

*-conformal η -Ricci soliton and almost *-conformal η -Ricci soliton on para-Sasakian manifolds

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Abstract. This article explores an intricate relationship between *-conformal η -Ricci solitons and paracontact geometry, focusing on the properties and implications of *-conformal η -Ricci solitons within para-Sasakian manifolds. Here we demonstrate that if the metric of a para-Sasakian manifold is a *-conformal η -Ricci soliton, then it must be η -Einstein with the soliton vector field either adhering to the Killing condition or preserving the structure tensor. Additionally, we introduce the notion of gradient almost *-conformal η -Ricci solitons, providing a comprehensive analysis of their properties and implications.

1 Introduction

The concept of an almost paracontact structure on a differentiable manifold was first introduced by Sato [11] as an analogue to the almost contact structure [13], with a fundamental distinction: while the almost contact structure [4] is closely related to the almost complex structure [14], the almost paracontact structure aligns more naturally with the almost product structure [30]. This foundational idea extends to the existence of a compatible Riemannian metric on any differentiable manifold equipped with an almost paracontact structure.

In subsequent developments, Kaneyuki and Konzai [23] formalized the almost paracontact structure specifically for pseudo-Riemannian manifolds of dimension $2m + 1$ associating it with an almost paracomplex structure on $M^{2m+1} \times \mathbb{R}$. This linkage underscores the flexibility of the almost paracontact structure [9] in higher-dimensional geometry.

Recently, Zamkovoy [26] provided further insights by associating this structure with pseudo-Riemannian metrics of signature $(m + 1, m)$. Zamkovoy's work demonstrated that any manifold equipped with an almost paracontact structure inherently supports such a metric, reinforcing the structural coherence and utility of these geometric frameworks in the study of pseudo-Riemannian geometry. Para-Sasakian manifolds (commonly referred to as p-Sasakian manifolds) have been extensively studied by numerous researchers, including Yildiz [3], Adati et al. [29], and Tripathi et al. [16], among others.

The Ricci flow, a fundamental tool in geometric analysis, was first introduced by Richard S. Hamilton in 1982 to study the evolution of Riemannian metrics on a manifold [22]. The Ricci flow equation, an intrinsic parabolic partial differential equation, is defined as

$$\frac{\partial g(t)}{\partial t} = -2Ric(t), \quad g_0 = g(0),$$

where g is the Riemannian metric, and Ric denotes the Ricci curvature tensor. This flow smooths

out irregularities in the geometry of a manifold and played a crucial role in the proof of the Poincaré conjecture by Grigori Perelman in the early 2000s.

Building upon this, Fischer (2003 – 2004) introduced the conformal Ricci flow, which modifies the volume constraint of the Ricci flow by incorporating a scalar curvature condition [2]. This flow is described by

$$\frac{\partial t}{\partial g} + 2(Ric + \frac{g}{m}) = -pg,$$

with the scalar curvature $\tau(g) = -1$, where m is the dimension of the manifold M , and p is a scalar non-dynamical field (time dependent scalar field). This formulation enables the study of geometric evolution while maintaining a prescribed scalar curvature, offering flexibility in applications to conformal geometry.

The concept of conformal Ricci solitons, introduced by Basu and Bhattacharyya in 2015 [19], extends the Ricci soliton framework to conformal Ricci flow. The governing equation is

$$\mathcal{L}_V g + 2Ric = \left[2\lambda - \left(p + \frac{2}{m} \right) \right] g,$$

where \mathcal{L}_V denotes the Lie derivative of the metric g along the vector field V , Ric is the Ricci tensor, p is a scalar field, λ is a constant, and m is the manifold's dimension.

In 2009, Cho and Kimura introduced the η -Ricci soliton [12], an extension of the Ricci soliton concept incorporating a 1-form,

$$\mathcal{L}_\xi g + 2Ric = 2\lambda g + 2\mu\eta \otimes \eta,$$

where ξ is a characteristic vector field, λ, μ are constants, and η is the associated 1-form. This framework adapts Ricci solitons to contact metric and almost contact geometry settings. The notion of a conformal η -Ricci soliton was later introduced by Siddiqi in 2018. [15]

$$\mathcal{L}_\xi g + 2Ric + \left[2\lambda - \left(p + \frac{2}{m} \right) \right] g + 2\mu\eta \otimes \eta = 0,$$

incorporating a time-dependent scalar field p to generalize the η -Ricci soliton under conformal Ricci flow.

In 2014, Kaimakamis and Panagiotidou [10] introduced a modification of the classical Ricci soliton by utilizing the *-Ricci tensor Ric^* , originally defined by Tachibana [24], in place of the Ricci tensor Ric , the *-Ricci tensor, Ric^* is taken and it is defined as follows

$$Ric^*(X, Y) = \frac{1}{2} trace\{\varphi \circ \mathfrak{R}(X, \varphi Y)\},$$

where X, Y in $\chi(M)$ \mathfrak{R} is the curvature tensor, and ϕ is a $(1, 1)$ -tensor field. Kaimakamis and Panagiotidou applied the concept of *-Ricci solitons within the framework of real hypersurfaces in a complex space form.

