Palestine Journal of Mathematics

Vol 14(4)(2025) , 790–799

First Order Differential Inclusions with Unbounded Perturbations of Subdifferentials of PLN Functions

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Communicated by Hichem Ben-El-Mechaiekh

MSC 2020 Classifications: Primary 49A52; Secondary 49J53, 34A60, 34G25, 34G20.

Keywords and phrases: differential inclusions, set-valued maps, subdifferentials, primal lower nice functions.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

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Abstract This paper is devoted to study the existence of solutions for the first order differential inclusion $-\dot{x}(t) \in \partial V(x(t)) + G(t, x(t))$ a.e. on $[T_0, T]$, $x(T_0) = x_0 \in domV$, where H is a real separable Hilbert space, V is a proper lower semicontinuous function which is primal lower nice (pln in short) at $x_0 \in domV$ with constants $s_0, c_0, Q_0 > 0$ and $G : [T_0, +\infty[\times H \rightrightarrows H$ is a set-valued unbounded perturbation with nonempty convex closed values satisfying standard regularity conditions.

1 Introduction

In this paper, we consider the first order differential inclusion

$$\begin{cases} -\dot{x}(t) \in \partial V(x(t)) + G(t, x(t)) \text{ a.e. on } [T_0, T], \\ x(T_0) = x_0 \in domV, \end{cases} \quad (\mathcal{P})$$

where H is a real separable Hilbert space, V is a proper lower semicontinuous function which is primal lower nice at $x_0 \in domV$ with constants s_0, c_0, $Q_0 > 0$ and $G : [T_0, +\infty[\times H \rightrightarrows H$ is a measurable set-valued mapping with nonempty closed convex values and upper hemicontinuous in x.

Such problems have been studied firstly by H. Attouch and A. Damlamian in 1972 with V being a proper lower semicontinuous (l.s.c. in short) convex function and G being a set-valued mapping with nonempty closed convex values and upper semicontinuous in the state variable x. Later, D. Kravvaritis and N. S. Papageoriou [8] studied the problem $(\mathcal{P})$ with the set-valued perturbation G being measurable, lower semicontinuous in x, and satisfying a sublinear growth condition. (For more on the convex case, see for instance [1].)

These results have been extended by S. Marcellin to the case where V is a proper lower semicontinuous so-called *primal lower nice function* (*pln* in short) and G is either a single-valued mapping or a set-valued mapping satisfying different assumptions (see [5], [9]).

Subsequently, an existence result has been established in [6] for the case where V is l.s.c. and pln with $\partial V(x(t))$ replaced by an upper semicontinuous selector $F(x(t)) \subset \partial V(x(t))$ on $[0, T]$ with nonempty compact values and single-valued Caratheodory perturbation G.

The aim in this paper is to establish an existence result for the problem $(\mathcal{P})$ when G is a set-valued unbounded perturbation with nonempty convex closed values satisfying standard regularity conditions. The paper is organized as follows: section 2 includes preliminary results needed for establishing, in section 3, the main theorem of this work.

2 Notations and preliminary results.

Throughout the paper, H denotes a real separable Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and associated norm $\|\cdot\|$; $B(x,r)$ (resp. $\overline{B}(x,r)$) is the open (resp. closed) ball centered at x with radius r.
For a given subset C of H, $\overline{co}C$ stands for the closed convex hull of C. The support function $\delta^*(\cdot,C)$ of C is defined by

$$\delta^*(\zeta,C) := \sup_{x\in C}\langle\zeta,x\rangle, \text{for all} \zeta\in H.$$

Recall that for a closed convex subset C in H, we have

$$d(x,C) = \sup_{\bar{x}\in\overline{B}(0,1)} [\langle\bar{x},x\rangle - \delta^*(\bar{x},C)].$$

The projection mapping onto the closed set C is

$$Proj(x,C) := \{u\in C : d(x,C) = \|u-x\|\}.$$

Definition 2.1. Let X,Y be topological spaces and $F : X \rightrightarrows Y$ a set-valued mapping with nonempty values in Y.

(i) We will say that F is upper semicontinuous $(u.s.c.)$ at $\bar{x}\in X$ if for every open neighborhood U of $F(\bar{x})$ in Y, there exists an open neighborhood V of $\bar{x}$ in X such that $F(x)\subset U$ for every $x\in V$. F is u.s.c. on X if it so at each point of X.

(ii) F is said to be upper hemicontinuous on X if, for any $y\in H$, the real-valued function $x\mapsto\delta^*(y,F(x))$ is upper semicontinuous (in the sense of real functionals) on X.

