Palestine Journal of Mathematics

Vol 14(4)(2025) , 826–848 © Palestine Polytechnic University-PPU 2025

CAUCHY PROBLEM FOR THE SYSTEM OF ELASTICITY THEORY

I. E. Niyozov, D. A. Juraev, R. Efendiev and A. Kashuri

Communicated by Geeta Arora

MSC 2020 Classifications: Primary 35J46; Secondary 35J56.

Keywords and phrases: The Cauchy problem, regularization, factorization, regular solution, fundamental solution.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper

Corresponding Author: A. Kashuri

Abstract. This article addresses the ill-posed Cauchy problem for the system of equations arising in the moment theory of elasticity. It explores the analytical continuation of solutions within a three-dimensional domain, utilizing data specified only on a portion of the boundary. To resolve the instability inherent in such problems, the authors develop solvability criteria and construct a Carleman matrix, enabling a stable and constructive approach to the problem. The study introduces integral representations and expansion techniques using fundamental solutions and harmonic function bases with double orthogonality. These methods lead to practical criteria and explicit formulas for solution reconstruction, which are of significant value in applications involving elasticity, inverse problems, and engineering diagnostics.

1 Introduction

The Cauchy problem for systems of partial differential equations, particularly those arising in the theory of elasticity, has long posed challenges due to its inherent ill-posed nature. This article focuses on the Cauchy problem within the framework of the moment theory of elasticity - a model that more accurately captures the behavior of elastic media under rotational and translational stresses. The primary difficulty in such problems lies in extending known values of the solution and its stresses from a part of the boundary into the entire spatial domain, where even small data errors can lead to significant deviations in the solution. Building upon foundational work by researchers such as Tikhonov, Lavrent'ev, and Yarmukhamedov, the authors develop a constructive method for solving this problem. They introduce a Carleman matrix tailored for the moment theory system, allowing for an explicit representation of solutions and a rigorous solvability criterion. The work also incorporates the use of fundamental solutions and Green-type integral formulas to represent the unknown functions in terms of boundary data. To ensure the analytic continuation of the solution, the authors derive a condition involving the convergence of a series based on harmonic functions with double orthogonality. The novelty of the article lies in the extension of classical Carleman-type techniques to a more complex elasticity system involving additional degrees of freedom. This leads to a new integral formula for solution reconstruction and stability estimates, making the proposed methods valuable not only in theoretical studies but also in applied fields such as geomechanics, materials science, and structural diagnostics. By addressing both the theoretical underpinnings and practical aspects of the problem, the article makes a meaningful contribution to the ongoing development of regularization techniques for ill-posed problems.

The present study builds upon the foundational concepts introduced by A. N. Tikhonov [1] and M.M. Lavrent'ev [2], who pioneered methods for addressing ill-posed problems through conditional well-posedness and regularization. These classical approaches serve as the theoretical basis for the stability analysis and approximation strategies developed in this paper. The techniques used for analytic continuation and solvability criteria draw from the ideas of V. A. Fock and F. M. Kuni [3], who established important conditions for reconstructing analytic functions from boundary data. In addition, the methods applied here reflect the analytical advancements of Lurie [4] in elasticity theory and the role of fundamental solutions discussed in the comprehensive framework presented in [5]. The article also benefits from the spectral and harmonic function expansions found in the works of Vekua [6], where boundary value problems for elliptic systems were systematically analyzed. The use of Carleman functions and integral representations is inspired by the ideas of Lavrent'ev and Romanov [7], as well as Mergelyan's approximation theorems [8], which support the construction of special kernel functions. Furthermore, the generalized potential theory methods and integral equation techniques are influenced by S.G. Mikhlin [9] and S.M. Nikol'skii [10], whose contributions to functional analysis and approximation in mathematical physics have been instrumental in shaping the mathematical tools employed in this paper. Altogether, these references form the theoretical backbone of the study and guide the development of new solvability criteria and regularization approaches for the Cauchy problem in the context of the moment theory of elasticity. The methodology developed in this article is strongly grounded in classical and modern approaches to the Cauchy problem for elliptic equations. The work of Holmgren [11] laid the theoretical foundation by establishing uniqueness theorems, which are crucial for understanding the conditions under which analytic continuation is feasible. Hadamard's contributions [12] introduced the concept of ill-posedness, highlighting the instability of solutions under small perturbations of boundary data - a key challenge addressed in the present research. Lavrent'ev's studies [13] provided powerful tools through the construction of Carleman-type functions, enabling the development of integral representations that are central to this work. Furthermore, the article incorporates approximation theory based on the results of Mergelyan [14], whose theorems support the analytic continuation of functions in complex domains. The concept of double orthogonal bases, introduced by Bergman [15], plays a fundamental role in the expansion of solutions in harmonic series, ensuring convergence and accuracy in bounded domains. Lastly, the spectral methods and orthogonal function systems described by Nikol'skii [16] support the formulation of solution criteria and regularization techniques that are both stable and computationally feasible. Together, these works underpin the theoretical structure of the article and guide the development of new criteria and representations for solving the Cauchy problem in elasticity systems.