Later, in 2019, Roy et al. [21] introduced the notion of a *-conformal η -Ricci soliton, characterized by the equation

$$\mathcal{L}_\xi g + 2Ric^* + \left[2\lambda - \left(p + \frac{2}{m} \right) \right] g + 2\mu\eta \otimes \eta = 0,$$

Motivated by this work, Sarkar and Dey[25] generalize the concept by considering V as an arbitrary vector field. We discuss a *-conformal η -Ricci soliton on a $(2m + 1)$ -dimensional para-Sasakian manifold as satisfying the following equation

$$\mathcal{L}_V g + 2Ric^* + \left[2\lambda - \left(p + \frac{2}{2m + 1} \right) \right] g + 2\mu\eta \otimes \eta = 0. \quad (1.1)$$

Further, if V is taken as the gradient of a smooth function f on M , the *-conformal η -Ricci soliton equation reduces to

$$Hess(f) + Ric^* + \left[\lambda - \left(\frac{p}{2} + \frac{1}{2m + 1} \right) \right] g + \mu\eta \otimes \eta = 0, \quad (1.2)$$

where $Hess(f)$ denotes the Hessian of f . In this context, we study the notion of a gradient almost $*$ -conformal η -Ricci soliton, defined as a gradient $*$ -conformal η -Ricci soliton in which λ is considered a smooth function.

The study of Ricci solitons in the context of paracontact geometry has garnered significant attention in the literature. In particular, Calvaruso and Perrone [8] conducted an explicit investigation of Ricci solitons on 3-dimensional almost paracontact manifolds, laying foundational insights into this area. In 2019, Patra [7] extended these studies to paracontact metric manifolds, proving that if such a manifold satisfies the Ricci soliton equation, it is necessarily either an Einstein manifold or an η -Einstein manifold. Furthermore, Naik and Venkatesha [6] studied para-Sasakian manifolds with η -Ricci and almost η -Ricci solitons, proving that such metrics lead to either Einstein manifolds or D -homothetically fixed η -Einstein manifolds. It also shows that higher-dimensional manifolds with gradient almost η -Ricci solitons are Einstein. However, Prakasha and Veerasha [5] investigates $*$ -Ricci solitons on para-Sasakian manifolds, demonstrating that if a para-Sasakian metric is a $*$ -Ricci soliton, then the manifold is either D -homothetic to an Einstein manifold or the Ricci tensor with respect to the canonical paracontact connection vanishes.

2 Preliminaries

Let M be a smooth manifold of dimension $(2m + 1)$, the manifold M is said to possess an almost paracontact structure if there exist a tensor field ϕ of type $(1, 1)$, a vector field ξ (called the characteristic or Reeb vector field), and a 1-form η such that the following conditions hold

$$\phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1. \quad (2.1)$$

An immediate consequence of these relations is that $\phi\xi = 0$ and $\eta \circ \phi = 0$ (see [6] for analogous results in the contact metric case). If M , with its (ϕ, ξ, η) -structure, is equipped with a pseudo-Riemannian metric g such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \quad (2.2)$$

for all $X, Y \in \chi(M)$, then (M, ϕ, ξ, η, g) is called an almost paracontact metric manifold. An almost paracontact metric manifold M is further termed as paracontact metric manifold, if the fundamental 2-form Φ , defined by $\Phi(X, Y) = g(X, \phi Y)$, satisfies

$$\Phi(X, Y) = d\eta(X, Y), \quad (2.3)$$

where η becomes a contact form, i.e., $\eta \wedge (d\eta)^n \neq 0$. In this case, M is also a contact manifold. The tensor field $h = \frac{1}{2}\mathcal{L}_\xi\phi$ is symmetric and satisfies the following relations:

$$h\phi = -\phi h, \quad h\xi = 0, \quad \nabla_X\xi = -\phi X + \phi hX.$$

An almost paracontact metric manifold M is called a para-Sasakian manifold if it satisfies

$$(\nabla_X\phi)Y = -g(X, Y)\xi + \eta(Y)X, \quad (2.4)$$

for all $X, Y \in \chi(M)$. In a para-Sasakian manifold, the tensor field h vanishes, and the following properties hold:

$$\nabla_X\xi = -\phi X, \quad (2.5)$$

$$\mathfrak{R}(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (2.6)$$

$$\mathfrak{R}(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \quad (2.7)$$

$$S\xi = -2m\xi. \quad (2.8)$$

Here, S denotes the Ricci operator such that, $Ric(X, Y) = g(SX, Y)$. In [6] (see Lemma 4), the authors proved that in a para-Sasakian manifold, the following result holds:

$$S\phi = \phi S. \quad (2.9)$$

Building upon this, Prakasha and Veerasha [5] established an elegant result for para-Sasakian manifolds (see Lemma 1).

$$(\nabla_{\xi} S)X = 0, \tag{2.10}$$

$$(\nabla_X S)\xi = S(\phi X) + 2m\phi X. \tag{2.11}$$

These results provide significant insights into the geometric properties of para-Sasakian manifolds. Taking the covariant derivative of equation $\eta(\xi) = 1$ along an arbitrary vector field Y and utilizing equation (2.5), we obtain $(\nabla_Y \phi)X = g(\phi X, Y)$.