Definition 2.2. The function $V : H\to\mathbb{R}\cup\{+\infty\}$ is said to be inf-ball compact if for every $\lambda>0$, the set $\{x\in H : V(x)\le\lambda\}$ is ball-compact, i.e., its intersection with any closed ball in H is compact.

Definition 2.3. Let $V : H\to\mathbb{R}\cup\{+\infty\}$ be a function and let $\overline{x}\in domV$, that is, $V(\overline{x})<+\infty$. The proximal subdifferential of V at $\overline{x}$ is the set $\partial_pV(\overline{x})$ of all elements $\xi\in H$ for which there exist $r>0$ and $\varepsilon>0$ such that

$$\langle\xi,x-\overline{x}\rangle \le V(x)-V(\overline{x})+r\|x-\overline{x}\|^2 \text{ for all } x\in B(\overline{x},\varepsilon).$$

The Fréchet subdifferential $\partial_FV(\overline{x})$ of V at $\overline{x}$ is defined by $\xi\in\partial_FV(\overline{x})$ provided that for each $\varepsilon>0$, there exists some $\eta>0$ such that for all $x\in B(\overline{x},\eta)$,

$$\langle\xi,x-\overline{x}\rangle \le V(x)-V(\overline{x})+\varepsilon\|x-\overline{x}\|.$$

The Clarke subdifferential of a l.s.c. function V at $\overline{x}$ is the set

$$\partial_CV(\overline{x}) \quad : \quad = \left\{\xi\in H : V^{\uparrow}(\overline{x};y)\ge\langle\xi,y\rangle,\ \forall y\in H\right\}.$$

where $V^{\uparrow}(\overline{x};y)$ is the generalized Rockafellar derivative given by

$$V^{\uparrow}(\overline{x};y) \quad = \quad \limsup_{\substack{x\xrightarrow{V}\overline{x}\\ t\to0^+}} \inf_{y'\to y} t^{-1}[V(x+ty')-V(x)],$$

where $x\xrightarrow{V}\overline{x}$ means $x\to\overline{x}$ et $V(x)\to V(\overline{x})$. If V is locally Lipschitz, the generalized Rockafellar derivative $V^{\uparrow}(\overline{x};y)$ coincides with the Clarke directional derivative $V^0(\overline{x},y)$ defined by

$$V^0(\overline{x},y) = \lim_{x\longrightarrow\overline{x}}\sup_{t\downarrow0}\frac{V(x+ty)-V(x)}{t}.$$

If $x\notin domV$, $\partial_CV(x):=\emptyset$.
When V is convex and lower semicontinuous, one has

$$\partial_pV = \partial_CV = \partial_FV = \partial V,$$

where ∂V denotes the subdifferential of convex analysis.

Definition 2.4. Let $V : H \to \mathbb{R} \cup \{+\infty\}$ be a proper function and consider $x_0 \in domV$. The function V is said to be primal lower nice at x_0, if there exist positive real numbers s_0, c_0, Q_0 such that for all $x \in \overline{B}(x_0, s_0)$, for all $q \geq Q_0$ and all $v \in \partial_p V(x)$ with $\|v\| \leq c_0 q$, one has

$$V(y) \geq V(x) + \langle v, y - x \rangle - \frac{q}{2} \|y - x\|^2 \text{ for each } y \in \overline{B}(x_0, s_0).$$

Remark 2.5. (i) Each extended real valued convex function is primal lower nice at each point of its domain.

(ii) Clearly, if V is pln at u_0 with constants s_0, c_0, Q_0, one has

$$\langle v_1 - v_2,\ x_1 - x_2 \rangle \geq -q \|x_1 - x_2\|^2$$

for any $v_i \in \partial_p V(x_i)$ with $\|v_i\| \leq c_0 q$ whenever $q \geq Q_0$ and $x_i \in \overline{B}(u_0, s_0)$, $i = 1, 2$.

(iii) If V is pln at $u_0 \in domV$ then for all x in an open neighborhood of u_0, the proximal subdifferential of V at x agrees with the Fréchet subdifferential and Clarke subdifferential of V at x, that is, $\partial_p V(x) = \partial_F V(x) = \partial_C V(x)$. In this case, we simply denote by $\partial V(x)$ the common subdifferential, and by $\partial^0 V(x)$ its element of minimal norm for $x \in domV$.

To learn more on the study of pln functions, we refer to [10], [12], [13], and [15].

The graph of the (proximal) subdifferential of a pln function enjoys the useful following closure property.