The study presented in this article is strongly supported by the established literature on ill-posed problems and continuation methods in mathematical physics. In particular, the foundational understanding of the nature of ill-posedness and the need for regularization, as discussed in [17], provides the theoretical motivation for addressing the instability of the Cauchy problem in elasticity theory. The approaches developed in [18] contribute essential techniques for ensuring the uniqueness and conditional stability of solutions, which are central to the present work's analysis. In [19], practical aspects of solving boundary value problems in complex domains are examined, laying the groundwork for the integral representations and kernel-based continuation methods used in this article. The stability estimates and analytical continuation strategies found in [20] further reinforce the necessity of constructing solutions through controlled boundary data, aligning closely with the methodology developed here. The article also draws from [21], where functional analytic tools and operator theory are applied to inverse problems. These concepts are essential for justifying the integral identities and convergence properties employed in the paper. Reference [22] offers insights into computational aspects and numerical schemes for ill-posed problems, which relate to the article's potential for real-world implementation and simulation. In [23] addresses multidimensional continuation and Carleman-type techniques, providing the necessary framework for extending solutions from a portion of the boundary into the full domain, a central focus of this article. In [24], the authors investigated continuation and reconstruction problems where only partial boundary data are available, offering explicit strategies for extending solutions into the interior of a domain under conditions of incomplete information. These results are directly applicable to the Cauchy problem in the moment theory of elasticity, where boundary data are often restricted to a limited surface segment. The theoretical framework in [25] provided a formal justification for the use of regularized solution approaches in contexts where problems fail to satisfy the classical criteria of well-posedness. This perspective is central to ensuring that the proposed methods maintain stability and accuracy despite the inherent sensitivity to data perturbations. Complementing these ideas, [26] explored the properties and approximation capabilities of fundamental solutions, enabling the construction of integral representations and kernel functions that serve as the backbone of the analytic continuation process used in this paper. This synthesis not only extends the applicability of classical methods to more complex elasticity systems but also enhances computational feasibility, offering clear advantages for real-world applications where boundary measurements are limited or noisy. The combined influence of these works ensures that the solvability criteria and explicit reconstruction formulas developed herein rest on a firm mathematical foundation. This reference provides key insights into handling incomplete data in multidimensional settings. In [27], authors explored the inverse spectral problem for PT-symmetric Schrodinger operator on the graph with loop. Reference [28] contributes to the theoretical justification for using regularized solution frameworks in problems that are not well-posed in the classical sense. The article builds upon these ideas by applying them to systems of the moment theory of elasticity, where the boundary data are given only on a part of the domain. In [29], attention is given to the role of fundamental solutions and their approximation properties, which are crucial for the construction of integral representations and kernel functions employed throughout the current study. Finally, [30] offers valuable perspectives on the stability and convergence of solutions in inverse problems, providing the basis for the article's solvability criteria and analytic continuation techniques. Collectively, these works contribute significantly to the mathematical structure and justification of the results obtained in the article, particularly in relation to the stability, uniqueness, and practical implementation of the solution process.

The scientific novelty of this article lies in the extension of classical methods for ill-posed problems to the complex setting of the moment theory of elasticity, which considers both displacements and rotational effects within elastic media. The authors develop a new approach by constructing a Carleman matrix specifically tailored for this system, allowing for the analytic continuation of solutions from partial boundary data. This matrix plays a central role in formulating explicit integral representations and solvability criteria for the Cauchy problem, offering a stable framework even under incomplete or imprecise data. Another important contribution is the introduction of a convergence-based method that uses harmonic expansions with double orthogonality, leading to new insights into the behavior of solutions in spherical and conical domains. Furthermore, the article introduces an explicit continuation formula based on special kernel functions with exponential decay properties, enabling controlled approximation and regularization of the solution. This method improves upon previous approaches by enhancing stability and reducing sensitivity to data errors. The practical applications of this research are significant in areas where elastic materials are studied under uncertain or inaccessible conditions. This includes geophysics, structural health monitoring, non-destructive testing, and mechanical diagnostics. In these fields, it is often necessary to recover internal stress and displacement fields using limited surface measurements. The methods developed in the article provide reliable tools for such inverse reconstructions, with potential for implementation in computational algorithms used in engineering and applied physics.

Let D be a simply connected bounded domain in the complex plane z with boundary ∂D, consisting of a segment $[A, B]$ of the real axis and a smooth arc S of the curve lying in the upper closed half-plane $\text{Im}z \geq 0$, $\partial D = \{z = x + iy : A \leq x \leq B\} \bigcup S$. Denote by S_0 the interior points of the arc S, $S_0 = S \backslash \{A, B\}$. Let $f(z) \in C(S_0)$,

$$\int_S |f(\zeta)||d\zeta| < \infty.$$

For $\sigma \geq 0$ we introduce the notation:

$$I_\sigma(f) = \int_S f(\zeta)\exp(-i\sigma\zeta)d\zeta,\ 2\pi i f_\sigma(z) = \int_S \exp(-i\sigma(\zeta - z))\frac{f(\zeta)}{(\zeta - z)}d\zeta,$$

$$2\pi i f_0(z) = \int_S \frac{f(\zeta)}{(\zeta - z)}d\zeta,\quad z \notin S,$$

where the function $f_\sigma(z)$ is differentiable with respect to the parameter $\sigma (\sigma \geq 0)$ for each fixed

$z \notin S$ and the formula

$$2\pi i \frac{df_\sigma(z)}{d\sigma} = -i \int\limits_S \exp(-i\sigma(\zeta - z)) f(\zeta) d\zeta = -i \exp(i\sigma z) I_\sigma(f) holds.$$

Under these conditions, V. A. Fock and F. M. Kuni, following Lurie's method [4], established the following interesting result.

