Triple higher Derivable maps on a class of nonassociative rings

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Abstract. Let us assume $\mathcal{R}$ be an alternative ring (not necessarily with the identity element). In the present paper, it is shown that if $\mathcal{R}$ admits a Jordan triple higher derivable mappings, then under certain assumptions it is additive.

1 Introduction

The study of nonassociative rings has received fair amount of attention during the last few decades. Many authors studied nonassociative algebras (see [12] and references therein), in particular, alternative rings after the discovery of their connection with the theory of projective planes. Let $\mathcal{R}$ be an alternative ring unless otherwise mentioned. For any $x, y \in \mathcal{R}$, $x \circ y = xy + yx$ will denote the Jordan product on $\mathcal{R}$. A ring $\mathcal{R}$ (not necessarily associative or commutative) is called an alternative ring if $\mathcal{R}$ satisfies $x^2y = x(xy)$ and $yx^2 = (yx)x$ for all $x, y \in \mathcal{R}$ and flexible if $x(yx) = (xy)x$ holds for all $x, y \in \mathcal{R}$. It can be easily seen that all associative rings are alternative and any alternative ring is flexible. Hence the product xyx will denote the product $x(yx)$ or $(xy)x$ for all $x, y \in \mathcal{R}$. An alternative ring $\mathcal{R}$ is said to be k-torsion free if $kx = 0$ implies that $x = 0$ for $k \in \mathbb{N}$ and for all $x \in \mathcal{R}$.

Remark 1.1. [5, Theorem 1.1] Let $\mathcal{R}$ be a 3-torsion free alternative ring. So $\mathcal{R}$ is a prime ring if and only if $x\mathcal{R} \cdot y = 0$ (or $x \cdot \mathcal{R}y = 0$) implies $x = 0$ or $y = 0$ for $x, y \in \mathcal{R}$.

In the remaining part of the paper, let $\mathcal{R}$ be an alternative ring with a nontrivial idempotent e_1 and formally set $e_0 = 1 - e_1$ ($\mathcal{R}$ need not have an identity element). It can be easily seen that $(e_ix)e_j = e_i(xe_j)$, where $i, j = 0, 1$ for all $x \in \mathcal{R}$. By Pierce decomposition $\mathcal{R} = \mathcal{R}_{11} + \mathcal{R}_{10} + \mathcal{R}_{01} + \mathcal{R}_{00}$, where $\mathcal{R}_{ij} = e_i\mathcal{R}e_j$ for $i, j \in \{0, 1\}$. The notion x_{ij} denote an arbitrary element of $\mathcal{R}_{ij}$ and any element $x \in \mathcal{R}$ can be expressed as $x = x_{11} + x_{10} + x_{01} + x_{00}$. Pierce decomposition of alternative rings satisfy the following relations:

(i) $\mathcal{R}_{ij}\mathcal{R}_{jk} \subseteq \mathcal{R}_{ik}$, when $i, j, k \in \{0, 1\}$,

(ii) $\mathcal{R}_{ij}\mathcal{R}_{ij} \subseteq \mathcal{R}_{ji}$ with $x_{ij}^2 = x_{ij}y_{ij} + y_{ij}x_{ij} = 0$,

(iii) $\mathcal{R}_{ij}\mathcal{R}_{kl} = 0$, $(j \neq k)$, $(i, j) \neq (k, l)$.

for all $x_{ij}, y_{ij} \in \mathcal{R}_{ij}$.

A map $d : \mathcal{R} \to \mathcal{R}$ (not necessarily additive) is said to be a derivable map if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in \mathcal{R}$. A map $d : \mathcal{R} \to \mathcal{R}$ (not necessarily additive) is said to be a Jordan derivable (resp. Jordan triple derivable) map if $d(x^2) = d(x)x + xd(x)$ (resp. $d(xyx) = d(x)(yx) + xd(y)x + (xy)d(x)$) for all $x, y \in \mathcal{R}$. If we consider $d : \mathcal{R} \to \mathcal{R}$ as an additive map in the above definitions, then a derivable map (resp. Jordan (triple) derivable map) is called a derivation (resp. Jordan (triple) derivation) of $\mathcal{R}$.