Proposition 2.6. *Let $V : H \to \mathbb{R} \cup \{+\infty\}$ be a proper l.s.c. function which is pln at $u_0 \in domV$ with constants $s_0, c_0, Q_0 > 0$, and let T_0, T, v_0, η_0 be positive real numbers such that $T > T_0$ and $v_0 + \eta_0 = s_0$. Let $v \in L^2_H([T_0, T])$ and u be a mapping from $[T_0, T]$ into H. Let $(u_n)_n$ be a sequence of mappings from $[T_0, T]$ into H and $(v_n)_n$ be a sequence in $L^2_H([T_0, T])$.*
Assume that:

(i) $\{u_n(t),\ n \in \mathbb{N}\} \subset \overline{B}(u_0, \eta_0) \cap domV$ for almost every $t \in [T_0, T]$;

(ii) $(u_n)_n$ converges almost everywhere to some mapping u with $u(t) \in domV$ for almost every $t \in [T_0, T]$;

(iii) $v_n \to v$ with respect to the weak topology of $L^2_H([T_0, T])$;

(iv) for each $n \geq 1$, $v_n(t) \in \partial V(u_n(t))$ for almost every $t \in [T_0, T]$.

Then, for almost all $t \in [T_0, T]$, $v(t) \in \partial V(u(t))$.

For the proof, we refer the reader to [10].

Let us recall also the existence result for the subdifferential operator of a pln function obtained in [5].

Theorem 2.7. *Let $V : H \to \mathbb{R} \cup \{+\infty\}$ be a proper l.s.c. function. Assume that V is pln at $x_0 \in domV$ with constants s_0, c_0, Q_0 and let some real numbers $\eta_0 \in]0, s_0[$ be such that*

$$\inf\{V(x) : x \in \overline{B}(x_0, \eta_0)\}$$

is finite (η_0 always exists due to the lower semicontinuity of V at x_0).
Consider also a real number $T_0 \geq 0$ and some $h \in L^2_{loc}([T_0, +\infty[)$.
Then, there exist some real number $\tau > T_0$ and a unique mapping $x : [T_0, \tau] \to \overline{B}(x_0, \eta_0)$ that is absolutely continuous on $[T_0, \tau]$ and such that

$$\begin{cases} 0 \in \dot{x}(t) + \partial V(x(t)) + h(t) \text{ for a.e. } t \in [T_0, \tau], \\ x(T_0) = x_0, \\ x(t) \in domV, \forall t \in [T_0, \tau]. \end{cases} \tag{2.1}$$

In addition, we have
(a) $-\dot{x}(t) = (\partial V(x(t)) - h(t))^0$ for a.e. $t \in]T_0, \tau[$,

(b) $\dot{x} \in L^2_{loc}([T_0, \tau])$,
(c) *for any* $s, t \in [T_0, \tau]$ *with* $s \leq t$,

$$\int_s^t \|\dot{x}(r)\|^2 dr \leq 2(V(x_0) - V(x(t))) + \int_{T_0}^t \|h(r)\|^2 dr.$$

Remark 2.8. [5] Under the notations of Theorem 2.7, note that, as $\eta_0 < s_0$, V is pln at any point of $\overline{B}(x_0, \frac{\eta_0}{2}) \cap domV$ with the same constants $\frac{\eta_0}{2}, c_0, Q_0$. So, given $M > 0$, for all $u_0 \in \overline{B}(x_0, \frac{\eta_0}{2})$ and all $h \in L^2([T_0, T])$ such that $V(u_0) \leq M$ and $\|h\|_{L^2([T_0,T])} \leq M$, for any real number $\tau \in]T_0, T]$ satisfying

$$(\tau - T_0)^{\frac{1}{2}}[2(M - \inf_{\overline{B}(x_0,\eta_0)} + M^2)] < \frac{\eta_0}{2},$$

there is an absolutely continuous mappings $x : [T_0, \tau] \to \overline{B}(x_0, \eta_0)$ such that

(i) $0 \in \dot{x}(t) + \partial V(x(t)) + h(t)$ a.e. in $[T_0, \tau]$, $x(T_0) = u_0$,

(ii) $x([T_0, \tau] \subset domV$,

(iii) $\dot{x} \in L^2([T_0, T])$, and for all $s, t \in [T_0, \tau]$ with $s \leq t$,

$$\begin{aligned}\int_s^t \|\dot{x}(r)\|^2 dr &\leq 2(V(x_0) - V(x(t))) + \int_{T_0}^t \|h(r)\|^2 dr \\ &\leq 2(M - \inf_{\overline{B}(x_0,\eta_0)} V) + M^2.\end{aligned}$$