3 *-Conformal Ricci soliton on para-Sasakian manifold

Lemma 3.1. [5] *The *-Ricci tensor on a $(2m+1)$ -dimensional para-Sasakian manifold (M, ϕ, ξ, η, g) is given by*

$$Ric^*(X, Y) = Ric(X, Y) - (2m - 1)g(X, Y) - \eta(X)\eta(Y), \tag{3.1}$$

for any vector fields X, Y on M .

Lemma 3.2. *If the metric g on a para-Sasakian manifold defines a *-conformal η -Ricci soliton, then*

$$\eta(\mathcal{L}_V \xi) = -(\mathcal{L}_V \eta)\xi = \lambda - \left[\frac{p}{2} + \frac{1}{2m + 1} \right] + \mu. \tag{3.2}$$

Proof. If the metric g is a *-conformal η -Ricci soliton, then from (2.8) and (1.1), we derive

$$(\mathcal{L}_V g)(X, \xi) = -2 \left\{ \lambda - \left[\frac{p}{2} + \frac{1}{2m + 1} \right] + \mu \right\} \eta(X), \tag{3.3}$$

for an arbitrary vector field X . Taking the Lie derivative of the relation $\eta(X) = g(X, \xi)$ along the soliton vector field V gives $(\mathcal{L}_V g)(X, \xi) = (\mathcal{L}_V \eta)X - g(X, \mathcal{L}_V \xi)$. Substituting this into the previous equation, we obtain

$$(\mathcal{L}_V \eta)X - g(X, \mathcal{L}_V \xi) = -2 \left\{ \lambda - \left[\frac{p}{2} + \frac{1}{2m + 1} \right] + \mu \right\} \eta(X). \tag{3.4}$$

Finally, incorporating the Lie derivative of $\eta(\xi) = 1$ along V , we arrive at the desired result. \square

Theorem 3.3. *Let M be a para-Sasakian manifold. If the metric g represents the *-conformal η -Ricci soliton, then M is necessarily η -Einstein. Furthermore, the soliton vector field V either satisfies the Killing condition or preserves the structure tensor ϕ , leaving it invariant.*

Proof. Substituting the value of Ric^* using (3.1) in (1.1) and taking the covariant derivative, we obtain

$$(\nabla_Z \mathcal{L}_V g)(X, Y) = 2(\nabla_Z Ric)(X, Y) + (2 - 2\mu)[g(\phi X, Z)\eta(Y) + g(\phi Y, Z)\eta(X)]. \tag{3.5}$$

From the following commutation formula, established by Yano [13],

$$\begin{aligned} (\mathcal{L}_V \nabla_X g - \nabla_X \mathcal{L}_V g - \nabla_{[V, X]} g)(Y, Z) = & - g((\mathcal{L}_V \nabla)(X, Y), Z) \\ & - g((\mathcal{L}_V \nabla)(X, Z), Y). \end{aligned} \tag{3.6}$$

From the above formula, we derive the relation

$$(\nabla_X \mathcal{L}_V g)(Y, Z) = g((\mathcal{L}_V \nabla)(X, Y), Z) + g((\mathcal{L}_V \nabla)(X, Z), Y). \tag{3.7}$$

Since $\mathcal{L}_V \nabla$ is symmetric, it follows from (3.7) that:

$$2g((\mathcal{L}_V \nabla)(X, Y), Z) = (\nabla_X \mathcal{L}_V g)(Y, Z) + (\nabla_Y \mathcal{L}_V g)(Z, X) - (\nabla_Z \mathcal{L}_V g)(X, Y). \tag{3.8}$$

Now, substituting the relation (3.5) into (3.8), we obtain

$$g((\mathcal{L}_V \nabla)(X, Y), Z) = (\nabla_X Ric)(Y, Z) + (\nabla_Y Ric)(X, Z) - (\nabla_Z Ric)(X, Y) + (2 - 2\mu)[g(\phi Z, X)\eta(Y) + g(\phi Y, Z)\eta(X)]. \tag{3.9}$$

Substituting $Y = \xi$ into equation (3.9) and applying equations (2.10) and (2.11), with the fact $(\nabla_Z Ric)(X, Y) = g((\nabla_Z S)X, Y)$, we get

$$(\mathcal{L}_V \nabla)(X, \xi) = 2S(\phi X) + 2(2m - 1 + \mu)\phi X. \tag{3.10}$$

By taking the covariant derivative of the previous equation with respect to an arbitrary vector field Y , and applying (2.4) and (2.5), we derive

$$(\nabla_Y \mathcal{L}_V \nabla)(X, \xi) = (\mathcal{L}_V \nabla)(X, \phi Y) + 2(\nabla_Y S)\phi X + 2\eta(X)SY + (2 - 2\mu)g(X, Y)\xi + 2(2m + \mu - 1)\eta(X)Y. \tag{3.11}$$