In recent years, characterizing the interrelation between the multiplicative and additive structure of a ring is an interesting topic and has received fair amount of attention of many mathematicians (see for reference [4, 5, 10, 11] where further references can be found). It was Martindale [10], who first studied this problem and raised the question : When is a multiplicative map additive? He answered this question for a multiplicative isomorphism of an associative ring with a family of idempotents under certain assumptions. More precisely, he proved the following result:

Theorem 1.2. *[10, Theorem 1] Let $\mathcal{R}$ be a ring (not necessarily with identity element) containing a family $\{e_\alpha : \alpha \in \Lambda\}$ of idempotents which satisfies :*

(i) $x\mathcal{R} = \{0\}$ *implies* $x = 0$,

(ii) *if* $e_\alpha \mathcal{R} x = \{0\}$ *for each* $\alpha \in \Lambda$, *then* $x = 0$ *(and hence* $\mathcal{R}x = \{0\}$ *implies* $x = 0$),

(iii) *for each* $\alpha \in \Lambda$, $e_\alpha x e_\alpha \mathcal{R}(1 - e_\alpha) = \{0\}$ *implies* $e_\alpha x e_\alpha = \{0\}$.

Then any multiplicative bijective map from $\mathcal{R}$ onto an arbitrary ring $\mathcal{R}'$ is additive.

Inspired by the above result due to Martindale [10], Daif [4] investigated this problem for multiplicative derivations on associative rings and obtained the following result:

Theorem 1.3. *[4, Theorem 1] Let $\mathcal{R}$ be an associative ring (not necessarily with identity element) containing a nontrivial idempotent e which satisfies:*

(i) $x\mathcal{R} = \{0\}$ *implies* $x = 0$,

(ii) $e\mathcal{R}x = \{0\}$ *implies* $x = 0$ *(hence* $\mathcal{R}x = \{0\}$ *implies* $x = 0$),

(iii) $exe\mathcal{R}(1 - e) = \{0\}$ *implies* $exe = 0$.

If d is a multiplicative derivation of $\mathcal{R}$, then d is additive.

Very recently, Rodrigues et al. [11] initiated the study of this problem for nonassociative rings named as alternative rings and obtained the result due to Daif [4] in this setting for derivable maps. Further this problem was studied by Ferreira and Ferreira [5] for Jordan triple derivable map on alternative rings.

The concept of derivation was extended to higher derivation by Hasse and Schmidt [9]. Let $\mathbb{N}$ be the set of all non-negative integers and $D = \{d_n\}_{n \in \mathbb{N}}$ be a family of additive maps on a ring $\mathcal{R}$ such that $d_0 = I_{\mathcal{R}}$, the identity map on $\mathcal{R}$. Then D is said to be a

(i) higher derivation if $d_n(xy) = \sum\limits_{i+j=n} d_p(x)d_q(y)$,

(ii) Jordan higher derivation if $d_n(x^2) = \sum\limits_{i+j=n} d_p(x)d_q(x)$,

(iii) Jordan triple higher derivation if $d_n(xyx) = \sum\limits_{p+q+r=n} d_p(x)d_q(y)d_r(x)$

on a ring $\mathcal{R}$ for each $n \in \mathbb{N}$ and holds for all $x, y \in \mathcal{R}$. Further, several results for derivation were extended for higher derivations (for reference see [2, 3, 6–8] and references therein).

Inspired by these works, Ashraf et al. [1, 2] studied the notion of Jordan triple (higher) derivable map on triangular algebras. Furthermore, they provided a sufficient condition on triangular algebras under which a Jordan triple (higher) derivable map becomes a (higher) derivation. Motivated by these observations, in this article, we discuss the question raised by Martindale [10] for alternative rings with Jordan triple higher derivable maps and prove that every Jordan triple higher derivable map is additive under certain assumptions.

2 Jordan triple higher derivable maps

We will use the following result throughout the rest of paper whenever needed without specific mention.

Theorem 2.1. *[5, Theorem 2.1] Let $\mathcal{R}$ be an alternative ring containing a non-trivial idempotent e_1 and $\mathcal{R} = \mathcal{R}_{11} \oplus \mathcal{R}_{10} \oplus \mathcal{R}_{01} \oplus \mathcal{R}_{00}$, the Peirce decomposition of $\mathcal{R}$, relative to e_1, satisfying:*

(i) If $(e_1 a e_1)x_{10} = 0$ for all $x_{10} \in \mathcal{R}_{10}$, then $(e_1 a e_1) = 0$;

(ii) If $x_{10}(e_0 a e_0) = 0$ for all $x_{10} \in \mathcal{R}_{10}$, then $(e_0 a e_0) = 0$;

(iii) If $x_{ij} a x_{ij} = 0$ for all $x_{ij} \in \mathcal{R}_{ij}$, then $(e_j a e_i) = 0$ for $i, j \in \{1, 0\}$.