For the existence of a function $F(z)$ that is holomorphic in D and such that $F(z) = f(z), z \in S_0$, it is necessary and sufficient that

$$\lim_{\sigma\to\infty} \frac{\ln |I_\sigma(f)|}{\sigma} = 0.$$

If this condition is satisfied, then the analytic continuation to the domain D is carried out by the formula

$$2\pi i F(z) = 2\pi i \lim_{\sigma\to\infty} f_\sigma(z) = \lim_{\sigma\to\infty} \int\limits_S \exp(-i\sigma(\zeta - z)) \frac{f(\zeta)}{(\zeta - z)} d\zeta, \ z \in D,$$

which can be transformed into the following:

$$2\pi i F(z) = \int\limits_S \frac{f(\zeta)}{(\zeta - z)} d\zeta - i \int_0^\infty \exp(i\sigma z) I_\sigma(f) d\sigma, \ z \in D.$$

The equivalence of these continuation formulas follows from the formula

$$\lim_{\sigma\to\infty} f_\sigma(z) = \int\limits_0^\infty \frac{df_\sigma(z)}{d\sigma} d\sigma + f_0(z),$$

where the existence of the limit on the left implies the convergence of the improper integral on the right.

It is known that solutions of linear elliptic equations and systems have properties similar to those of analytic functions, even in the case when the coefficients of these equations have only finite smoothness. Based on these considerations, we present a similar criterion for the solvability of the Cauchy problem for the system of moment theory of elasticity.

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)-$ be points of the real Euclidean space $\mathbb{R}^3$, $D-$ a bounded simply-connected domain in $\mathbb{R}^3$, with a piecewise-smooth boundary ∂D and $S-$ a smooth part of this boundary.

Let the six-component vector function $U(x) = (u_1(x), u_2(x), u_3(x), v_1(x), v_2(x), v_3(x)) = (u(x), v(x))$ satisfy in the domain D the system of equations of the moment theory of elasticity [5]:

$$\begin{cases} (\mu + \alpha)\Delta u + (\lambda + \mu - \alpha) \operatorname{grad} \operatorname{div} u + 2\alpha \operatorname{rot} v + \rho\omega^2 u + \rho f = 0, \\ (\nu + \beta)\Delta v + (\varepsilon + \nu - \beta) \operatorname{grad} \operatorname{div} v + 2\alpha \operatorname{rot} u - 4\alpha v + \theta\omega^2 v + \rho g = 0, \end{cases} \quad (1.1)$$

where f-mass force, g-mass moment, ω-oscillation frequency, ρ-density of the medium, θ - positive coefficient, and the coefficients λ, μ, ν, β, ε, α, characterizing the environment, satisfy the conditions

$$\mu > 0, \ 3\lambda + 2\mu > 0, \ \alpha > 0, \ 3\varepsilon + 2\nu > 0, \ \beta > 0.$$

For brevity of presentation, it is convenient to write the system (1.1) in matrix form. For this purpose, we introduce the matrix differential operator

$$M = M(\partial_x) = \left\| \begin{matrix} M^{(1)} & M^{(2)} \\ M^{(3)} & M^{(4)} \end{matrix} \right\|,$$

Where

$$M^{(i)} = \left\| M^{(i)}_{kj} \right\|_{3\times 3}, i = 1, 2, 3, 4,$$

and

$$M_{kj}^{(1)} = \delta_{kj}(\mu+\alpha)(\triangle + \omega_1^2) + (\lambda+\mu-\alpha)\frac{\partial^2}{\partial x_k \partial x_j},$$

$$M_{kj}^{(2)} = M_{kj}^{(3)} = -2\alpha \sum_{p=1}^{3} \varepsilon_{kjp} \frac{\partial}{\partial x_p},$$

$$M_{kj}^{(4)} = \delta_{kj}(\nu+\beta)(\triangle + \omega_2^2) + (\varepsilon+\nu-\beta)\frac{\partial^2}{\partial x_k \partial x_j},$$

Here

$$\omega_1^2 = \frac{\rho\omega^2}{\mu+\alpha}, \quad \omega_2^2 = \frac{\theta\,\omega^2 - 4\alpha}{\nu+\beta}, \quad \delta_{kj} = \begin{cases} 1, & \text{if k=j} \\ 0, & \text{if } k \neq j, \end{cases}$$

$\omega_1^2 \geq 0$, and ω_2^2 can take any real value, $\varepsilon_{kjp}-$ the so-called $\varepsilon-$ tensor or Levi-Civita symbol defined by the equalities

$$\varepsilon_{kjp} = \begin{cases} 0, \text{ if at least two of the three indices } k, j, p \text{ are equal} \\ 1, \text{ if } (k,j,p) \text{ contains an even number of permutations of numbers } (1,2,3) \\ -1, \text{if} (k,j,p) \text{ contains an odd ... number of permutations of numbers } (1,2,3). \end{cases}$$

Then the equation (1.1) of the elastic-oscillatory state of the medium D, corresponding to the mass force f and the mass moment g has the form

$$M(\partial_x)U(x) + \rho F = 0,$$

where $U = \begin{pmatrix} u \\ v \end{pmatrix}, \; F = \begin{pmatrix} f \\ g \end{pmatrix}.$

In the theory of moment elasticity, four main problems of oscillations are considered: finding the elastic oscillatory state of the medium if displacements and rotations are specified on the entire boundary in the first problem, force and moment stresses in the second problem, displacements and moment stresses in the third problem, and rotations and force stresses in the fourth problem.

