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$*$-conformal η-Ricci-Yamabe solitons on LP-Sasakian manifolds

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Abstract. In the present article, we study $*$-conformal η-Ricci-Yamabe solitons on LP-Sasakian manifolds. Moreover, we explore LP-Sasakian manifolds admitting $*$-conformal η-Ricci-Yamabe solitons satisfying Ricci semi-symmetric, Einstein semi-symmetric and $S(\xi, X) \cdot R = 0$ conditions and discuss the cases for the $*$-conformal η-Ricci-Yamabe solitons to be shrinking, expanding or steady. Futher, we characterize M-projectively flat and quasi-M-projectively flat LP-Sasakian manifolds admitting $*$-conformal η-Ricci-Yamabe solitons. Also, we study $*$-conformal η-Ricci-Yamabe solitons on LP-Sasakian manifolds admitting torse forming vector fields. At last, we give an example of LP-Sasakian manifolds to verify our result.

1 Introduction

A very long time ago Hamilton introduced the notion of Ricci and Yamabe flows ([17], [18]). Recently in 2019, a scalar combination of Ricci and Yamabe flows was introduced by the authors Güler and Crasmareanu [15], this advanced class of geometric flows called Ricci-Yamabe (RY) flow of type (σ, ρ) and is defined by

$$\frac{\partial}{\partial t}g(t) + 2\sigma S(g(t)) + \rho r(t)g(t) = 0, \quad g(0) = g_0,$$

for some scalars σ and ρ.

A solution to the RY flow is called Ricci-Yamabe soliton (RYS) if it depends only on one parameter group of diffeomorphism and scaling. A Riemannian (or semi-Riemannian) manifold $\mathcal{M}$ is said to have a RYS if

$$\pounds_K g + 2\sigma S + (2\lambda - \rho r)g = 0,$$

where $\pounds_K$ represents the Lie derivative operator along the smooth vector field K on $\mathcal{M}$ and $\lambda \in \mathbb{R}$ (the set of real numbers) and $r(t)$ is the scalar curvature of the metric $g(t)$. Please, see [7, 19, 30].

The concept of conformal Ricci flow was introduced by Fischer [11], which is defined on an n-dimensional Riemannian manifold $\mathcal{M}$ by the equations

$$\frac{\partial g}{\partial t} = -2(S + \frac{g}{n}) - pg, \quad r(g) = -1,$$

where p defines a time dependent non-dynamical scalar field (also called the conformal pressure), g is the Riemannian metric; r and S represent the scalar curvature and the Ricci tensor of $\mathcal{M}$, respectively. The conformal Ricci flow equations are analogous to the Navier-Stokes equations of fluid mechanics, and because of this analogy, the time-dependent scalar field p is called a conformal pressure, and as for the real physical pressure in fluid mechanics that serves to maintain the incompressibility of the fluid, the conformal pressure serves as a Lagrange multiplier to conformally deform the metric flow so as to maintain the scalar curvature constraint. The term -pg plays a role of constraint force to maintain r in the above equation.

In [2], Basu and Bhattacharyya proposed the concept of conformal Ricci soliton on $\mathcal{M}$ and is defined by

$$\pounds_K g + 2S + (2\lambda - (p + \frac{2}{n}))g = 0,$$

where $\lambda \in \mathbb{R}$.

A Riemannian (or semi-Riemannian) manifold $\mathcal{M}$ is said to have a conformal Ricci-Yamabe soliton (CRYS) if [37]

$$\pounds_K g + 2\sigma S + (2\lambda - \rho r - (p + \frac{2}{n}))g = 0, \tag{1.1}$$

where $\sigma, \rho, \lambda \in \mathbb{R}$.

In 2018, Deshmukh and Chen ([4], [6]) briefly studied Yamabe solitons to find sufficient conditions on the soliton vector field so that the metric of the Yamabe soliton is of constant scalar curvature. Yamabe solitons have also been studied by ([5], [8], [31], [36]) and many others.

The notion of $*$-Ricci solitons has been introduced by Kaimakamis and Panagiotidou [22] within the framework of real hypersurfaces of complex space forms. Here, it is to be mentioned that the notion of $*$-Ricci tensor was first introduced by Tachibana [32] on almost Hermitian manifolds and further studied by Hamada [16] on real hypersurfaces of non-flat complex space forms. A Riemannian metric g on a smooth manifold $\mathcal{M}$ is called a $*$-Ricci soliton if there exists a smooth vector field V and a real number λ, such that

$$(\pounds_K g)(X, Y) + 2S^*(X, Y) + 2\lambda g(X, Y) = 0,$$

where

$$S^*(X, Y) = g(Q^* X, Y) = Trace\{\phi \circ R(X, \phi Y)\},$$

for all vector fields X, Y on $\mathcal{M}$. Here, ϕ is a tensor field of type $(1, 1)$ and Q^* is the $*$-Ricci operator. In this connection, we recommend the papers ([12], [13], [14], [20], [21], [27], [28], [33]) and the references therein for more detailed studies on Ricci solitons, η-Ricci solitons and $*$-Ricci solitons.