If $d : \mathcal{R} \to \mathcal{R}$ is a Jordan triple multiplicative derivation, then d is additive.

Throughout this section, We assume that the family $D = \{d_n\}_{n \in \mathbb{N}}$ of mappings $d_n : \mathcal{R} \to \mathcal{R}$ such that $d_0 = I_{\mathcal{R}}$ is a Jordan triple higher derivable map, i.e.,

$$d_n(xyx) = \sum_{p+q+r=n} d_p(x)d_q(y)d_r(x) \text{ for all } x, y \in \mathcal{R}. \tag{2.1}$$

We may see that for n = 1, (2.1) reduces to $d_1(xyx) = d_1(x)yx + xd_1(y)x + xyd_1(x)$ for all $x, y \in \mathcal{R}$. In view of Theorem 2.1, it is clear that d_1 is additive. At this moment, we extend Theorem 2.1 for higher derivations by induction on n and obtained the following result:

Theorem 2.2. *Let $\mathcal{R}$ be an alternative ring containing a non-trivial idempotent e_1 and $\mathcal{R} = \mathcal{R}_{11} \oplus \mathcal{R}_{10} \oplus \mathcal{R}_{01} \oplus \mathcal{R}_{00}$, the Peirce decomposition of $\mathcal{R}$, relative to e_1, satisfying:*

(i) If $(e_1 a e_1)x_{10} = 0$ for all $x_{10} \in \mathcal{R}_{10}$, then $(e_1 a e_1) = 0$;

(ii) If $x_{10}(e_0 a e_0) = 0$ for all $x_{10} \in \mathcal{R}_{10}$, then $(e_0 a e_0) = 0$;

(iii) If $x_{ij} a x_{ij} = 0$ for all $x_{ij} \in \mathcal{R}_{ij}$, then $(e_j a e_i) = 0$ for $i, j \in \{1, 0\}$.

If a family $D = \{d_n\}_{n \in \mathbb{N}}$ of mappings $d_n : \mathcal{R} \to \mathcal{R}$ such that $d_0 = I_{\mathcal{R}}$ satisfies

$$d_n(xyx) = \sum_{p+q+r=n} d_p(x)d_q(y)d_r(x)$$

for all $x, y \in \mathcal{R}$, then d_n is additive for each $n \in \mathbb{N}$ and hence $D = \{d_n\}_{n \in \mathbb{N}}$ is a Jordan triple higher derivation on $\mathcal{R}$.

In order to prove the above result, we start with the following sequence of facts :

Fact 2.1. $d_n(0) = 0$.

Proof. For any $x \in \mathcal{R}$, we find that

$$d_n(0) = d_n(0x0) = d_n(0)(x0) + 0d_n(x)0 + (0x)d_n(0) + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(0)d_q(x)d_r(0)$$

$$= 0.$$

$\square$

Fact 2.2. For any $x_{00} \in \mathcal{R}_{00}, y_{01} \in \mathcal{R}_{01}, z_{10} \in \mathcal{R}_{10}$ and $w_{11} \in \mathcal{R}_{11}$,

$$d_n(x_{00} + y_{01} + z_{10} + w_{11}) = d_n(x_{00}) + d_n(y_{01}) + d_n(z_{10}) + d_n(w_{11}).$$

