$\Longrightarrow \mu^-_{R_A}((x_1\diamond(y_1\diamond z_1))\diamond z_1,(x_2\diamond(y_2\diamond z_2))\diamond z_2)$$
$$\leq max\{max\{\mu^-_A(x_1),\mu^-_A(x_2)\},max\{\mu^-_A(y_1),\mu^-_A(y_2\ )\}\}$$
$$\Longrightarrow \max\{\mu^-_A((x_1\diamond(y_1\diamond z_1))\diamond z_1\ ),\mu^-_A((x_2\diamond(y_2\diamond z_2))\diamond z_2)\}$$
$$\leq max\{max\{\mu^-_A(x_1),\mu^-_A(y_1)\},max\{\mu^-_A(x_2),\mu^-_A(y_2\ )\}\}$$

If we put $x_2=y_2=1$,we get $\mu^-_A((x_1\diamond(y_1\diamond z_1))\diamond z_1)\leq max\{\mu^-_A(x_1),\mu^-_A(y_1)\}$

Hence A is a bipolar fuzzy ideal of a BE-algebra X. □

6 Conclusion

In this manuscript, we applied the theory of bipolar fuzzy sets (BFS) to the ideals of BE-algebras. We introduced the concept of bipolar fuzzy ideals (BFIs) in BE-algebras and explored several of their fundamental properties. We also provided characterizations of BFIs in BE-algebras through their level subsets. Furthermore, we demonstrated that both the homomorphic image and the inverse image of bipolar fuzzy ideals in a BE-algebra are again bipolar fuzzy ideals. Moreover, we studied the Cartesian products of BFIs in BE-algebras and investigated some important results. Finally, we discussed the relationship between the strongest bipolar fuzzy relations in BE-algebras and the bipolar fuzzy ideals in BE-algebras. The main results of this study are presented in Sections 3, 4 and 5. Since every fuzzy set can be viewed as a bipolar fuzzy set, it follows that our results on bipolar fuzzy ideals of BE-algebras are generalizations of fuzzy ideals of BE-algebras.

We believe that the results of this work will provide the groundwork for future research in the theory of bipolar fuzzy BE-algebras and offer more dimensions to the structure of bipolar fuzzy ideals in BE-algebras based on bipolar fuzzy sets. In our future work, we will extend this concept to bipolar fuzzy filters, bipolar pseudo-fuzzy ideals, bipolar pseudo-fuzzy filters, and bipolar anti-fuzzy ideals in BE-algebras and other algebraic structures to obtain additional novel results. Furthermore, we will study the applications of bipolar fuzzy ideals in BE-algebras, particularly in dealing with preference relations and decision-making.

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Author information

Beza Lamesgin Derseh, Department of Mathematics, College of Natural and Computational Science, Debre Markos University, Debre Markos, Ethiopia.
E-mail: beza_lamesgin@dmu.edu.et, dbezalem@gmail.co

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