In 2020, Dey and Roy [9] introduced and studied the notion of $*$-η-Ricci soliton on an n-dimensional Sasakian manifold. A Riemannian manifold $(\mathcal{M}, g)$ is called $*$-η-Ricci soliton (g, ξ, λ, μ) if

$$(\pounds_\xi g)(X, Y) + 2S^*(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{1.2}$$

A Riemannian manifold is said to have $*$-conformal η-Ricci-Yamabe soliton if [38]

$$(\pounds_K g)(X, Y) + 2\sigma S^*(X, Y) + (2\lambda - \rho r^* - (p + \frac{2}{n}))g(X, Y) + 2\mu\eta(X)\eta(Y) = 0, \tag{1.3}$$

where S^* is the $*$-Ricci tensor, λ, μ, σ and ρ are constants. A $*$-conformal η-Ricci-Yamabe soliton is said to be shrinking, steady or expanding if it admits a soltion vector for which λ is negative, zero or positive, respectively. In particular, if $\mu = 0$, then the notion of $*$-conformal η-Ricci-Yamabe soliton $(g, V, \lambda, \mu, \sigma, \rho)$ reduces to the notion of $*$-conformal Ricci-Yamabe soliton $(g, V, \lambda, \sigma, \rho)$.

We present our work as follows: In section 2, we give the basic results and definitions of LP-Sasakian manifolds. In Section 3, we study $*$-conformal η-Ricci-Yamabe solitons on LP-Sasakian manifolds. In section 4, we consider the Ricci semi-symmetic and Einstein semi-symmetric LP-Sasakian manifolds admitting $*$-conformal η-Ricci-Yamabe solitons. Section 5 deals with the study of LP-Sasakian manifolds admitting $*$-conformal η-Ricci-Yamabe solitons satisfying $S(\xi, X) \cdot R = 0$. Section 6 concerned with the study of M- projectively flat and quasi-M-projectively flat LP-Sasakian manifolds admitting $*$-conformal η-Ricci-Yamabe soliton. In section 7, we proved that an LP-Sasakian manifold admitting $*$-conformal η-Ricci-Yamabe solitons with the torse forming vector is a generalized η-Einstein manifold. At last, we give a 3-dimensional example of LP-Sasakian manifolds to verify our result.

2 Preliminaries

Let $\mathcal{M}$ be an n-dimensional smooth manifold equipped with the structure (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1, 1)$, ξ is the unit timelike vector field, η is a 1-form and g is a Lorentzian metric on $\mathcal{M}$ such that [3, 10, 25]

$$\phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \tag{2.1}$$

which implies

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad rank(\phi) = n - 1 \tag{2.2}$$

for all $X \in \chi(\mathcal{M})$; where $\chi(\mathcal{M})$ denotes the set of all smooth vector fields on $\mathcal{M}$. The manifold $\mathcal{M}$ is said to have an almost para-contact metric structure (ϕ, ξ, η, g) when it admits a Lorentzian metric g, such that

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \;\; \eta(X) = g(X, \xi), \tag{2.3}$$

for all $X, Y \in \chi(\mathcal{M})$. If moreover,

$$(\nabla_X \phi)Y = \eta(X)\phi^2 X + g(\phi X, \phi Y)\xi, \tag{2.4}$$

$$\nabla_X \xi = \phi X \iff (\nabla_X \eta)Y = g(\phi X, Y) = g(X, \phi Y), \tag{2.5}$$

then $(\mathcal{M}, \phi, \xi, \eta, g)$ is called a Lorentzian para-Sasakian manifold (briefly, LP-Sasakian manifold), where ∇ denotes the Levi-Civita connection of the manifold [23].
For an n-dimensional LP-Sasakian manifold, the following relations hold [1, 29]:

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \quad R(\xi, X)\xi = X + \eta(X)\xi, \tag{2.6}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.7}$$

$$S(X, \xi) = (n - 1)\eta(X) \iff Q\xi = (n - 1)\xi, \tag{2.8}$$

for any $X, Y \in \chi(\mathcal{M})$.