Proof. For any $t_{ij} \in \mathcal{R}_{ij}$, where $i, j = 0, 1$, we have

$$
\begin{aligned}
&t_{ij}[d_n(x_{00} + y_{01} + z_{10} + w_{11}) - d_n(x_{00}) - d_n(y_{01}) - d_n(z_{10}) - d_n(w_{11})]t_{ij} \\
&= d_n\big(t_{ij}(x_{00} + y_{01} + z_{10} + w_{11})t_{ij}\big) - d_n(t_{ij}x_{00}t_{ij}) - d_n(t_{ij}y_{01}t_{ij}) - d_n(t_{ij}z_{10}t_{ij}) \\
&\quad - d_n(t_{ij}w_{11}t_{ij}) - d_n(t_{ij})\big((x_{00} + y_{01} + z_{10} + w_{11})t_{ij}\big) + d_n(t_{ij})(x_{00}t_{ij}) \\
&\quad + d_n(t_{ij})(z_{10}t_{ij}) + d_n(t_{ij})(w_{11}t_{ij}) - \big(t_{ij}(x_{00} + y_{01} + z_{10} + w_{11})\big)d_n(t_{ij}) \\
&\quad + (t_{ij}x_{00})d_n(t_{ij}) + (t_{ij}y_{01})d_n(t_{ij}) + (t_{ij}z_{10})d_n(t_{ij}) + (t_{ij}w_{11})d_n(t_{ij}) \\
&\quad - \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(t_{ij})d_q(x_{00} + y_{01} + z_{10} + w_{11})d_r(t_{ij}) + d_n(t_{ij})(y_{01}t_{ij}) \\
&\quad + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(t_{ij})d_q(x_{00})d_r(t_{ij}) + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(t_{ij})d_q(y_{01})d_r(t_{ij}) \\
&\quad + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(t_{ij})d_q(z_{10})d_r(t_{ij}) + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(t_{ij})d_q(w_{11})d_r(t_{ij}) \\
&= 0.
\end{aligned}
$$

In view of our assumption the above relation yields

$$
\begin{aligned}
e_0[d_n(x_{00} + y_{01} + z_{10} + w_{11}) - d_n(x_{00}) - d_n(y_{01}) - d_n(z_{10}) - d_n(w_{11})]e_0 &= 0, \\
e_0[d_n(x_{00} + y_{01} + z_{10} + w_{11}) - d_n(x_{00}) - d_n(y_{01}) - d_n(z_{10}) - d_n(w_{11})]e_1 &= 0, \\
e_1[d_n(x_{00} + y_{01} + z_{10} + w_{11}) - d_n(x_{00}) - d_n(y_{01}) - d_n(z_{10}) - d_n(w_{11})]e_0 &= 0, \\
e_1[d_n(x_{00} + y_{01} + z_{10} + w_{11}) - d_n(x_{00}) - d_n(y_{01}) - d_n(z_{10}) - d_n(w_{11})]e_1 &= 0.
\end{aligned}
$$

Therefore, we have

$$
d_n(x_{00} + y_{01} + z_{10} + w_{11}) = d_n(x_{00}) + d_n(y_{01}) + d_n(z_{10}) + d_n(w_{11}).
$$

$\square$

Fact 2.3. For any $x_{10}, y_{10} \in \mathcal{R}_{10}$; $x_{01}, z_{01} \in \mathcal{R}_{01}$ and $y_{00}, z_{00} \in \mathcal{R}_{00}$,

(i) $d_n(x_{10} + y_{10}z_{00}) = d_n(x_{10}) + d_n(y_{10}z_{00})$,

(ii) $d_n(x_{01} + y_{00}z_{01}) = d_n(x_{01}) + d_n(y_{00}z_{01})$.

Proof. (i) For any $x_{10}, y_{10} \in \mathcal{R}_{10}$ and $z_{00} \in \mathcal{R}_{00}$,

$$
\begin{aligned}
&d_n(e_1) + d_n(x_{10} + y_{10}z_{00}) \\
&= d_n(e_1 + x_{10} + y_{10}z_{00}) \\
&= d_n\big((e_1 + x_{10} + z_{00})(e_1 + y_{10})(e_1 + x_{10} + z_{00})\big) \\
&= d_n(e_1 + x_{10} + z_{00})\big((e_1 + y_{10})(e_1 + x_{10} + z_{00})\big) \\
&\quad + (e_1 + x_{10} + z_{00})d_n(e_1 + y_{10})(e_1 + x_{10} + z_{00}) \\
&\quad + \big((e_1 + x_{10} + z_{00})(e_1 + y_{10})\big)d_n(e_1 + x_{10} + z_{00}) \\
&\quad + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(e_1 + x_{10} + z_{00})d_q(e_1 + y_{10})d_r(e_1 + x_{10} + z_{00})
\end{aligned}
$$

$$d_n(e_1) + d_n(x_{10} + y_{10}z_{00})$$

$$= [d_n(e_1) + d_n(x_{10}) + d_n(z_{00})]((e_1 + y_{10})(e_1 + x_{10} + z_{00}))$$