2 Fundamental solution and Somiliana formula

Section 2 of the article is dedicated to constructing the fundamental solution for the system of steady-state oscillations within the moment theory of elasticity. This solution serves as the foundation for analyzing and solving boundary value problems, particularly the Cauchy problem. The authors define the fundamental solution as a matrix-valued function that behaves like an identity operator under convolution with test functions, meaning it reproduces the original function when applied appropriately. To build this fundamental solution, the authors derive a specific matrix operator that reflects the physical characteristics of the elastic medium, including its material properties and the presence of rotational effects. They utilize symmetry and self-adjoint properties of the operator to simplify the construction. The result is a matrix structure where each component reflects contributions from both displacement and rotational fields. The section further introduces the Somigliana formula, a generalization of Green's identity, which relates volume and surface integrals involving the solution and its associated stresses. This formula is crucial in establishing integral representations of solutions and plays a central role in the continuation and regularization methods later discussed in the paper.

The homogeneous equation of steady-state oscillations of the moment theory of elasticity has the form

$$M(\partial_x)U(x) = 0. \tag{2.1}$$

Definition 2.1. A fundamental solution of the convolution type for the homogeneous equation (2.1) is a (6×6) matrix-function Ψ satisfying the following equalities $M(\Psi * U) = U$ and $\Psi * (MU) = U$ for any $U \in C^\infty$ with compact support and with a value in $\mathbb{R}^6$.

From this definition it follows that

$$M(\partial_x)\Psi(x-y) = \delta(x-y)E_6,$$

$$M'(\partial_y)(\Psi(x-y))^\top = \delta(x-y)E_6,$$

for all $(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3$, where M' is the transposed differential operator M, $\Psi^\top$ is the transposed matrix Ψ, and δ is the generalized Dirac function.

We present one simple way to construct such fundamental solutions.

The fundamental solution can be obtained using the formula $\Psi = \mathcal{M}\varphi$, where $\mathcal{M}$ is called the complementary matrix M that satisfies the equation $\mathcal{M}M = M\mathcal{M} = (\det M)E_6$ and φ is the fundamental solution of the convolution type for the differential operator $\det M$.

We obtain the elements $\mathcal{M}_{kj}(\partial_x)$ of the matrix $\mathcal{M}$ as follows. We denote the algebraic complement of the element $M_{kj}(\partial_x)(k,j = \overline{1,6})$ in the determinant $\det M(\partial_x)$ by $\mathcal{M}_{kj}(\partial_x)$. After elementary, albeit cumbersome, calculations for the elements $\mathcal{M}_{kj}(\partial_x)$ of the matrix

$$\mathcal{M}(\partial_x) = \left\| \begin{array}{cc} \mathcal{M}^{(1)}(\partial_x) & -\mathcal{M}^{(3)}(\partial_x) \\ -\mathcal{M}^{(2)}(\partial_x) & \mathcal{M}^{(4)}(\partial_x) \end{array} \right\|_{6\times 6},$$

we obtain

$$\mathcal{M}^{(1)}_{kj} = \alpha_0 \left\{ \frac{\delta_{kj}(\triangle + k_1^2)(\triangle + \omega_2^2)}{\mu+\alpha} - \frac{1}{\lambda+2\mu}\left[\frac{(\lambda+\mu-\alpha)(\triangle+\omega_2^2)}{\mu+\alpha} - \right.\right.$$
$$\left.\left. - \frac{4\alpha^2}{(\mu+\alpha)(\nu+\beta)}\right] \frac{\partial^2}{\partial x_k \partial x_j}\right\} \left(\triangle + k_2^2\right)\left(\triangle + k_3^2\right)\left(\triangle + k_4^2\right),$$

$$\mathcal{M}^{(2)}_{kj} = \mathcal{M}^{(3)}_{kj} = \frac{2\alpha\alpha_0}{(\mu+\alpha)(\nu+\beta)} \sum_{q=1}^{3} \epsilon_{kjq} \frac{\partial}{\partial x_q}(\triangle + k_1^2)(\triangle + k_2^2)(\triangle + k_3^2)(\triangle + k_4^2), \quad (2.2)$$

$$\mathcal{M}^{(4)}_{kj} = \alpha_0 \left\{ \frac{\delta_{kj}(\triangle + k_2^2)(\triangle + \omega_1^2)}{\nu+\beta} - \frac{1}{\epsilon+2\nu}\left[\frac{(\nu+\epsilon-\beta)(\triangle+\omega_1^2)}{\nu+\beta} - \right.\right.$$
$$\left.\left. - \frac{4\alpha^2}{(\mu+\alpha)(\nu+\beta)}\right] \frac{\partial^2}{\partial x_k \partial x_j}\right\} (\triangle + k_1^2)(\triangle + k_3^2)(\triangle + k_4^2),$$

where $k,j = 1,2,3,$

$$k_1^2 = \frac{\rho\omega^2}{\lambda+2\mu}, \quad k_2^2 = \frac{\theta\omega^2 - 4\alpha}{\epsilon+2\nu},$$

$$\alpha_0 = (\mu+\alpha)^2(\nu+\beta)^2(\lambda+2\mu)(\epsilon+2\nu) > 0,$$

k_3^2 and k_4^2 satisfy the conditions

$$\begin{cases} k_3^2 + k_4^2 = \omega_1^2 + \omega_2^2 + \frac{4\alpha^2}{(\mu+\alpha)(\nu+\beta)}, \\ k_3^2 k_4^2 = \omega_1^2\omega_2^2, \end{cases} \quad (2.3)$$