Definition 2.1. The M-projective curvature tensor $\mathcal{H}$ in an n-dimensional LP-Sasakian manifold is defined by [26]

$$\begin{aligned} \mathcal{H}(X, Y)Z &= R(X, Y)Z - \frac{1}{2(n-1)}[S(Y, Z)X - S(X, Z)Y \\ &\quad + g(Y, Z)QX - g(X, Z)QY], \end{aligned} \tag{2.9}$$

where R and S are the curvature tensor and the Ricci tensor of $\mathcal{M}$, respectively; and Q is the Ricci operator such that $S(Y, Z) = g(QY, Z)$.

Definition 2.2. An LP-Sasakian manifold $\mathcal{M}$ is said to be a generalized η-Einstein if its non-vanishing Ricci tensor S is of the form [35]

$$S(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y) + \gamma g(\phi X, Y), \tag{2.10}$$

where α, β and γ are smooth functions on $\mathcal{M}$. If $\gamma = 0$ ($resp., \beta = \gamma = 0$), then $\mathcal{M}$ is called an η-Einstein (resp,. Einstein) manifold.

Lemma 2.3. *The $*$-Ricci tensor of an n-dimensional LP-Sasakian manifold is given by [20]*

$$S^*(X, Y) = S(X, Y) + (n - 2)g(X, Y) - g(X, \phi Y)\theta + (2n - 3)\eta(X)\eta(Y), \tag{2.11}$$

where θ is the trace of ϕ and $X, Y \in \chi(M)$.

From (2.11) it follows that

$$r^* = r + n^2 - 4n + 3 - \theta^2, \tag{2.12}$$

where θ is the trace of ϕ.

3 $*$-Conformal η-Ricci-Yamabe solitons on LP-Sasakian manifolds

Let the metric g of an n-dimensional LP-Sasakian manifold be a $*$-conformal η-Ricci-Yamabe soliton $(g, \xi, \lambda, \mu, \sigma, \rho)$, then (1.3) turns to

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2\sigma S^*(X, Y) + (2\lambda - \rho r^* - (p + \frac{2}{n}))g(X, Y) + 2\mu\eta(X)\eta(Y) = 0$$

for any vector fields X and Y on $\mathcal{M}$.

By making use of (2.5) and (2.11), the above equation leads to the form

$$S(X, Y) = \alpha g(X, Y) + \beta g(X, \phi Y) + \gamma\eta(X)\eta(Y), \tag{3.1}$$

where $\alpha = -[n - 2 + \frac{1}{\sigma}(\lambda - \frac{\rho r^*}{2} - \frac{1}{2}(p + \frac{2}{n}))]$, $\beta = (\theta - \frac{1}{\sigma})$ and $\gamma = -(2n - 3 + \frac{\mu}{\sigma})$, $\sigma \neq 0$. By putting $Y = \xi$ in (3.1), then using (2.1) and (2.2), we get

$$S(X, \xi) = (\alpha - \gamma)\eta(X), \tag{3.2}$$

where $\alpha - \gamma = n - 1 + \frac{1}{\sigma}(\mu - \lambda + \frac{\rho r^*}{2} + \frac{1}{2}(p + \frac{2}{n}))$ and $\sigma \neq 0$.

In view of (2.6), from (3.2) it follows that

$$\lambda - \mu = \frac{\rho r^*}{2} + \frac{1}{2}(p + \frac{2}{n}). \tag{3.3}$$

Thus, from (3.1) and (3.3), we have the following Theorem:

Theorem 3.1. *If $(\mathcal{M}, g)$ is an n-dimensional LP-Sasakian manifold admitting a $*$-conformal η-Ricci-Yamabe soliton $(g, \xi, \lambda, \mu, \sigma, \rho)$, then $\mathcal{M}$ is a generalized η-Einstein manifold. Moreover, the scalars λ and μ are related by $\lambda - \mu = \frac{\rho r^*}{2} + \frac{1}{2}(p + \frac{2}{n})$.*

On contracting (3.1), we find

$$r = -(n - 1)(n - 3 + \frac{\mu}{\sigma}) + \theta^2 - \frac{\theta}{\sigma}, \tag{3.4}$$

where μ and $\sigma(\neq 0)$ are constants. In particular, if we take $\theta = \sigma = 1$ and $\mu = 0$ in (3.1) and (3.3), then we have $S(X, Y) = -(n-2)g(X, Y) - (2n-3)\eta(X)\eta(Y)$ and $\lambda = -\frac{\rho}{2} + \frac{1}{2}(p + \frac{2}{n})$, respectively; being $r = -(n - 1)(n - 3)$ and $r^* = \theta^2 = 1$. Thus, we have