$$+(e_1 + x_{10} + z_{00})[d_n(e_1) + d_n(y_{10})](e_1 + x_{10} + z_{00})$$

$$+((e_1 + x_{10} + z_{00})(e_1 + y_{10}))[d_n(e_1) + d_n(x_{10}) + d_n(z_{00})]$$

$$+ \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} [d_p(e_1) + d_p(x_{10}) + d_p(z_{00})][d_q(e_1) + d_q(y_{10})][d_r(e_1) + d_r(x_{10}) + d_r(z_{00})]$$

$$= d_n((e_1 + x_{10} + z_{00})e_1(e_1 + x_{10} + z_{00})) + d_n((e_1 + x_{10} + z_{00})y_{10}(e_1 + x_{10} + z_{00}))$$

$$= d_n(e_1 + x_{10}) + d_n(y_{10}z_{00})$$

$$= d_n(e_1) + d_n(x_{10}) + d_n(y_{10}z_{00}).$$

This implies that $d_n(x_{10} + y_{10}z_{00}) = d_n(x_{10}) + d_n(y_{10}z_{00})$.

(ii) Similar to (i).   □

Fact 2.4. For any $x_{10}, y_{10} \in \mathcal{R}_{10}$ and $x_{01}, y_{01} \in \mathcal{R}_{01}$,

(i) $d_n(x_{10} + y_{10}) = d_n(x_{10}) + d_n(y_{10})$,

(ii) $d_n(x_{01} + y_{01}) = d_n(x_{01}) + d_n(y_{01})$.

Proof. (i) For any $x_{10}, y_{10} \in \mathcal{R}_{10}$, using Fact 2.3, we obtain

$$t_{ij}[d_n(x_{10} + y_{10}) - d_n(x_{10}) - d_n(y_{10})]t_{ij}$$

$$= d_n(t_{ij}(x_{10} + y_{10})t_{ij}) - d_n(t_{ij}x_{10}t_{ij}) - d_n(t_{ij}y_{10}t_{ij}) - d_n(t_{ij})((x_{10} + y_{10})t_{ij})$$

$$+d_n(t_{ij})(x_{10}t_{ij}) + d_n(t_{ij})(y_{10}t_{ij}) - (t_{ij}(x_{10} + y_{10}))d_n(t_{ij}) + (t_{ij}x_{10})d_n(t_{ij})$$

$$+(t_{ij}y_{10})d_n(t_{ij}) - \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(t_{ij})d_q(x_{10} + y_{10})d_r(t_{ij})$$

$$+ \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(t_{ij})d_q(x_{10})d_r(t_{ij}) + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(t_{ij})d_q(y_{10})d_r(t_{ij})$$

$$= 0.$$

By our assumption the above relation yields that

$$e_1[d_n(x_{10} + y_{10}) - d_n(x_{10}) - d_n(y_{10})]e_0 = 0,$$
$$e_0[d_n(x_{10} + y_{10}) - d_n(x_{10}) - d_n(y_{10})]e_0 = 0,$$
$$e_1[d_n(x_{10} + y_{10}) - d_n(x_{10}) - d_n(y_{10})]e_0 = 0,$$
$$e_1[d_n(x_{10} + y_{10}) - d_n(x_{10}) - d_n(y_{10})]e_1 = 0.$$

Therefore,

$$d_n(x_{10} + y_{10}) = d_n(x_{10}) + d_n(y_{10}).$$

(ii) Similar to (i).   □

Fact 2.5. For any $x_{10} \in \mathcal{R}_{10}; y_{01} \in \mathcal{R}_{01}$ and $x_{00}, y_{00} \in \mathcal{R}_{00}$,

(i) $d_n(x_{11}y_{10}) = d_n(e_1)(x_{11}y_{10}) + d_n(y_{10})x_{11} + e_1d_n(x_{11})y_{10} + y_{10}d_n(x_{11})e_1$

$$+x_{11}d_n(y_{10}) + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(e_1)d_q(x_{11})d_r(y_{10}) + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(y_{10})d_q(x_{11})d_r(e_1),$$

(ii) $d_n(x_{00}y_{01}) = d_n(e_0)(x_{00}y_{01}) + d_n(y_{01})x_{00} + e_0d_n(x_{00})y_{01} + y_{01}d_n(x_{00})e_0$

$$+x_{00}d_n(y_{01}) + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(e_0)d_q(x_{00})d_r(y_{01}) + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(y_{01})d_q(x_{00})d_r(e_0).$$