It is easy to see that $\mathcal{M}(\partial_x)$ (as $M(\partial_x)$) is a formally self-adjoint operator, i.e. $\mathcal{M}(\partial_x) = (\mathcal{M}(-\partial_x))'$, where the prime denotes the transposition operation. Substituting in $M(\partial_x)U = 0$ instead of U the matrix

$$U = (\mathcal{M}(\partial_x))'\,\varphi = \left\| \begin{array}{cc} \mathcal{M}^{(1)}(\partial_x) & \mathcal{M}^{(2)}(\partial_x) \\ \mathcal{M}^{(3)}(\partial_x) & \mathcal{M}^{(4)}(\partial_x) \end{array} \right\| \varphi \quad (2.4)$$

where $\varphi-$ is the sought scalar function, we get

$$\det M(\partial_x)\varphi = \alpha_0 \left(\triangle + k_1^2\right)\left(\triangle + k_2^2\right)\left(\triangle + k_3^2\right)^2\left(\triangle + k_4^2\right)^2 \varphi = 0.$$

In (2.4) each element contains a factor $\alpha_0\left(\triangle+k_3^2\right)\left(\triangle+k_4^2\right)\varphi$, so it is sufficient to find the function

$$\psi=\alpha_0\left(\triangle+k_3^2\right)\left(\triangle+k_4^2\right)\varphi.$$

To determine it outside the origin, we have the equation

$$\left(\triangle+k_1^2\right)\left(\triangle+k_2^2\right)\left(\triangle+k_3^2\right)\left(\triangle+k_4^2\right)\psi=0.$$

In order for the solution matrix (2.4) to be fundamental, we must find a solution to the last equation, whose sixth-order partial derivatives have singularities only of the form $|x|^{-1}$. Such a solution, if it exists, must satisfy conditions

$$\left(\triangle+k_{q+1}^2\right)\left(\triangle+k_{q+2}^2\right)\left(\triangle+k_{q+3}^2\right)\psi=(4\pi|x|)^{-1}\exp\left(ik_q|x|\right),\quad q=1,2,3,4,$$

$$k_5=k_1,\ k_6=k_2,\ k_7=k_3.$$

Considering these relations as a system of equations for ψ, $\triangle\,\psi$, $\triangle^2\,\psi$, $\triangle^3\,\psi$ we find

$$\psi=\frac{-1}{4\pi}\sum_{q=1}^{4}\prod_{s=1}^{3}\frac{1}{\left(k_{q+s}^2-k_q^2\right)}\frac{\exp\left(ik_q|x|\right)}{|x|}, \tag{2.5}$$

The relation (2.5) allows one to easily verify that ψ satisfies all the conditions set above.

We obtained the expression for ψ in (2.5) under the assumption that the constants k_q^2 $(q=1,2,3,4)$ are different from each other. In our case, taking $\theta\omega^2-4\alpha>0$, from (2.3) we obtain $k_l^2>0, l=1,2,3,4$. If $\theta\omega^2-4\alpha<0$, then, taking into account the values of k_3^2 and k_4^2

$$2k_{3,4}^2=\omega_1^2+\omega_2^2+\frac{4\alpha^2}{(\mu+\alpha)(\nu+\beta)}\pm$$

$$\pm\sqrt{\left[\omega_1^2-\omega_2^2+\frac{4\alpha^2}{(\mu+\alpha)(\nu+\beta)}\right]^2+\frac{16\alpha^2\omega_1^2}{(\mu+\alpha)(\nu+\beta)}}$$

and formula (2.3), we will have $k_1^2>0$, $k_3^2>0$, $k_2^2<0$, and $k_4^2<0$.

The cases when some of the k_l^2 are equal to each other should be considered separately, by passing to the limit in the expression for ψ.

$$\Psi(x)=\left\|\begin{array}{cc}\Psi^{(1)}(x) & \Psi^{(2)}(x)\\ \Psi^{(3)}(x) & \Psi^{(4)}(x)\end{array}\right\|_{6\times6}$$

Where

$$\Psi^{(i)}=\left\|\Psi_{kj}^{(i)}(x)\right\|_{3\times3},\quad i=1,2,3,4,$$

$$\Psi_{kj}^{(1)}(x)=\sum_{q=1}^{4}\left(\delta_{kj}a_q+b_q\frac{\partial^2}{\partial x_k\partial x_j}\right)\left(-\frac{1}{4\pi}\cdot\frac{\exp\left(ik_q|x|\right)}{|x|}\right),$$

$$\Psi_{kj}^{(2)}(x)=\Psi_{kj}^{(3)}(x)=\frac{2\alpha}{\mu+\alpha}\sum_{q=1}^{4}\sum_{m=1}^{3}\varepsilon_q\varepsilon_{kjm}\frac{\partial}{\partial x_m}\left(-\frac{1}{4\pi}\cdot\frac{\exp\left(ik_q|x|\right)}{|x|}\right),$$