According to Yano [13], the following relation holds

$$(\mathcal{L}_V \mathfrak{R})(X, Y)\xi = (\nabla_X \mathcal{L}_V \nabla)(Y, \xi) - (\nabla_Y \mathcal{L}_V \nabla)(X, \xi). \tag{3.12}$$

Substituting the expression (3.11) in (3.12), we derive

$$\begin{aligned} (\mathcal{L}_V \mathfrak{R})(X, Y)\xi &= ((\mathcal{L}_V \nabla)(Y, \phi X) - (\mathcal{L}_V \nabla)(X, \phi Y)) + 2((\nabla_X S)\phi Y \\ &- a_Y S)\phi X + 2(\eta(Y)SX - \eta(X)SY) \\ &+ 2(2m + \mu - 1)(\eta(Y)X - \eta(X)Y). \end{aligned} \tag{3.13}$$

Plugging $Y = \xi$ in (3.13) and using (2.10), (2.11) and (3.10), we get

$$(\mathcal{L}_V \mathfrak{R})(X, \xi)\xi = 4SX + 4(2m + \mu - 1)X + 4(1 - \mu)\eta(X)\xi. \tag{3.14}$$

This holds for an arbitrary vector field X . Furthermore, from equation (2.6), we obtain

$$\mathfrak{R}(X, \xi)\xi = \eta(X)\xi - X. \tag{3.15}$$

Using (3.4) and (3.2), we take the Lie derivative of the last relation along V , yielding

$$(\mathcal{L}_V \mathfrak{R})(X, \xi)\xi = 2 \left\{ \lambda - \left[\frac{p}{2} + \frac{1}{2m+1} \right] + \mu \right\} \phi^2 X. \tag{3.16}$$

Substituting (3.16) in (3.14) and taking the inner product with an arbitrary vector field Y , we arrive at

$$\begin{aligned} Ric(X, Y) &= \frac{1}{2} \left\{ \lambda - \left[\frac{p}{2} + \frac{1}{2m+1} \right] - 2(2m - 1) - \mu \right\} g(X, Y) \\ &- \frac{1}{2} \left\{ \lambda - \left[\frac{p}{2} + \frac{1}{2m+1} \right] + 2 - \mu \right\} \eta(X)\eta(Y). \end{aligned} \tag{3.17}$$

Plugging the above equation in (1.1), we obtain

$$(\mathcal{L}_V g)(X, Y) = - \left\{ \lambda - \left[\frac{p}{2} + \frac{1}{2m+1} \right] + \mu \right\} [g(X, Y) + \eta(X)\eta(Y)]. \tag{3.18}$$

Differentiating the (3.17) covariantly with respect to Z , we get

$$\begin{aligned} (\nabla_Z Ric)(X, Y) &= -\frac{1}{2} \left\{ \lambda - \left[\frac{p}{2} + \frac{1}{2m+1} \right] + 2 + \mu \right\} [g(\phi X, Z)\eta(Y) \\ &+ g(\phi Y, Z)\eta(X)]. \end{aligned} \tag{3.19}$$

Differentiating (3.18) covariantly along Z and use the resultant equation in (3.8), one would get

$$(\mathcal{L}_V \nabla)(X, Y) = \left\{ \lambda - \left[\frac{p}{2} + \frac{1}{2m+1} \right] + \mu \right\} [\eta(Y)\phi X + \eta(X)\phi Y]. \tag{3.20}$$

For arbitrary vector fields X and Y on M , taking the covariant derivative of the above equation along an arbitrary vector field Z and after rearrangement, we derive

$$(\nabla_Z \mathcal{L}_V \nabla)(X, Y) = \left\{ \lambda - \left[\frac{p}{2} + \frac{1}{2m+1} \right] + \mu \right\} [g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y - g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + 2\eta(X)\eta(Y)Z]. \tag{3.21}$$

Again from Yano [13], we have

$$(\mathcal{L}_V \mathfrak{R})(X, Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_V \nabla)(X, Z). \tag{3.22}$$

Feeding the relation (3.21) in (3.22) and contracting along X , we get

$$(\mathcal{L}_V Ric)(Y, Z) = 2 \left\{ \lambda - \left[\frac{p}{2} + \frac{1}{2m+1} \right] + \mu \right\} [(2m+1)\eta(Y)\eta(Z) - g(Y, Z)]. \tag{3.23}$$

Taking the Lie derivative of equation (3.17) and employing equation (3.18), then equating the resulting expression with equation (3.23), we ultimately obtain

$$\begin{aligned} & \left(1 - 2m + \frac{1}{2} \left\{ \lambda - \left[\frac{p}{2} + \frac{1}{2m+1} \right] - \mu \right\} \right) \left(\lambda - \left[\frac{p}{2} + \frac{1}{2m+1} \right] + \mu \right) [g(X, Y) + \eta(X)\eta(Y)] \\ & + \left(1 + \frac{1}{2} \left\{ \lambda - \left[\frac{p}{2} + \frac{1}{2m+1} \right] - \mu \right\} \right) [(\mathcal{L}_V \eta)(X)\eta(Y) + \eta(X)(\mathcal{L}_V \eta)(Y)] \\ & = 2 \left\{ \lambda - \left[\frac{p}{2} + \frac{1}{2m+1} \right] + \mu \right\} ((2m+1)\eta(X)\eta(Y) - g(X, Y)). \end{aligned} \tag{3.24}$$