Corollary 3.2. *Let $\mathcal{M}$ be an n-dimensional LP-Sasakian manifold. If the manifold admits a $*$-conformal Ricci-Yamabe soliton $(g, \xi, \lambda, 0, 1, \rho)$ with $\theta = 1$, then $\mathcal{M}$ is an η-Einstein manifold. Moreover, the soliton is shrinking, steady or expanding according to $\rho > (p + \frac{2}{n})$, $\rho = 0$ or $\rho < (p + \frac{2}{n})$.*

4 Ricci semi-symmetric and Einstein semi-symmetric LP-Sasakian manifolds admitting $*$-conformal η-Ricci-Yamabe solitons

In 1992, Mirzoyan [24] introduced the notion of Ricci semi-symmetric for the Riemann spaces. In this section, first, we consider a $*$-conformal η-Ricci-Yamabe soliton $(g, \xi, \lambda, \mu, \sigma, \rho)$, in n-dimensional LP-Sasakian manifold, which is Ricci semi-symmetric, i.e., $R(X, Y) \cdot S = 0$. This leads to

$$S(R(X, Y)Z, W) + S(Z, R(X, Y)W) = 0 \tag{4.1}$$

for any X, Y, Z and W on $\mathcal{M}$.
Putting $Y = \xi$ in (4.1) and then using (2.6), we have

$$\eta(Z)S(X, W) - g(X, Z)S(\xi, W) + \eta(W)S(X, Z) - g(X, W)S(Z, \xi) = 0. \tag{4.2}$$

Now, putting $Z = \xi$ in (4.2) and using (2.2), (3.2) and (3.3), we obtain

$$S(X, W) = [n - 1 + \frac{1}{\sigma}(\mu - \lambda + \frac{\rho r^*}{2} + \frac{1}{2}(p + \frac{2}{n}))]g(X, W). \tag{4.3}$$

By means of the fact that in an LP-Sasakian manifold admitting a $*$-conformal η-Ricci-Yamabe soliton, equation (3.3) holds. Thus (4.3) turns to

$$S(X,W) = (n-1)g(X,W). \tag{4.4}$$

By contracting (4.4) over X and W, we obtain $r = n(n-1)$. Thus, we have

Theorem 4.1. *Let $\mathcal{M}$ be a Ricci semi-symmetric LP-Sasakian manifold and admit a $*$-conformal η-Ricci-Yamabe soliton. Then $\mathcal{M}$ is an Einstein manifold with the scalar curvature $r = n(n-1)$.*

In particular, for $\mu = 0$, (3.3) reduces to $\lambda = \frac{1}{2}[\rho((n-1)(2n-3)-\theta^2) + (p+\frac{2}{n})]$, where $r^* = (n-1)(2n-3) - \theta^2$. Thus, we have

Corollary 4.2. *Let $\mathcal{M}$ be a Ricci semi-symmetric LP-Sasakian manifold and admit a $*$-conformal Ricci-Yamabe soliton. Then, the manifold is shrinking, steady or expanding according to $p < \rho(\theta^2-(n-1)(2n-3))-\frac{2}{n}$, $p = \rho(\theta^2-(n-1)(2n-3))-\frac{2}{n}$, or $p > \rho(\theta^2-(n-1)(2n-3))-\frac{2}{n}$.*

Definition 4.3. An n-dimensional LP-Sasakian manifold is called Einstein semi-symmetric if $R \cdot E = 0$, where E is the Einstein tensor given by

$$E(X,Y) = S(X,Y) - \frac{r}{n}g(X,Y), \tag{4.5}$$

where r is the scalar curvature of the manifold.

Now, we consider a $*$-conformal η-Ricci-Yamabe soliton $(g,\xi,\lambda,\mu,\sigma,\rho)$, in an n-dimensional LP-Sasakian manifold, which is Einstein semi-symmetric, i.e., $R \cdot E = 0$. Thus we have

$$E(R(X,Y)Z,W) + E(Z,R(X,Y)W) = 0,$$

which in view of (4.5) becomes

$$\begin{aligned} S(R(X,Y)Z,W) + S(Z,R(X,Y)W) &= \frac{r}{n}\{g(R(X,Y)Z,W) \\ &\quad + g(Z,R(X,Y)W)\}. \end{aligned} \tag{4.6}$$