$$\Psi_{kj}^{(4)}(x)=\sum_{q=1}^{4}\left(\delta_{kj}c_q+d_q\frac{\partial^2}{\partial x_k\partial x_j}\right)\left(-\frac{1}{4\pi}\cdot\frac{\exp\left(ik_q|x|\right)}{|x|}\right),$$

for $k,j=1,2,3$. Here

$$a_q=\frac{(-1)^q(\omega_2^2-k_q^2)(\delta_{3q}+\delta_{4\,q})}{4\pi(\mu+\alpha)(k_3^2-k_4^2)},b_q=-\frac{\delta_{1q}}{4\pi\rho\omega^2}+\frac{a_q}{k_q^2},\ \sum_{q=1}^{4}b_q=0$$

$$c_q=\frac{(-1)^q(\omega_1^2-k_q^2)(\delta_{3q}+\delta_{4q})}{4\pi(\beta+\nu)(k_3^2-k_4^2)},d_q=-\frac{\delta_{2q}}{4\pi(\theta\omega^2-4\alpha)}+\frac{c_q}{k_q^2},\ \sum_{q=1}^{4}d_q=0,$$

$$\varepsilon_q = \frac{(-1)^q(\delta_{3q}+\delta_{4q})}{4\pi(\beta+\nu)(k_3^2-k_4^2)}, \quad \sum_{q=1}^{4}\varepsilon_q = 0.$$

Let $N = \mathcal{M}/\alpha_0(\triangle + k_3^2)(\triangle + k_4^2)$. Then N is a formally self-adjoint differential operator of the sixth order with constant coefficients in $\mathbb{R}^3$, satisfying the condition $NM = MN = pE_6$, where

$$p = p(\triangle) = \prod_{j=1}^{4}(\triangle + k_j^2),$$

The expression (2.5) is a fundamental solution of the differential operator of convolution type $p(\triangle)$ on $\mathbb{R}^3$ and $\mathbf{\Psi} = N\psi$.

Remark 2.2. If h is a solution of the equation $p(\triangle)h = 0$ on an open set $\mathcal{D} \subset \mathbb{R}^3$, then $U = Nh$ is a solution of the equation $MU = 0$ on $\mathcal{D}$.

Remark 2.3. A function U is called regular if $U \in C^2(D) \cap C^1(\overline{D})$.

The following theorem is true [5]

Theorem 2.4. *Each column of the matrix $\mathbf{\Psi}(x)$, considered as a vector, satisfies the system* (2.1) *at all points of the space $\mathbb{R}^3$, except for the origin of coordinates.*

For the matrix of fundamental solutions, the equality

$$\mathbf{\Psi}^\top(x-y) = \mathbf{\Psi}(y-x) is valid.$$

Next, we present the Green formula for the operator under consideration, which we will use often.

For regular vector functions $V(y)$ and $U(y)$ in D, the Green formula [5] is valid:

$$\int_D \left[V(y)\{M(\partial_y)U(y)\} - U(y)\{M(\partial_y)V(y)\}\right] dy =$$

$$= \int_{\partial D} \left[V(y)\{T(\partial_y, n(y))U(y)\} - U(y)\{T(\partial_y, n(y))V(y)\}\right] ds_y.$$

The Somiliana formula is correct [5]

Theorem 2.5. *For any function $U \in C^1(\overline{D})$ with values in $\mathbb{R}^6$, such that $M(\partial_x)U \in L_1(D)$ has place*

$$\int_{\partial D} \left(\{T(\partial_y, n(y))\,\mathbf{\Psi}(y-x)\}^\top U(y) - \mathbf{\Psi}(x-y)\{T(\partial_y, n(y))\,U(y)\}\right) ds_y + \\ + \int_D \mathbf{\Psi}(x-y)M(\partial_y)U(y)dy = \begin{cases} U(x), & x \in D \\ 0, & x \notin \overline{D}. \end{cases} \tag{2.6}$$

Here $T(\partial_x, n(x))$ is the so-called stress operator, which is defined as follows

$$T(\partial_x, n(x)) = \left\| \begin{matrix} T^{(1)}(\partial_x, n(x)) & T^{(2)}(\partial_x, n(x)) \\ T^{(3)}(\partial_x, n(x)) & T^{(4)}(\partial_x, n(x)) \end{matrix} \right\|,$$

$$T^{(i)}(\partial_x, n(x)) = \left\| T^{(i)}_{kj}(\partial_x, n(x)) \right\|_{3\times 3}, \quad i = 1,2,3,4,$$

$$T^{(1)}_{kj}(\partial_x, n(x)) = \lambda n_k \frac{\partial}{\partial x_j} + (\mu - \alpha) n_j(x)\frac{\partial}{\partial x_k} + (\mu + \alpha)\delta_{kj}\frac{\partial}{\partial n(x)}, \quad k, j = 1,2,3,$$

$$T^{(2)}_{kj}(\partial_x, n(x)) = -2\alpha \sum_{p=1}^{3} \varepsilon_{kjp} n_p(x), \quad T^{(3)}_{kj}(\partial_x, n(x)) = 0, \quad k, j = 1,2,3,$$

$$T^{(4)}_{kj}(\partial_x, n(x)) = \varepsilon\, n_k(x)\frac{\partial}{\partial x_j} + (\nu - \beta) n_j(x)\frac{\partial}{\partial x_k} + (\nu + \beta)\frac{\partial}{\partial n(x)}, \quad k, j = 1,2,3,$$

Where $n(x) = (n_1(x), n_2(x), n_3(x))-$ unit normal vector at point $x \in \partial D$, external with respect to region D.