By substituting X and Y with $\phi^2 X$ and ϕY respectively, and applying (2.1) by considering the fact that $d\eta(X, Y) = g(\phi X, Y) \neq 0$, we deduce

$$\left\{ \lambda - \left[\frac{p}{2} + \frac{1}{2m+1} \right] + \mu \right\} \left(2m+1 - \frac{1}{2} \left\{ \lambda - \left[\frac{p}{2} + \frac{1}{2m+1} \right] - \mu \right\} \right) = 0. \tag{3.25}$$

From the previous equation we can conclude that, $\lambda = \frac{p}{2} + \frac{1}{2m+1} - \mu$ or $\lambda = 2(2m+1)\left(\frac{p}{2} + \frac{1}{2m+1}\right) + \mu$.

Case I: Suppose $\lambda = \frac{p}{2} + \frac{1}{2m+1} - \mu$. By plugging the value in (3.17), we obtain $Ric(X, Y) = -(2m-1+\mu)g(X, Y) + (\mu-1)\eta(X)\eta(Y)$. Which means the manifold M is η -Einstein manifold. Also by substituting the value of λ in (3.18) we get $\mathcal{L}_V g = 0$. Hence V is Killing.

Case II: Again, substituting $\lambda = 2(2m+1) + \left(\frac{p}{2} + \frac{1}{2m+1}\right) + \mu$, into (3.17), we derive $Ric(X, Y) = 2g(X, Y)(2m+2)\eta(X)\eta(Y)$. This confirms that the manifold is η -Einstein. Replacing X with ϕX and setting $Y = \xi$ in (3.24) with $\lambda = 2(2m+1) + \left(\frac{p}{2} + \frac{1}{2m+1}\right) + \mu$, we arrive at $(\mathcal{L}_V \eta)(\phi X) = 0$. Further, replacing X with ϕX and applying (2.1), we find $\mathcal{L}_V \eta = -2(2m+1+\mu)\eta$. Taking the exterior derivative of the last equation and using the well-known relation $d(\mathcal{L}_V \eta) = \mathcal{L}_V d\eta$ along with (2.3), we obtain

$$(\mathcal{L}_V d\eta)(X, Y) = -2(2m+1+\mu)g(X, \phi Y), \tag{3.26}$$

for all $X, Y \in \chi(M)$. Taking the Lie derivative of (2.3) along V and using (3.18), we deduce

$$(\mathcal{L}_V d\eta)(X, Y) = -2(2m+1+\mu)g(X, \phi Y) + g(X, (\mathcal{L}_V \phi)Y). \tag{3.27}$$

Comparing equations (3.26) and (3.27), we find $\mathcal{L}_V \phi = 0$. Hence, V preserves ϕ , leaving it invariant. □

Theorem 3.4. *Let (M, g) be a para-Sasakian manifold. If (g, V) represents a *-conformal η -Ricci soliton where the soliton vector field $V = \vartheta\xi$, with $\vartheta\xi \in C^\infty(M)$, then M is an η -Einstein manifold with the constant curvature.*

Proof. Since the soliton vector field V is aligned pointwise with the reeb vector field ξ , we express $V = \vartheta\xi$, where ϑ is a smooth function defined on M^{2m+1} . Substituting $V = \vartheta\xi$ into the Lie derivative condition

$$(\mathcal{L}_V g)(X, Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V), \tag{3.28}$$

and applying equation (2.5), we have

$$(\mathcal{L}_V g)(X, Y) = (X\vartheta)\eta(Y) + (Y\vartheta)\eta(X). \tag{3.29}$$

Using (3.29) in (1.1), we obtain

$$(X\vartheta)\eta(Y) + (Y\vartheta)\eta(X) - 2Ric(X, Y) - \left(2(2m - 1) - 2\lambda + \left[p + \frac{2}{2m + 1}\right]\right) g(X, Y) - (2 - 2\mu)\eta(X)\eta(Y) = 0. \tag{3.30}$$

Plugging $Y = \xi$ in (3.30), we get

$$D\vartheta = \xi(a)\xi. \tag{3.31}$$

Differentiating (3.32) along X , applying equation (2.5), we derive

$$g(\nabla_X D\vartheta, Y) = X(\xi(\vartheta))\eta(Y) + \xi(\vartheta)g(X, \phi Y). \tag{3.32}$$

Applying anti-symmetric axiom the above expressions with the fact $g(\nabla_X D\vartheta, Y) = g(\nabla_Y D\vartheta, X)$ yields

$$X(\xi(\vartheta))\eta(Y) - Y(\xi(\vartheta))\eta(X) + 2\xi(\vartheta)g(X, \phi Y) = 0. \tag{3.33}$$