Proof. (i) For any $y_{01} \in \mathcal{R}_{10}$ and $x_{00} \in \mathcal{R}_{00}$, we find that

$$d_n(x_{11}) + d_n(x_{11}y_{10})$$
$$= d_n(x_{11} + x_{11}y_{10})$$
$$= d_n\big((e_1 + y_{10})x_{11}(e_1 + y_{10})\big)$$
$$= d_n\big(e_1 + y_{10}\big)(x_{11}(e_1 + y_{10})) + (e_1 + y_{10})d_n(x_{11})(e_1 + y_{10})$$
$$\quad + \big((e_1 + y_{10})x_{11}\big)d_n(e_1 + y_{10}) + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(e_1 + y_{10})d_q(x_{11})d_r(e_1 + y_{10})$$
$$= [d_n(e_1) + d_n(y_{10})]\big(x_{11}(e_1 + y_{10})\big) + (e_1 + y_{10})d_n(x_{11})(e_1 + y_{10})$$
$$\quad + \big((e_1 + y_{10})x_{11}\big)[d_n(e_1) + d_n(y_{10})]$$
$$\quad + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} [d_p(e_1) + d_p(y_{10})]d_q(x_{11})[d_r(e_1) + d_r(y_{10})]$$
$$= d_n(e_1 x_{11} e_1) + d_n(y_{10}x_{11}y_{10}) + d_n(e_1)(x_{11}y_{10}) + e_1 d_n(x_{11})y_{10}$$
$$\quad + y_{10}d_n(x_{11})e_1 + d_n(y_{10})x_{11} + x_{11}d_n(y_{10})$$
$$\quad + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(e_1)d_q(x_{11})d_r(y_{10}) + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(y_{10})d_q(x_{11})d_r(e_1)$$
$$= d_n(x_{11}) + d_n(e_1)(x_{11}y_{10}) + d_n(y_{10})x_{11} + e_1 d_n(x_{11})y_{10} + y_{10}d_n(x_{11})e_1$$
$$\quad + x_{11}d_n(y_{10}) + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(e_1)d_q(x_{11})d_r(y_{10}) + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(y_{10})d_q(x_{11})d_r(e_1).$$

Hence

$$d_n(x_{11}y_{10}) = d_n(e_1)(x_{11}y_{10}) + d_n(y_{10})x_{11} + e_1 d_n(x_{11})y_{10} + y_{10}d_n(x_{11})e_1 + x_{11}d_n(y_{10})$$
$$+ \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(e_1)d_q(x_{11})d_r(y_{10}) + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(y_{10})d_q(x_{11})d_r(e_1).$$

(ii) Similar to (i). □

Fact 2.6. For any $x_{00}, y_{00} \in \mathcal{R}_{00}$ and $x_{11}, y_{11} \in \mathcal{R}_{11}$,

(i) $d_n(x_{11} + y_{11}) = d_n(x_{11}) + d_n(y_{11})$,

(ii) $d_n(x_{00} + y_{00}) = d_n(x_{00}) + d_n(y_{00})$.

Proof. (i) For any $t_{ij} \in \mathcal{R}_{ij}$ where $i, j = 0, 1$ but $(i, j) \neq (1, 1)$, we have

$$t_{ij}[d_n(x_{11} + y_{11}) - d_n(x_{11}) - d_n(y_{11})]t_{ij}$$
$$= d_n\big(t_{ij}(x_{11} + y_{11})t_{ij}\big) - d_n(t_{ij}x_{11}t_{ij}) - d_n(t_{ij}y_{11}t_{ij}) - d_n(t_{ij})\big((x_{11} + y_{11})t_{ij}\big)$$
$$\quad + (t_{ij}y_{11})d_n(t_{ij}) + d_n(t_{ij})(x_{11}t_{ij}) + (t_{ij}x_{11})d_n(t_{ij}) + d_n(t_{ij})(y_{11}t_{ij})$$
$$\quad - \big(t_{ij}(x_{11} + y_{11})\big)d_n(t_{ij}) - \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(t_{ij})d_q(x_{11} + y_{11})d_r(t_{ij})$$
$$\quad + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(t_{ij})d_q(x_{11})d_r(t_{ij}) + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(t_{ij})d_q(y_{11})d_r(t_{ij})$$
$$= 0.$$