By putting $X = Z = \xi$ in (4.6), we have

$$S(R(\xi,Y)\xi,W) + S(\xi,R(\xi,Y)W) = \frac{r}{n}\{g(R(\xi,Y)\xi,W) + g(\xi,R(\xi,Y)W)\}.$$

By making use of (2.6), and (3.2), the foregoing equation leads to

$$S(Y,W) = [n-1+\frac{1}{\sigma}(\mu-\lambda+\frac{\rho r^*}{2}+\frac{1}{2}(P+\frac{2}{n}))]g(Y,W). \tag{4.7}$$

By using (3.3) then (4.7) turns to

$$S(Y,W) = (n-1)g(Y,W). \tag{4.8}$$

By contracting (4.8) over Y and W, we obtain $r = n(n-1)$. Thus, we have

Theorem 4.4. *Let $\mathcal{M}$ be an Einstein semi-symmetric LP-Sasakian manifold of dimension n and admit a $*$-conformal η-Ricci-Yamabe soliton. Then $\mathcal{M}$ is an Einstein manifold with the scalar curvature $r = n(n-1)$.*

In particular, for $\mu = 0$, (3.3) reduces to $\lambda = \frac{1}{2}[\rho((n-1)(2n-3)-\theta^2) + (p+\frac{2}{n})]$, where $r^* = (n-1)(2n-3) - \theta^2$. Thus, we have

Corollary 4.5. *Let $\mathcal{M}$ be an Einstein semi-symmetric LP-Sasakian manifold of dimension n and admit a $*$-conformal Ricci-Yamabe soliton. Then, the manifold is shrinking, steady or expanding according to $p < \rho(\theta^2-(n-1)(2n-3))-\frac{2}{n}$, $p = \rho(\theta^2-(n-1)(2n-3))-\frac{2}{n}$, or $p > \rho(\theta^2-(n-1)(2n-3))-\frac{2}{n}$.*

5 $*$-Conformal η-Ricci-Yamabe solitons in LP-Sasakian manifolds satisfying $(S(X,Y)\cdot R)(U,V)W = 0$

Let the manifold $\mathcal{M}$ admit a $*$-conformal η-Ricci-Yamabe soliton and satisfies $(S(X,Y)\cdot R)(U,V)W = 0$. Then we have

$$\begin{aligned} S(Y,R(U,V)W)X - S(X,R(U,V)W)Y + S(Y,U)R(X,V)W \\ -S(X,U)R(Y,V)W + S(Y,V)R(U,X)W - S(X,V)R(U,Y)W \\ +S(Y,W)R(U,V)X - S(X,W)R(U,V)Y = 0, \end{aligned}$$

which by taking the inner product with ξ takes the form

$$\begin{aligned} S(Y,R(U,V)W)\eta(X) - S(X,R(U,V)W)\eta(Y) + S(Y,U)\eta(R(X,V)W) \\ -S(X,U)\eta(R(Y,V)W) + S(Y,V)\eta(R(U,X)W) - S(X,V)\eta(R(U,Y)W) \\ +S(Y,W)\eta(R(U,V)X) - S(X,W)\eta(R(U,V)Y) = 0. \end{aligned} \tag{5.1}$$

Putting $U = W = \xi$ in (5.1), then using (2.6) and (3.2) we find

$$S(Y,V)\eta(X) = S(X,V)\eta(Y) + (\alpha-\gamma)g(V,X)\eta(Y) - (\alpha-\gamma)g(V,Y)\eta(X). \tag{5.2}$$

Now putting $X = \xi$ in (5.2), and using (2.1) and (3.1), we obtain

$$S(Y,V) = -(\alpha-\gamma)g(Y,V) - 2(\alpha-\gamma)\eta(Y)\eta(V),$$

which by using (3.3) takes the form

$$S(Y,V) = -(n-1)g(Y,V) - 2(n-1)\eta(Y)\eta(V). \tag{5.3}$$

By contracting (5.3) over Y and V, we obtain $r = -(n-1)(n-2)$. This helps us to state the following theorem:

Theorem 5.1. *Let an n dimensional LP-Sasakian manifold $\mathcal{M}$ admitting a $*$-conformal η-Ricci-Yamabe soliton satisfies $S(X,\xi)\cdot R = 0$. Then $\mathcal{M}$ is an η-Einstein manifold with the scalar curvature $r = -(n-1)(n-2)$.*

In particular, for $\mu = 0$, (3.3) reduces to $\lambda = \frac{1}{2}[-\rho(n-1+\theta^2) + (p+\frac{2}{n})]$, where $r^* = -(n-1+\theta^2)$. Thus we have