Now, let $X \perp \xi$ be a unit vector field, i.e., $g(X, X) = 1$. Then ϕX is also a unit vector field satisfying $g(\phi X, \phi X) = -1$ and $\phi X \perp \xi$. Substituting $Y = \phi X$ into (3.33), we get $\xi(a) = 0$. Consequently, Equation (3.31) implies that a is constant. So V is constant multiple of ξ . Equation (3.30) simplifies to

$$Ric(X, Y) = -2 \left((2m - 1) - \lambda + \left[\frac{p}{2} + \frac{1}{2m + 1}\right] \right) g(X, Y) + (\mu - 1)\eta(X)\eta(Y). \tag{3.34}$$

Taking the trace of this equation, we obtain

$$\tau = -2(2m + 1) \left((2m - 1) - \lambda + \left[\frac{p}{2} + \frac{1}{2m + 1}\right] \right) + (\mu - 1), \tag{3.35}$$

where τ is the scalar curvature of the manifold. This completes the proof. □

4 On Almost *-Conformal η -Ricci Solitons

In this section, we delve into the concept of almost *-conformal η -Ricci solitons within the setting of para-Sasakian manifolds. As inferred from equation (1.1), the notion of an almost *-conformal η -Ricci soliton represents a broader generalization of the almost Ricci solitons [20], [18] and [17], distinguished by its dependence on two smooth functions, τ and μ .

Definition 4.1. [28] A vector field V is termed an infinitesimal contact transformation if there exists a smooth function $\varpi \in C^1(M)$ such that

$$\mathcal{L}_V \eta = \varpi \eta. \tag{4.1}$$

If ϖ vanishes identically ($\varpi = 0$), the vector field V is specifically referred to as a strictly infinitesimal contact transformation.

Theorem 4.2. *Let M be a para-Sasakian manifold with $n > 1$. If g defines an almost *-conformal η -Ricci soliton and the associated soliton vector field V acts as an infinitesimal contact transformation, then M is η -Einstein. Additionally, the soliton vector field V either possesses the properties of a Killing vector field or remains invariant under the structure tensor ϕ .*

Proof. The exterior derivative of equation (4.1) leads to

$$\mathcal{L}_V d\eta(X, Y) = d\varpi(X)\eta(Y) + \varpi d\eta(X, Y), \quad (4.2)$$

By employing equation (2.3), this expression can be reformulated as

$$(\mathcal{L}_V d\eta)(X, Y) = \frac{1}{2}[(X\varpi)\eta(Y) - \eta(X)(Y\varpi)] + \varpi g(X, \phi(Y)). \quad (4.3)$$

Next, differentiating equation (2.3) along the soliton vector field V using the Lie derivative and considering (3.28) and (4.3) yields

$$\begin{aligned} 2(\mathcal{L}_V \phi)Y &= (D\varpi)\eta(Y) - (Y\varpi)\xi - 4S\phi Y \\ &+ \left(2\varpi - 4(2m - 1) + 4\lambda - 2p - \frac{4}{2m + 1}\right)\phi Y. \end{aligned} \quad (4.4)$$

Plugging $Y = \xi$ in (4.4), we get

$$2(\mathcal{L}_V \phi)\xi = D\varpi - (\xi\varpi)\xi. \quad (4.5)$$

Taking Lie derivative of $g(\xi, \xi) = 1$, by considering (1.2) and (3.1) we finally arrived at

$$\eta(\mathcal{L}_V \xi) = -(\mathcal{L}_V \eta)\xi = \lambda - \left[\frac{p}{2} + \frac{1}{2m + 1}\right] + \mu. \quad (4.6)$$

Also considering the Lie derivative of $\eta(X) = g(X, \xi)$, with the help of (4.1) and (4.6), we get

$$\varpi = -\lambda + \frac{p}{2} + \frac{1}{2m + 1} - \mu. \quad (4.7)$$

Again, by applying the Lie derivative for $\phi\xi = 0$ we arrived at $(\mathcal{L}_V \phi)\xi = 0$, using this in (4.5), we obtain $D\varpi = (\xi\varpi)\xi$. i.e., $d\varpi = (\xi\varpi)\xi$. Operating d in the previous equation with the fact $d^2 = 0$ gives

$$d(\xi\varpi) \wedge \eta + (\xi\varpi) \wedge d\eta = 0.$$

Taking the wedge product of the preceding equation with η , and noting that $\eta \wedge \eta = 0$, we obtain

$$(\xi\varpi)\eta \wedge d\eta = 0,$$

since M is contact manifold, which implies $\eta \wedge d\eta \neq 0$. Hence $D\varpi = 0$, which means ϖ is constant. Since ϖ is constant, using (4.7) in (4.6) we have

$$(\mathcal{L}_V \phi)Y = -2S\phi Y + \left(\lambda - \frac{p}{2} - \frac{1}{2m + 1} - \mu - (2m - 1)\right)\phi Y. \quad (4.8)$$