3 The Main result

Our main result is the following:

Theorem 3.1. *Let* $V : H \to \mathbb{R} \cup \{+\infty\}$ *be proper l.s.c. and pln on* $domV$ *such that*
(H_1) *for some positive number* α, $V(x) \geq -\alpha(1 + \|x\|)$, $\forall x \in H$;
(H_2) V *is inf-ball compact around each point of* $domV$.
Let $G : [T_0, +\infty[\times H \rightrightarrows H$ *be a set-valued mapping with nonempty closed convex values and satisfying the following assumptions:*
(H_3) G *is upper hemicontinuous with respect to the second variable;*
(H_4) *for any* $(t, x) \in [T_0, +\infty[\times H$ *the mapping* $Proj(0, G(t, x))$ *is measurable on* $[T_0, +\infty[$, *and for some nonnegative function* $\beta \in L^2([T_0, +\infty[)$, G *verifies the following growth condition*

$$d(0, G(t, x)) \leq \beta(t)(1 + \|x\|)$$

for all $t \in [T_0, +\infty[$ *and* $x \in H$.
Then, for each $x_0 \in domV$, *there exists a locally absolutely continuous mapping* $x : [T_0, +\infty[\to H$ *that satisfies*

$$\begin{cases} 0 \in \dot{x}(t) + \partial V(x(t)) + G(t, x(t)) \text{ a.e. } t \in [T_0, +\infty[, \\ x(T_0) = x_0, \\ x(t) \in domV, \forall t \in [T_0, +\infty[, \end{cases} \qquad (\mathcal{P})$$

and such that, for all $r, t \in [T_0, +\infty[$, $r \leq t$ *one has*

$$\int_r^t \|\dot{x}(s)\|^2 ds \leq 2(V(x_0) - V(x(t))) + \int_{T_0}^t (\beta(s) + 1)^2 (1 + \|x(s)\|)^2 ds.$$

Proof. **I) Existence of a local solution for** $(\mathcal{P})$.
Let T be a fixed number greater than T_0, and $I = [T_0, T]$. For each $n \in \mathbb{N}$ and for each $k = 1, ..., n + 1$, define

$$t_k^n := T_0 + (k - 1)\frac{T - T_0}{n},$$

and consider for $k \in \{1,...,n\}$, $\delta_k^n \in [t_k^n, t_{k+1}^n]$ such that

$$\beta(\delta_k^n) \leq \inf_{t\in[t_k^n, t_{k+1}^n[} \beta(t) + 1. \tag{3.1}$$

Now, fix an arbitrary $n \in \mathbb{N}$ and define $x_1^n(t_1^n) = x_0$. Taking $h(t) = g_1^n = Proj(0, G(\delta_k^n, x_0))$ in Theorem 2.7, let $x_1^n : [t_1^n, T] \to H$ be the absolutely continuous solution on $[t_1^n, T]$ to the problem

$$\begin{cases} 0 \in \dot{y}(t) + \partial V(y(t)) + g_1^n \text{ a.e. } t \in [t_1^n, T], \\ y(t_1^n) = x_1^n(t_1^n) = x_0. \end{cases}$$

Next, for each $k \in \{2,...,n\}$, choose $g_k^n = Proj(0, G(\delta_k^n, x_{k-1}^n(t_k^n)))$, and let $x_k^n : [t_k^n, T] \to H$ be the absolutely continuous solution of

$$\begin{cases} 0 \in \dot{y}(t) + \partial V(y(t)) + g_k^n \text{ a.e. } t \in [t_k^n, T], \\ y(t_k^n) = x_{k-1}^n(t_k^n). \end{cases}$$

In view of Theorem 2.7, we have for any $k \in \{1, ...n\}$,

$$\int_r^t \|\dot{x}_k^n(s)\|^2 ds \leq 2(V(x_k^n(t_k^n)) - V(x_k^n(t))) + (t - t_k^n)\|g_k^n\|^2 \tag{3.2}$$

whenever $r, t \in [t_k^n, T], r \leq t$. Now, we define $u_n : [T_0, T] \to H$ by

$$u_n(t) = \begin{cases} x_k^n(t) \text{ if } t \in [t_k^n, t_{k+1}^n[\text{ for some } k \in \{1,...,n\}, \\ x_n^n(T) \text{ if } t = T. \end{cases}$$