Corollary 5.2. *Let an n dimensional LP-Sasakian manifold $\mathcal{M}$ admitting a $*$-conformal Ricci-Yamabe soliton satisfies $S(X,\xi)\cdot R = 0$. Then, the manifold is shrinking, steady or expanding according to $p < \rho(n-1+\theta^2) - \frac{2}{n}$, $p = \rho(n-1+\theta^2) - \frac{2}{n}$, or $p > \rho(n-1+\theta^2) - \frac{2}{n}$.*

6 M-projectively flat and quasi-M-projectively flat LP-Sasakian manifolds admitting $*$-conformal η-Ricci-Yamabe solitons

Let an n-dimensional LP-Sasakian manifold admit $*$-conformal η-Ricci-Yamabe solitons, and the manifold be an M-projectively flat, i.e., $\mathcal{H}(X,Y)Z = 0$. Thus, from (2.9), we have

$$R(X,Y)Z = \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY],$$

which by putting $Z = \xi$, and using (2.7) and (3.2) takes the form

$$[n-1+\frac{1}{\sigma}\{\lambda-\mu-\frac{\rho r^*}{2}-\frac{1}{2}(p+\frac{2}{n})\}](\eta(Y)X-\eta(X)Y) = \eta(Y)QX-\eta(X)QY. \tag{6.1}$$

By putting $Y = \xi$ in (6.1), we have

$$[n-1+\frac{1}{\sigma}\{\lambda-\mu-\frac{\rho r^*}{2}-\frac{1}{2}(p+\frac{2}{n})\}]X + \frac{2}{\sigma}[\lambda-\mu-\frac{\rho r^*}{2}-\frac{1}{2}(p+\frac{2}{n})]\eta(X)\xi = QX. \tag{6.2}$$

Now taking the inner product of (6.2) with Y, we find

$$\begin{aligned} S(X,Y) &= [n-1+\frac{1}{\sigma}\{\lambda-\mu-\frac{\rho r^*}{2}-\frac{1}{2}(p+\frac{2}{n})\}]g(X,Y) \\ &+\frac{2}{\sigma}[\lambda-\mu-\frac{\rho r^*}{2}-\frac{1}{2}(p+\frac{2}{n})]\eta(X)\eta(Y). \end{aligned} \tag{6.3}$$

By means of the fact that in LP-Sasakian manifold admitting a $*$-conformal η-Ricci-Yamabe equation (3.3) holds. Thus (6.3) turns to

$$S(X,Y) = (n-1)g(X,Y). \tag{6.4}$$

By contracting (6.4) over X and Y, we get $r = n(n-1)$. Thus, we have the following theorem:

Theorem 6.1. *Let $(\mathcal{M}, g)$ be an n-dimensional M-projectively flat LP-Sasakian manifold admitting a $*$-conformal η-Ricci-Yamabe soliton $(g, \xi, \lambda, \mu, \sigma, \rho)$, then $\mathcal{M}$ is an Einstein manifold with the scalar curvature $r = n(n-1)$.*

Now, let an n-dimensional LP-Sasakian manifold admit $*$-conformal η-Ricci-Yamabe solitons, and the manifold be a quasi-M-projectively flat, i. e., $g(\mathcal{H}(X,Y)Z, \phi W) = 0$. Then, from (2.9) it follows that

$$\begin{aligned} g(R(X,Y)Z,\phi W) &= \frac{1}{2(n-1)}[S(Y,Z)g(Y,\phi XW)-S(X,Z)g(Y,\phi W) \\ &+g(Y,Z)S(X,\phi W)-g(X,Z)S(Y,\phi W)], \end{aligned}$$

which by putting $Y = Z = \xi$, and using (2.1), (2.2), (2.7) and (3.2) reduces to

$$[n-1+\frac{1}{\sigma}\{\lambda-\mu-\frac{\rho r^*}{2}-\frac{1}{2}(p+\frac{2}{n})\}]g(Y,\phi X) = S(Y,\phi X). \tag{6.5}$$

By replacing X by ϕX in (6.5) and using (2.1), we obtain

$$\begin{aligned} S(Y,X) &= [n-1+\frac{1}{\sigma}\{\lambda-\mu-\frac{\rho r^*}{2}-\frac{1}{2}(p+\frac{2}{n}\}]g(Y,X) \\ &+\frac{2}{\sigma}[\lambda-\mu-\frac{\rho r^*}{2}-\frac{1}{2}(p+\frac{2}{n})]\eta(Y)\eta(X), \quad \sigma\neq 0. \end{aligned} \tag{6.6}$$