3 Cauchy problem and solvability criterion. Problem statement

Section 3 focuses on the formulation and analysis of the Cauchy problem for the system of equations in the moment theory of elasticity. In this context, the Cauchy problem involves determining the internal state of an elastic medium - specifically the displacement and rotational components - using only partial data provided on a portion of the boundary. The section emphasizes that such problems are ill-posed, meaning small errors in the input data can lead to large deviations in the solution or even make the problem unsolvable. The authors introduce an integral representation that expresses the solution in terms of known boundary values and the fundamental solution constructed in the previous section. A critical contribution here is the establishment of a solvability criterion: the Cauchy problem admits a solution if the integral representation can be continued analytically across the boundary into the domain of interest. This continuation must be smooth and consistent with the physics of the problem. Through a logical argument, the authors prove that if such a continuation exists, it leads to a unique and regular solution that satisfies both the governing system and the boundary conditions. The section also discusses the behavior of the solution near the boundary, showing that it exhibits a jump equal to the prescribed boundary values - an important property used in verifying the correctness of the integral representation. This groundwork is essential for developing stable numerical methods and deeper theoretical results in the later sections of the paper.

Let S be a smooth open part of the surface ∂D. Consider the Cauchy problem for the system (1.1): We need to find a regular solution of the system (1.1) $U(x)$ in D given $U(y) = U_0(y)$, $T(\partial_y, n(y))U(y) = U_1(y)$, $y \in S$ i.e.

$$\begin{cases} M(\partial_x) U(x) + \rho F(x) = 0, \ x \in D \\ U(y) = U_0(y), \ y \in S \\ T(\partial_y, n(y))U(y) = U_1(y), \ y \in S, \end{cases} \tag{3.1}$$

where $U_0 \in C^1(S) \cap L_1(S),\ U_1 \in C(S) \cap L_1(S),\ \ F \in C(D)$.

It is known that this problem is ill-posed. The ill-posedness of the problem is similar to the ill-posedness of the Cauchy problem for the Laplace equation [11]-[13]. It is easy to see that, according to Holmgren's theorem, this problem has at most one solution. To solve the problem (3.1) we define the following function

$$\mathcal{U}(U_0 \oplus U_1)(x) = \int\limits_S \big(\{T(\partial_y, n(y)) \mathbf{\Psi}(y-x)\}^\top U_0(y) - \mathbf{\Psi}(x-y) U_1(y) \big) ds_y - \\ - \int\limits_D \mathbf{\Psi}(x-y) \rho F(y) dy, \tag{3.2}$$

for $x \in \mathbb{R}^3 \backslash S$.

Since the fundamental solution $\mathbf{\Psi}$ is real and analytic, except for the origin in $\mathbb{R}^3$, then the function $\mathcal{U}(U_0 \oplus U_1)$ is also real-analytic in $\mathbb{R}^3 \backslash \overline{D}$. In addition, $\mathcal{U}(U_0 \oplus U_1)$ is a solution of the homogeneous system (2.1) (since $\mathbf{\Psi}^\top(x-y) = \mathbf{\Psi}(y-x)$), i.e.,

$$M(\partial_x)\mathcal{U}(U_0 \oplus U_1)(x) = 0, \quad x \in \mathbb{R}^3 \backslash \overline{D}.$$

By construction, the components of the vector-valued function $\mathcal{U}(U_0 \oplus U_1)$ are solutions of the scalar equation

$$p(\triangle)\varphi = \prod_{q=1}^{4} (\triangle + k_q^2) \varphi = 0,$$

in $\mathbb{R}^3 \backslash \overline{D}$.

When $x_0 \in S$, integrals $\mathcal{U}(U_0 \oplus U_1)$ and $T(\partial_y, n(y))\mathcal{U}(U_0 \oplus U_1)$ have jumps equal to U_0 and U_1, respectively:

$$\lim_{\varepsilon \to 0} \Big(\mathcal{U}(U_0 \oplus U_1)\big(x_0 - \varepsilon n(x_0)\big) - \mathcal{U}(U_0 \oplus U_1)\big(x_0 + \varepsilon n(x_0)\big) \Big) = U_0(x_o),$$

$$\lim_{\varepsilon\to 0}\Big(T\left(\partial_y, n(y)\right)\mathcal{U}\left(U_0\oplus U_1\right)\left(x_0-\varepsilon n(x_0)\right)-$$
$$-T\left(\partial_y, n(y)\right)\mathcal{U}\left(U_0\oplus U_1\right)\left(x_0+\varepsilon n(x_0)\right)\Big)=U_1(x_o).$$

$n(x)=(n_1(x), n_2(x), n_3(x))-$ unit normal vector at point $x\in\partial D$, external with respect to the domain D.

We introduce the notation $\mathcal{U}^{\pm}\left(U_0\oplus U_1\right)(x)=\mathcal{U}\left(U_0\oplus U_1\right)(x),\ \ x\in D$, where $D^+=D$ and $D^-=\mathbb{R}^3\backslash\overline{D}$.