Each (u_n) is absolutely continuous on $[T_0, T]$. Consider the mappings $\Theta_n, \Lambda_n : [T_0, T] \to [T_0, T]$ given by

$$\Theta_n(t) = \begin{cases} t_k^n \text{ if } t \in [t_k^n, t_{k+1}^n[\text{ for some } k \in \{1,...,n\}, \\ T \text{ if } t = T. \end{cases}$$

and

$$\Lambda_n(t) = \begin{cases} \delta_k^n \text{ if } t \in [t_k^n, t_{k+1}^n[\text{ for some } k \in \{1,...,n\}, \\ \delta_n^n \text{ if } t = T. \end{cases}$$

Next, define $g_n : [T_0, T] \to H$ by

$$g_n(t) = \begin{cases} g_k^n \text{ if } t \in [t_k^n, t_{k+1}^n[\text{ for some } k \in \{1,...,n\}, \\ g_n^n \text{ if } t = T. \end{cases}$$

Then, for each $n \in \mathbb{N}$, we have the following
(a) $\forall t \in [T_0, T], g_n(t) \in G(\Lambda_n(t), u_n(\Theta_n(t)))$;
(b) $d(0, G(\Lambda_n(t), u_n(\Theta_n(t))) \leq \beta(\Lambda_n(t))(1 + \|u_n(\Theta_n(t))\|)$, that is
$\|g_n(t)\| \leq \beta(\Lambda_n(t))(1 + \|u_n(\Theta_n(t))\|)$;
(c) $u_n(T_0) = x_0$;
(d) $0 \in \dot{u}_n(t) + \partial V(u_n(t)) + g_n(t)$ a.e. $t \in [T_0, T]$, and hence:

$$0 \in \dot{u}_n(t) + \partial V(u_n(t)) + G(\Lambda_n(t), u_n(\Theta_n(t))) \text{ a.e. } t \in [T_0, T].$$

Further, by (3.2) it is not difficult to see that for all $T_0 \leq r \leq t \leq T$,

$$\int_r^t \|\dot{u}_n(s)\|^2 ds \leq 2(V(x_0) - V(u_n(t))) + \int_{T_0}^t \|g_n(s)\|^2 ds, \tag{3.3}$$

thus, using (H_1), (H_4) and (3.1), it comes

$$\int_r^t \|\dot{u}_n(s)\|^2 ds \leq 2(V(x_0) + \alpha(1 + \|u_n(t)\|)) + \int_{T_0}^t (\beta(s) + 1)^2 (1 + \|u_n(\Theta_n(s))\|)^2 ds. \tag{3.4}$$

Let us denote by s_0, c_0, Q_0 some positive constants associated with the pln property of V at x_0, and fix $\eta_0 \in]0, s_0[$. Then, we fix a real number $\varsigma \in]T_0, T]$ such that

$$(\varsigma - T_0)^{\frac{1}{2}}[2(V(x_0) + \alpha(1 + s_0 + \|x_0\|)) + 2(1 + s_0 + \|x_0\|)^2(\|\beta\|_{L^2}^2 + T - T_0))]^{\frac{1}{2}} < \eta_0. \quad (3.5)$$

Then, by estimation (3.4) and (3.5), we obtain

$$u_n([T_0, \varsigma]) \subset \overline{B}(x_0, \eta_0), \forall n \in \mathbb{N}. \quad (3.6)$$

While (3.4) and (3.6) ensure that

$$\sup_{n\in\mathbb{N}} \|\dot{u}_n\|_{L^2_H([T_0,\varsigma])} < +\infty. \quad (3.7)$$

Set

$$A := \sup_{n\in\mathbb{N}} \|\dot{u}_n\|_{L^2_H([T_0,\varsigma])} < +\infty. \quad (3.8)$$

We will show that some subsequence of (u_n) converges uniformly to some local solution of the differential inclusion $(\mathcal{P})$.
By (3.6), the set (u_n) is bounded. Using the Cauchy-Schwartz inequality and (3.8), for any $n \in \mathbb{N}$, and any $s, t \in [T_0, \varsigma]$ such that $s \le t \le \varsigma$ we have

$$\|u_n(t) - u_n(s)\| = \|\int_s^t \dot{u}_n(r)dr\|^2 \le (t-s)^{\frac{1}{2}}(\int_s^t \|\dot{u}_n(r)\|^2 dr)^{\frac{1}{2}} \le (t-s)^{\frac{1}{2}} A,$$