Applying the Lie derivative \mathcal{L}_V to the first equation of (2.1), and employing the relations (4.1) and (4.6), we find that $(\mathcal{L}_V \phi^2)X = 0$. This result translates to the equation

$$\phi(\mathcal{L}_V \phi)X + (\mathcal{L}_V \phi)\phi X = 0.$$

Substituting the expression from (4.4) into the above equation, we derive

$$\begin{aligned} SX &= \left(-\frac{\varpi}{2} + (2m - 1) - \lambda + \frac{p}{2} + \frac{1}{2m + 1}\right) X \\ &+ \left(\frac{\varpi}{2} - (4m - 1) + \lambda - \frac{p}{2} - \frac{1}{2m + 1}\right) \eta(X)\xi. \end{aligned} \quad (4.9)$$

Differentiating (4.9) covariantly along an arbitrary vector field Y , one would get

$$\begin{aligned}
 (\nabla_Y S)X &= -\left(\frac{\varpi}{2} + \lambda - \left[\frac{p}{2} + \frac{1}{2m+1}\right] - (4m-1)\right) [g(X, \phi Y)\xi \\
 &\quad - \eta(X)\phi Y] - (Y\lambda)[X - \eta(X)\xi].
 \end{aligned}
 \tag{4.10}$$

Taking inner product of (4.10) with Z and contracting along Y and X separately, yields

$$(X\tau) = -2(X\lambda) + 2(\xi\lambda)\eta(X), \tag{4.11}$$

$$(X\tau) = -2m(X\lambda). \tag{4.12}$$

Given that ξ is a Killing vector field in a para-Sasakian manifold, it directly implies $(\xi\tau) = 0$. Consequently, (4.12) leads to $(\xi\lambda) = 0$. Substituting this result into equation (4.11), we find $X\tau = -2(X\lambda)$. Using this with equation (4.12) yields $(1 - m)(X\lambda) = 0$. Since $m > 1$, it follows that λ must be constant. Since ϖ is also constant equation (4.7), we deduce that μ is also constant. Therefore, the metric g represents an η -Ricci soliton, and the result follows directly from Theorem 3.1. \square

Lemma 4.3. *Let (M, g) be a para-Sasakian manifold. If g is a gradient almost *-conformal η -Ricci soliton, then the Riemannian curvature tensor and associated soliton satisfy the following relation*

$$\begin{aligned}
 \mathfrak{R}(X, Y)Df &= (\nabla_X S)Y - (\nabla_Y S)X - ((X\lambda)Y - (Y\lambda)X) \\
 &\quad - ((X\mu)\eta(Y) - (Y\mu)\eta(X))\xi + 2(1 - \mu)g(X, \phi Y)\xi \\
 &\quad + (\mu - 1)[\eta(Y)\phi X - \eta(X)\phi Y].
 \end{aligned}
 \tag{4.13}$$

Proof. From the gradient almost *-conformal η -Ricci soliton (1.2) and (3.1), we directly derive

$$\nabla_X Df = SX + \left((2m - 1) + \left[\frac{p}{2} + \frac{1}{2m+1}\right] - \lambda\right) X + (1 - \mu)\eta(X)\xi. \tag{4.14}$$

Differentiating the above equation with respect to Y and using the resultant equation in the formula $\mathfrak{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z$, we get the desired result. \square

Theorem 4.4. *Let M be a para-Sasakian manifold of dimension > 3 . If g represents the gradient almost *-conformal η -Ricci soliton, with the non-zero soliton vector field then the manifold M is Einstein, with the constant scalar curvature $-2m(2m + 1)$.*

Proof. Setting $Y = \xi$ in (4.13), and applying (2.10) and (2.11), we obtain the following simplified expression

$$\mathfrak{R}(X, \xi)Df = S\phi X + (2m + \mu - 1)\phi X - ((X\lambda)\xi - (\xi\lambda)X) - ((X\mu) - (\xi\mu)\eta(X))\xi. \tag{4.15}$$

Using (2.4) and (2.7) in (4.15), one can easily get

$$\begin{aligned}
 g((\nabla_X \phi)Y), Df &= g(S\phi X, Y) + (2m + \mu - 1)g(\phi X, Y) \\
 &\quad - (X\lambda)\eta(Y) + (\xi\lambda)g(X, Y) - ((X\mu) - (\xi\mu)\eta(X))\eta(Y).
 \end{aligned}
 \tag{4.16}$$

By substituting X and Y with ϕX and ϕY respectively in the above equation, we obtain

$$\begin{aligned}
 g((\nabla_{\phi X} \phi)\phi Y, Df) &= -g(S(\phi X), Y) - (2m + \mu - 1)g(\phi X, Y) \\
 &\quad - (\xi\lambda)(g(X, Y) - \eta(X)\eta(Y)).
 \end{aligned}
 \tag{4.17}$$

Subtracting (4.16) in (4.17) yields

$$\begin{aligned}
 g((\nabla_{\phi X} \phi)\phi Y - (\nabla_X \phi)Y, Df) &= -2g(S\phi X, Y) - 2(2m + \mu - 1)g(\phi X, Y) \\
 &\quad - 2(\xi\lambda)g(X, Y) + (\xi\lambda)\eta(X)\eta(Y) + (X\lambda)\eta(Y) \\
 &\quad + (X\mu)\eta(Y) - (\xi\mu)\eta(X)\eta(Y).
 \end{aligned}
 \tag{4.18}$$