By assumption this gives that

$$
\begin{aligned}
e_1[d_n(x_{11} + y_{11}) - d_n(x_{11}) - d_n(y_{11})]e_1 &= 0, \\
e_0[d_n(x_{11} + y_{11}) - d_n(x_{11}) - d_n(y_{11})]e_1 &= 0, \\
e_1[d_n(x_{11} + y_{11}) - d_n(x_{11}) - d_n(y_{11})]e_0 &= 0.
\end{aligned}
$$

Also, for any $t_{10} \in \mathcal{R}_{10}$ and $x_{11}, y_{11} \in \mathcal{R}_{11}$, we have

$$
\begin{aligned}
d_n&\big((x_{11} + y_{11})t_{10}\big) \\
&= d_n(e_1)\big((x_{11} + y_{11})t_{10}\big) + d_n(t_{10})(x_{11} + y_{11}) + e_1 d_n(x_{11} + y_{11})t_{10} \\
&\quad + t_{10} d_n(x_{11} + y_{11})e_1 + (x_{11} + y_{11})d_n(t_{10}) \\
&\quad + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(e_1)d_q(x_{11} + y_{11})d_r(t_{10}) + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(t_{10})d_q(x_{11} + y_{11})d_r(e_1).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
d_n&\big((x_{11} + y_{11})t_{10}\big) \\
&= d_n(x_{11}t_{10} + y_{11}t_{10}) \\
&= d_n(x_{11}t_{10}) + d_n(y_{11}t_{10}) \\
&= d_n(e_1)(x_{11}t_{10}) + d_n(t_{10})x_{11} + e_1 d_n(x_{11})t_{10} + t_{10}d_n(x_{11})e_1 + x_{11}d_n(t_{10}) \\
&\quad + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(e_1)d_q(x_{11})d_r(t_{10}) + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(t_{10})d_q(x_{11})d_r(e_1) \\
&\quad + d_n(e_1)(y_{11}t_{10}) + d_n(t_{10})y_{11} + e_1 d_n(y_{11})t_{10} + t_{10}d_n(y_{11})e_1 + y_{11}d_n(t_{10}) \\
&\quad + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(e_1)d_q(y_{11})d_r(t_{10}) + \sum_{\substack{p+q+r=n \\ 0<p,q,r<n}} d_p(t_{10})d_q(y_{11})d_r(e_1).
\end{aligned}
$$

Now comparing the above two expressions, we find that

$$
e_1[d_n(x_{11} + y_{11}) - d_n(x_{11}) - d_n(y_{11})]e_1 = 0.
$$

Therefore,

$$
d_n(x_{11} + y_{11}) = d_n(x_{11}) + d_n(y_{11}).
$$

(ii) Similar to (i).   □

Proof of Theorem 2.2. For any $x, y \in \mathcal{R}$, let $x = x_{11} + x_{10} + x_{01} + x_{00}$ and $y = y_{11} + y_{10} + y_{01} + y_{00}$ for all $x_{ij}, y_{ij} \in \mathcal{R}_{ij}$ where $i, j \in \{0, 1\}$. Using Facts 2.2, 2.4, 2.6, we have

$$
\begin{aligned}
d_n(x + y) &= d_n(x_{11} + x_{10} + x_{01} + x_{00} + y_{11} + y_{10} + y_{01} + y_{00}) \\
&= d_n(x_{11} + y_{11}) + d_n(x_{01} + y_{01}) + d_n(x_{01} + y_{01}) + d_n(x_{00} + y_{00}) \\
&= d_n(x_{11}) + d_n(y_{11}) + d_n(x_{01}) + d_n(y_{01}) + d_n(x_{01}) + d_n(y_{01}) \\
&\quad + d_n(x_{00}) + d_n(y_{00}) \\
&= d_n(x_{11} + x_{10} + x_{01} + x_{00}) + d_n(y_{11} + y_{10} + y_{01} + y_{00}) \\
&= d_n(x) + d_n(y).
\end{aligned}
$$

This implies that $D = \{d_n\}_{n \in \mathbb{N}}$ is additive on $\mathcal{R}$ and hence $D = \{d_n\}_{n \in \mathbb{N}}$ is a Jordan triple higher derivation on $\mathcal{R}$.   □

Immediately in view of Remark 1.1, we can obtain the following result:

Corollary 2.3. *Let $\mathcal{R}$ be a 3-torsion free prime unital alternative ring with a nontrivial idempotent. Then every Jordan triple higher derivable map is additive.*

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