Using (3.3) in (6.6), we lead to

$$S(X,Y) = (n-1)g(X,Y). \tag{6.7}$$

By contracting (6.7) over X and Y, we get $r = n(n-1)$. Thus, we have the following theorem:

Theorem 6.2. *If $(\mathcal{M}, g)$ is an n-dimensional quasi-M-projectively flat LP-Sasakian manifold admitting a $*$-conformal η-Ricci-Yamabe soliton $(g, \xi, \lambda, \mu, \sigma, \rho)$, then $\mathcal{M}$ is an Einstein manifold. with the scalar curvature $r = n(n-1)$.*

7 $*$-Conformal η-Ricci-Yamabe solitons in LP-Sasakian manifolds with torse-forming vector field

Definition 7.1. A vector field N in an LP-Sasakian manifold $\mathcal{M}$ is said to be torse-forming vector field if [34]

$$\nabla_X N = fX + \omega(X)N, \tag{7.1}$$

where f is a smooth function and ω is a 1-form.

Let us consider an LP-Sasakian manifold admitting a $*$-conformal η-Ricci-Yamabe soliton, further considering the Reeb vector field ξ as a torse-forming vector field. Thus, from (7.1) we have

$$\nabla_X \xi = fX + \omega(X)\xi \tag{7.2}$$

for all $X \in \chi(M)$.
Taking the inner product of (7.2) with ξ and using (2.3), we have

$$g(\nabla_X \xi, \xi) = f\eta(X) - \omega(X). \tag{7.3}$$

Also from (2.5), we obtain

$$g(\nabla_X \xi, \xi) = 0. \tag{7.4}$$

Thus, from the equations (7.3) and (7.4), it follows that $\omega = f\eta$, and hence (7.2) turns to

$$\nabla_X \xi = f(X + \eta(X)\xi). \tag{7.5}$$

Now, by using (7.5), we have

$$(\pounds_\xi g)(X,Y) = 2f\{g(X,Y) + \eta(X)\eta(Y)\}. \tag{7.6}$$

Using (7.6), (1.3) turns to

$$S^*(X,Y) = -\frac{1}{\sigma}(\lambda - \frac{\rho r^*}{2} - \frac{1}{2}(p + \frac{2}{n}) + f)g(X,Y) - \frac{1}{\sigma}(f+\mu)\eta(X)\eta(Y),$$

which by using (2.11) becomes

$$\begin{aligned} S(X,Y) &= -\frac{1}{\sigma}[\lambda - \frac{\rho r^*}{2} - \frac{1}{2}(p + \frac{2}{n}) + f + \sigma(n-2)]g(X,Y) \\ &\quad + ag(X,\phi X) - \frac{1}{\sigma}[f + \mu + \sigma(2n-3)]\eta(X)\eta(Y), \end{aligned}$$

By recalling (3.3) in the foregoing equation, we arrive at

$$\begin{aligned} S(X,Y) &= -\frac{1}{\sigma}[\mu + f + \sigma(n-2)]g(X,Y) \\ &\quad + g(X,\phi Y)\theta - \frac{1}{\sigma}[f + \mu + \sigma(2n-3)]\eta(X)\eta(Y), \end{aligned} \tag{7.7}$$

which is a generalized η-Einstein manifold. Thus, we have the following theorem:

Theorem 7.2. *Let $\mathcal{M}$ be an n-dimensional LP-Sasakian manifold admitting a $*$-conformal η-Ricci-Yamabe soliton with a torse-forming vector field ξ. Then $\mathcal{M}$ is a generalized η-Einstein manifold.*

On contracting (7.7), we obtain $r = -\frac{(n-1)}{\sigma}(f + \mu + \sigma(n-3)) + \theta^2$. In particular, for $\mu = 0$, from (2.12) and (3.3) we obtain $\lambda = -\frac{\rho(n-1)f}{2\sigma} + \frac{1}{2}(p + \frac{2}{n})$. Thus, we have

Corollary 7.3. *An n-dimensional LP-Sasakian manifold admitting a $*$-conformal Ricci-Yamabe soliton with a torse-forming vector field ξ is the manifold is shrinking, steady or expanding according to $p < \frac{\rho n(n-1)f - 2\sigma}{n\sigma}$, $p = \frac{\rho n(n-1)f - 2\sigma}{n\sigma}$, or $p > \frac{\rho n(n-1)f - 2\sigma}{n\sigma}$.*