According to Zamkovoy [27], the following relation holds on a para-Sasakian manifold

$$(\nabla_{\phi X}\phi)Y - (\nabla_X\phi)Y = 2g(X, Y)\xi - (X + \eta(X)\xi)\eta(Y), \quad (4.19)$$

where X and Y are arbitrary vector fields. (This equation is derived by substituting $h = 0$ into relation (2.8) as presented in [27]). Considering (4.19), equation (4.18) can be reformulated as

$$\begin{aligned} 2(\xi f)g(X, Y) - \eta(Y)((Xf) - (\xi f)\eta(X)) &= 2g(S\phi X, Y) \\ &- 2(2m + \mu - 1)g(\phi X, Y) - 2(\xi\lambda)g(X, Y) \\ &+ (\xi\lambda)\eta(X)\eta(Y) + (X\lambda)\eta(Y) + (X\mu)\eta(Y) \\ &- (\xi\mu)\eta(X)\eta(Y). \end{aligned} \quad (4.20)$$

By anti-symmetrizing the previous equation and substituting X and Y with ϕX and ϕY , respectively, we obtain $S\phi X = (2m + \mu - 1)\phi X = 0$. Furthermore, replacing X with ϕX in the preceding relation gives

$$SX = -(2m + \mu - 1)X + (\mu - 1)\phi(X)\xi. \quad (4.21)$$

Taking covariant derivative of last equation we get

$$(\nabla_Y S)X = (Y\mu)(\eta(X)\xi - X) + (\mu - 1)\{g(\phi X, Y)\xi - \eta(X)\phi Y\}. \quad (4.22)$$

Using (4.22) in (4.13), we obtain

$$\mathfrak{R}(X, Y)Df = (Y\mu)X - (X\mu)Y + (Y\lambda)X - (X\lambda)Y. \quad (4.23)$$

Tracing (4.13) and (4.23), gives

$$S(Df) = -\frac{1}{2}D\tau + 2mD\lambda + D\mu - (\xi\mu)\xi \quad (4.24)$$

$$S(Df) = 2m(D\mu + D\lambda). \quad (4.25)$$

Equating previous two equations we get

$$\frac{1}{2}(D\tau) = -(2m - 1)D\mu - (\xi\mu)\xi. \quad (4.26)$$

Contracting (4.21) gives

$$\tau = 2m(1 - \mu) - 2m(2m + 1). \quad (4.27)$$

On differentiating this, we get $D\tau = -2mD\mu$. By (2.10), $g((\nabla_{e_i}S)\xi, e_i) = \frac{1}{2}(\xi\tau) = 0$. Hence, we deduce $D\mu = 0$, it follows that μ is constant.

Now plugging $X = \xi$ in (4.23), using (2.7) we get

$$Y(f + \lambda) = \xi(f + \lambda)Y. \quad (4.28)$$

Taking inner product of (4.25) with ξ , we find

$$\xi(f + \lambda) = 0. \quad (4.29)$$

From (4.29) in (4.28), we obtain $Y(f + \lambda) = 0$, which implies $f + \lambda$ is constant. Setting $X = Df$ in (4.21) and comparing with (4.25) gives,

$$(1 - \mu)(Df - (\xi f)\xi) = 0, \quad (4.30)$$

which means $\mu = 1$ or $\mu \neq 1$.

Suppose $\mu = 1$, from (4.21), M is Einstein manifold. If suppose $\mu \neq 1$, then by (4.30) we have $Df - (\xi f)\xi = 0$. Plugging the value of Df from last equation and also putting (4.21) in (4.14), we have

$$X(\xi f)\xi - (\xi f)\phi X = -\left(\lambda - \left[\frac{p}{2} + \frac{1}{2m + 1}\right] + \mu\right)X. \quad (4.31)$$

After taking the inner product with ξ in (4.31), yields

$$X(\xi f) = -\left(\lambda - \left[\frac{p}{2} + \frac{1}{2m+1}\right] + \mu\right)\eta(X). \quad (4.32)$$

Tracing the resultant equation after substituting (4.32) in (4.31), we get $\lambda + \mu = \frac{p}{2} + \frac{1}{2m+1}$. Since μ is constant, implies λ is constant, hence f is also constant. This result contradicts the assumption that the soliton vector field V is non-zero, as it leads to $V = Df = 0$. Proof completes with (4.27). □

5 Conclusion

By investigating the intricate relationships between geometric structures, curvature conditions, and soliton equations, we establish significant results that contribute to the broader understanding of Ricci solitons in the context of paracontact geometry. The findings emphasize that if a para-Sasakian manifold satisfies certain soliton conditions, it must exhibit properties of an η -Einstein manifold, illustrating how these solitons interact with the manifold's curvature. Furthermore, establishing the conditions under which such manifolds can be classified as Einstein or exhibit gradient properties deepens our understanding of the geometric implications of Ricci flow in higher dimensions.

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