consequently, (u_n) is equicontinuous in $C_H([T_0, \varsigma])$, since by (H_2), V is inf-ball compact, the set $\{u_n(t), n \in \mathbb{N}\}$ is relatively compact in H, in light of the Ascoli's theorem, we can extract a subsequence of (u_n) that converges uniformly on $[T_0, \varsigma]$ to some $u \in C_H([T_0, \varsigma])$. From (3.8), $(\dot{u}_n)$ is bounded in $L^2_H([T_0, \varsigma])$, thus we can extract a subsequence that converges weakly to some function $v \in L^2_H([T_0, \varsigma])$. The absolute continuity of (u_n) implies that $\dot{u}_n = v$.
By (a) and (H_4) one has

$$\sup_{n\in\mathbb{N}} \|g_n\|^2_{L^2_H([T_0,\varsigma])} \le (1 + s_0 + \|x_0\|)^2 \int_{T_0}^{\varsigma} (\beta(s) + 1)^2 ds < +\infty.$$

Hence, we may assume that (g_n) converges weakly in $L^2_H([T_0, \varsigma])$ to a map g.
From the inclusion

$$-\dot{u}_n(t) - g_n(t) \in \partial V(u_n(t)) \text{ for a.e. } t \in [T_0, \varsigma],$$

and the preceding convergence result, invoking Proposition 2.6, we conclude that

$$-\dot{u}(t) - g(t) \in \partial V(u(t)) \text{ for a.e. } t \in [T_0, \varsigma]. \quad (3.9)$$

It remains to show that

$$g(t) \in G(t, u(t)) \text{ for a.e. } t \in [T_0, \varsigma].$$

As (g_n) converges weakly in $L^2_H([T_0, \varsigma])$, using the Mazur's lemma for (g_n) provides a sequence (ξ_n) such that

$$\xi_n \in \bigcap_n \overline{co}\{g_q, q \ge n\}, \quad (3.10)$$

and (ξ_n) converges strongly in $L^2_H([T_0, \varsigma])$ to g. We can extract from (ξ_n) a subsequence which converges a.e. to g. Thus, there is a Lebesgue negligible set $\mathcal{N} \subset [T_0, \varsigma]$ such that for every $t \in [T_0, \varsigma] \backslash \mathcal{N}$

$$g(t) \in \bigcap_{n\ge 0} \overline{\{\xi_m(t) : m \ge n\}} \subset \bigcap_{n\ge 0} \overline{co}\{g_q(t) : q \ge n\}. \quad (3.11)$$

As

$$g_n(t) \in G(\Lambda_n(t), u_n(\Theta_n(t))),$$

for every $n \in \mathbb{N}, t \in [T_0, \varsigma] \backslash \mathcal{N}$ and any $\xi \in H$, one has

$$\langle \xi, g_n(t) \rangle \leq \delta^*(\xi, G(\Lambda_n(t), u_n(\Theta_n(t)))), \tag{3.12}$$

moreover, for each $n \in \mathbb{N}$ and any $t \in [T_0, \varsigma] \backslash \mathcal{N}$, by (3.10) we have

$$\langle \xi, \xi_k(t) \rangle \leq \sup_{q \geq n} \langle \xi, g_q(t) \rangle \ \forall k \geq n. \tag{3.13}$$

Taking the limit in (3.13) as $k \to +\infty$ and using (3.12) one finds

$$\langle \xi, g(t) \rangle \leq \sup_{q \geq n} \langle \xi, g_q(t) \rangle \leq \sup_{q \geq n} \delta^*(\xi, G(\Lambda_q(t), u_q(\Theta_q(t)))),$$

which implies that

$$\langle \xi, g(t) \rangle \leq \limsup_{n \to +\infty} \delta^*(\xi, G(\Lambda_n(t), u_n(\Theta_n(t))),$$

by the convergence of (u_n), (Λ_n), (Θ_n) and the upper hemicontinuity of $G(t, \cdot)$ we get for all $t \in [T_0, \varsigma] \backslash \mathcal{N}$,

$$\langle \xi, g(t) \rangle \leq \delta^*(\xi, G(t, u(t)),$$

by the convexity of $G(t, u(t))$ we can write

$$d(g(t), G(t, w(t)) \leq 0,$$

then we obtain $g(t) \in G(t, w(t))$ a.e. $t \in [T_0, \varsigma]$ because the set $G(t, u(t))$ is closed. The proof is thus completed. Taking (3.9) into account, we conclude that u is an absolutely continuous solution of

$$0 \in \dot{u}(t) + \partial V(u(t)) + G(t, u(t)) \text{ for a.e. } t \in [T_0, \varsigma], \ u(T_0) = x_0,$$

and is a local solution of $(\mathcal{P})$.
Now, letting $n \to +\infty$ in (3.4) yields