8 Example

We consider a 3-dimensional manifold $\mathcal{M} = \{(x,y,z) \in R^3\}$, where (x,y,z) are the standard coordinates in R^3. Let ϱ_1, ϱ_2 and ϱ_3 be the vector fields on $\mathcal{M}$ given by

$$\varrho_1 = coshz\frac{\partial}{\partial x} + sinhz\frac{\partial}{\partial y}, \quad \varrho_2 = sinhz\frac{\partial}{\partial x} + coshz\frac{\partial}{\partial y}, \quad \varrho_3 = \frac{\partial}{\partial z} = \xi,$$

which are linearly independent at each point of $\mathcal{M}$. Let g be the Lorentzian like (semi-Riemannian) metric defined by

$$g(\varrho_1,\varrho_1) = g(\varrho_2,\varrho_2) = 1,\ \ g(\varrho_3,\varrho_3) = -1,\ \ g(\varrho_1,\varrho_2) = g(\varrho_1,\varrho_3) = g(\varrho_2,\varrho_3) = 0.$$

Let η be the 1-form on $\mathcal{M}$ defined by $\eta(X) = g(X,\varrho_3)$ for all $X \in \chi(M)$. Let ϕ be the $(1,1)$ tensor field on $\mathcal{M}$ defined by

$$\phi\varrho_1 = -\varrho_2,\ \ \phi\varrho_2 = -\varrho_1,\ \ \phi\varrho_3 = 0.$$

The linearity property of ϕ and g yields

$$\eta(\varrho_3) = -1,\ \ \phi^2 X = X + \eta(X)\xi,\ \ g(\phi X,\phi Y) = g(X,Y) + \eta(X)\eta(Y)$$

for all $X,Y \in \chi(\mathcal{M})$.
Now, by direct computations, we obtain

$$[\varrho_1,\varrho_2] = 0, \quad [\varrho_2,\varrho_3] = -\varrho_1, \quad [\varrho_1,\varrho_3] = -\varrho_2.$$

Using Koszul's formula, we can easily calculate

$$\begin{aligned}
&\nabla_{\varrho_1}\varrho_1 = 0, && \nabla_{\varrho_2}\varrho_1 = -\varrho_3, && \nabla_{\varrho_3}\varrho_1 = 0,\\
&\nabla_{\varrho_1}\varrho_2 = -\varrho_3, && \nabla_{\varrho_2}\varrho_2 = 0, && \nabla_{\varrho_3}\varrho_2 = 0,\\
&\nabla_{\varrho_1}\varrho_3 = -\varrho_2, && \nabla_{\varrho_2}\varrho_3 = -\varrho_1, && \nabla_{\varrho_3}\varrho_3 = 0.
\end{aligned}$$

Also, one can easily verify that

$$\nabla_X\xi = \phi X, \quad (\nabla_X\phi)Y = \eta(Y)\phi^2 X + g(\phi X,\phi Y)\xi.$$

Thus, the manifold $\mathcal{M}$ is an LP-Sasakian manifold. It is known that

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

By using the above results, we can easily obtain the components of the curvature tensor as follows:

$$\begin{aligned}
&R(\varrho_1,\varrho_2)\varrho_1 = \varrho_2, && R(\varrho_1,\varrho_2)\varrho_2 = -\varrho_1, && R(\varrho_1,\varrho_2)\varrho_3 = 0,\\
&R(\varrho_2,\varrho_3)\varrho_1 = 0, && R(\varrho_2,\varrho_3)\varrho_2 = -\varrho_3, && R(\varrho_2,\varrho_3)\varrho_3 = -\varrho_2,\\
&R(\varrho_1,\varrho_3)\varrho_1 = -\varrho_3, && R(\varrho_1,\varrho_3)\varrho_2 = 0, && R(\varrho_1,\varrho_3)\varrho_3 = -\varrho_1.
\end{aligned}$$

By using the above components of the curvature tensor, we can easily calculate

$$S(\varrho_1,\varrho_1) = S(\varrho_2,\varrho_2) = 0, \quad S(\varrho_3,\varrho_3) = -2. \tag{8.1}$$

Putting $X = \xi$ in (3.2) and using (8.1), we lead to $\alpha - \gamma = 2$, which by using the values of α and γ for three dimensional LP-Sasakian manifold gives

$$\lambda - \mu = \frac{\rho r^*}{2} + \frac{1}{2}(p + \frac{2}{3}), \qquad \sigma \neq 0.$$

Hence Λ and μ satisfies the equation (3.3), and so the metric g defines a $*$-conformal-η-Ricci-Yamabe soliton on the given 3-dimensional LP-Sasakian manifold.

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