$$\int_r^t \|\dot{u}(s)\|^2 ds \leq 2(V(x_0) + \alpha(1 + \|u(t)\|)) + \int_{T_0}^t (\beta(s) + 1)^2 (1 + \|u(s)\|)^2 ds$$

for any $r, t \in [T_0, \varsigma], r \leq t$. Analogously, passing to the limit when $n \to +\infty$ in (3.3) we obtain the estimate

$$V(u(t)) \leq V(x_0) + \frac{1}{2} \int_{T_0}^t (\beta(s) + 1)^2 (1 + \|u(s)\|)^2 ds$$

for any $t \in [T_0, t]$.
II) Existence of a global solution for $(\mathcal{P})$,
Denote by $x : [T_0, T_1[\to H$ with $T_1 \leq +\infty$ the maximal locally absolutely continuous solution of the problem

$$\begin{cases} 0 \in \dot{x}(t) + \partial V(x(t)) + G(t, x(t)) \text{ a.e. } t \in [T_0, T_1[, \\ x(T_0) = x_0, \\ x(t) \in domV, \forall t \in [T_0, T_1[, \end{cases}$$

such that

(i) $V(x(t)) \leq V(x_0) + \frac{1}{2} \int_{T_0}^t (\beta(s) + 1)^2 (1 + \|x(s)\|)^2 ds,$

(ii) $\int_r^t \|\dot{x}(s)\|^2 ds \leq 2(V(x_0) - V(x(t)) + \int_{T_0}^t (\beta(s) + 1)^2 (1 + \|x(s)\|)^2 ds$

for any $r, t \in [T_0, T_1[, r \leq t$.
We have to show that $T_1 = +\infty$. Fix any $t \in [T_0, T_1[$. By virtue of (H_1) and (2), one obtains

$$\int_{T_0}^t \|\dot{x}(s)\|^2 ds \leq 2(V(x_0) + \alpha(1 + \|x(t)\|)) + \int_{T_0}^t (\beta(s) + 1)^2 (1 + \|x(s)\|)^2 ds,$$

and hence

$$\|x(t)-x_0\|^2 \le 2(t-T_0)[V(x_0)+\alpha+\int_{T_0}^{t}(\beta(s)+1)^2 ds]+2(t-T_0)[\alpha\|x(t)\|+\int_{T_0}^{t}(\beta(s)+1)^2\|x(s)\|^2 ds].$$

This entails

$$\|x(t)\|^2-4\alpha(t-T_0)\|x(t)\| \le 2\|x_0\|^2+4(t-T_0)[V(x_0)+\alpha+\int_{T_0}^{t}(\beta(s)+1)^2 ds+\int_{T_0}^{t}(\beta(s)+1)^2\|x(s)\|^2 ds].$$

We hence conclude that

$$\|x(t)\| \le 4\alpha(t-T_0)+2[2\|x_0\|^2+4(t-T_0)(V(x_0)+\alpha+\int_{T_0}^{t}(\beta(s)+1)^2 ds+\int_{T_0}^{t}(\beta(s)+1)^2\|x(s)\|^2 ds]^{\frac{1}{2}},$$

and hence

$$\begin{aligned} \|x(t)\|^2 &\le 8(4\alpha^2(t-T_0)^2+2\|x_0\|^2+4(t-T_0)(V(x_0)+\alpha+\int_{T_0}^{t}(\beta(s)+1)^2 ds)) && (3.14)\\ &+ 32(t-T_0)\int_{T_0}^{t}(\beta(s)+1)^2\|x(s)\|^2 ds. && (3.15)\end{aligned}$$

Applying Gronwall's inequality yields

$$\|x(t)\|^2 \le r(t)+32(t-T_0)\int_{T_0}^{t}\alpha(s)(\beta(s)+1)^2\exp(32\int_{s}^{t}(\beta(\tau)+1)^2(\tau-T_0)d\tau)ds, \quad (3.16)$$

where

$$r(t) := 8(4\alpha^2(t-T_0)^2+2\|x_0\|^2+4(t-T_0)(V(x_0)+\alpha+\int_{T_0}^{t}(\beta(s)+1)^2 ds))$$

for each $t \in [T_0, T_1[$.
Now, to show that $T_1 = +\infty$, we proceed by contradiction. Assume that $T_1 < +\infty$. Then we easily deduce that

$$R_{T_1} := \sup_{t\in[T_0,T_1[}\|x(t)\| < +\infty. \quad (3.17)$$

Then, by (3.16) and (3.17), for any $r, t \in [T_0, T_1[$ with $r \le t$,

$$\|x(t)-x(r)\| \le (t-r)^{\frac{1}{2}}[2V(x_0)+\alpha(1+R_{T_1}))+(1+R_{T_1})^2\int_{T_0}^{T_1}(\beta(s)+1)^2 ds]^{\frac{1}{2}},$$

which implies, by Cauchy's criterion, that $\bar{x} := \lim_{t\uparrow T_1} x(t)$ exists in $(H, \|\cdot\|)$.
As

$$\forall t \in [T_0, T_1[, V(x(t)) \le V(x_0)+\frac{1}{2}(1+R_{T_1})^2\int_{T_0}^{\theta}(\beta(s)+1)^2 ds,$$

in view of (1), the lower semi-continuity of V ensures that $\bar{x} \in domV$ and hence V is pln at $\bar{x}$. Taking T_1 as initial time and $\bar{x}$ as initial value, under our assumption, results obtained in the step $I)$ above guarantees that there exist $\delta > 0$ and an absolutely continuous mapping $y : [T_1, T_1+\delta] \to H$ satisfying

$$\begin{cases} 0 \in \dot{y}(t)+\partial V(y(t))+G(t,y(t)) \text{ a.e. } t \in [T_1, T_1+\delta],\\ y(T_1)=\bar{x},\\ y(t) \in domV, \forall t \in [T_1, T_1+\delta],\end{cases}$$

for any $r, t \in [T_1, T_1+\delta], r \le t$,

$$\int_{r}^{t}\|\dot{y}(s)\|^2 ds \le 2(V(\bar{x})-V(y(t)))+\int_{T_1}^{t}(\beta(s)+1)^2(1+\|y(s)\|^2)ds,$$

$$V(y(t) \leq V(\bar{x}) + \frac{1}{2}\int_{T_1}^{t} (\beta(s)+1)^2(1+\|y(s)\|^2)ds.$$

As a result, defining $\widetilde{x} : [T_0, T_1+\delta[\to H$ as

$$\widetilde{x}(t) = \begin{cases} x(t) \text{ if } t \in [T_0, T_1[, \\ y(t) \text{ if } t \in [T_1, T_1+\delta[. \end{cases}$$

we see that $\widetilde{x}$ is absolutely continuous on $[T_0, T_1+\delta[$ and satisfies

$$\begin{cases} 0 \in \dot{\widetilde{x}}(t) + \partial V(\widetilde{x}(t)) + G(t, \widetilde{x}(t)) \text{ a.e. } t \in [T_0, T_1+\delta], \\ \widetilde{x}(T_0) = x_0, \end{cases}$$

with

$$\int_r^t \|\dot{\widetilde{x}}(s)\|^2 ds \leq 2(V(x_0) - V(\widetilde{x}(t))) + \int_{T_0}^{t} (\beta(s)+1)^2(1+\|\widetilde{x}(s)\|^2)ds,$$

and

$$V(\widetilde{x})(t) \leq V(x_0) + \frac{1}{2}\int_{T_0}^{t} (\beta(s)+1)^2(1+\|\widetilde{x}(s)\|^2)ds,$$

for any $r, t \in [T_0, T_1+\delta], r \leq t$. This means that $\widetilde{x}$ is an extension of x on $[T_0, T_1+\delta]$ which contradicts the fact that x is maximal. We conclude that $T_1 = +\infty$, and x is a global solution of $(\mathcal{P})$ on $[T_0, +\infty[$. □

Remark 3.2. Note that

(i) when $V(t,\cdot)$ is the indicator function of some ρ-prox-regular set $C(t)$, that is,

$$V(t,x) := \begin{cases} 0 \text{ if } & x \in C(t), \\ +\infty & \text{otherwise}, \end{cases} \tag{3.18}$$

we obtain that $V(t,\cdot)$ is pln and problem $(\mathcal{P})$ becomes the well known perturbed sweeping process.

(ii) Composite functions defined on finite dimensional spaces are pln whenever a qualification condition is fulfilled. More precisely: letting $f : \mathbb{R}^q \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function and $F : \mathbb{R}^p \to \mathbb{R}^q$ be a mapping twice continuously differentiable near a point $\bar{x}$ with $F(\bar{x}) \in dom f$, if the qualification condition

$$N(dom f; F(\bar{x})) \cap \ker(\nabla F(\bar{x})^*) = \{0\}$$

is satisfied (where $\nabla F(\bar{x})^*$ denotes the adjoint of $\nabla F(\bar{x})$ and $N(dom f;\cdot)$ is the normal cone to $dom f$), then $f \circ F$ is pln at $\bar{x}$.

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