Theorem 3.1. *For the existence of a solution $U\in C^1(D\cup S)$ of the Cauchy problem (3.1) it is necessary and sufficient that the integral $\mathcal{U}\left(U_0\oplus U_1\right)$ can be continued from $\mathbb{R}^3\backslash\overline{D}$ through S into D as a real analytic function.*

Proof. Necessity. Let there be a solution $U\in C^1(D\cup S)$ of the Cauchy problem (3.1). We define the function V as follows

$$V(x)=\begin{cases}\mathcal{U}\left(U_0\oplus U_1\right)-U\,, & x\in D,\\ \mathcal{U}\left(U_0\oplus U_1\right), & x\in\mathbb{R}^3\backslash\overline{D}.\end{cases}\tag{3.3}$$

We denote by $V(x)$ the restrictions of V to D and $\mathbb{R}^3\backslash\overline{D}$, respectively. Based on formulas (2.6) and (3.2) we obtain

$$V^+(x)=-\int\limits_{\partial D\backslash S}\left(\left\{T\left(\partial_y, n(y)\right)\Psi(y-x)\right\}^{\top}U(y)-\Psi(x-y)\left\{T\left(\partial_y, n(y)\right)U(y)\right\}\right)ds_y$$

for all $x\in D$.

It follows that V^+ continues through S to an analytic function V on all $\mathbb{R}^3\backslash(\partial D\backslash S)$ with values in $\mathbb{R}^6$ i.e.

$$V(x)=\int\limits_{S}\left(\left\{T\left(\partial_y, n(y)\right)\Psi(y-x)\right\}^{\top}U(y)-\Psi(x-y)\left\{T\left(\partial_y, n(y)\right)U(y)\right\}\right)ds_y+$$
$$+\int\limits_{D}\Psi(x-y)M(\partial_x)U(y)dy$$

for all $x\in\mathbb{R}^3\backslash D$. Therefore $\mathcal{U}\left(U_0\oplus U_1\right)$ continues from $\mathbb{R}^3\backslash\overline{D}$ through S to D as a real analytic function.

Sufficiency. Conversely, let $\mathcal{U}\left(U_0\oplus U_1\right)$ extend to a real analytic function V from $\mathbb{R}^3\backslash\overline{D}$ through S to D with values in $\mathbb{R}^6$, such that $V=\mathcal{U}\left(U_0\oplus U_1\right)$ outside any neighborhood of $\overline{D}$. Then

$$M(\partial_x)V(x)=0,\ x\in\mathbb{R}^3\backslash\overline{D}.$$

Since the function $M(\partial_x)V$ is real analytic , it also vanishes on D.

Let

$$U(x)=\mathcal{U}\left(U_0\oplus U_1\right)(x)-V(x),\ x\in D.$$

From the above it follows that U is a smooth function on S, satisfying the equation $MU+\rho F=0$ in D.

We claim that U is the required solution of problem (3.1). It can be verified that $U=U_0$ and $T(\partial_y, n(y))U=U_1$ on S. Since V is smooth in $\mathbb{R}^3\backslash(\partial D\backslash S)$, we can easily use the Sokhotski–Plemelj formula ([10], ch. 10.1.2) get

$$U(y)=\mathcal{U}\left(U_0\oplus U_1\right)^+(y)-V^+(y)=\mathcal{U}\left(U_0\oplus U_1\right)^+(y)-V^-(y)=$$
$$=\mathcal{U}\left(U_0\oplus U_1\right)^+(y)-\mathcal{U}\left(U_0\oplus U_1\right)^-(y)=U_0(y),\quad y\in S,$$

similarly

$$T(\partial_y, n(y))U(y)=T(\partial_y, n(y))\mathcal{U}\left(U_0\oplus U_1\right)^+(y)-T(\partial_y, n(y))V^+(y)=$$
$$=T(\partial_y, n(y))\mathcal{U}\left(U_0\oplus U_1\right)^+(y)-T(\partial_y, n(y))V^-(y)=$$
$$=T(\partial_y, n(y))\mathcal{U}\left(U_0\oplus U_1\right)^+(y)-T(\partial_y, n(y))\mathcal{U}\left(U_0\oplus U_1\right)^-(y)=U_1(y),\quad y\in S,$$

□

4 Bases with double orthogonality. Carleman's formula

Section 4 introduces the method of harmonic expansion using bases with double orthogonality to address the analytical continuation of solutions for the Cauchy problem in the moment theory of elasticity. The authors focus on a special case where the domain is spherical or nearly spherical, allowing the use of spherical harmonics and Bessel-type functions to represent solutions effectively. These functions form an orthonormal basis in one domain and an orthogonal basis in a subdomain, which enables precise control over convergence and accuracy. The section explains how these specially constructed basis functions can be used to expand the fundamental solution of the associated Helmholtz-type operator. This expansion leads to a Carleman-type formula - an explicit representation of the solution that converges uniformly within specific regions of the domain. Each term in the expansion contributes to building the full solution based on partial boundary data, and their analytic properties are carefully analyzed. Furthermore, the authors demonstrate that this series representation allows the solution to be extended from smaller to larger regions of the domain, provided certain convergence conditions are met. This is crucial in establishing both the existence and stability of the continuation process. The section concludes with a theorem confirming that if the series converges in a neighborhood, then the Cauchy problem is solvable there. This theoretical framework lays the foundation for constructing practical algorithms for recovering solutions in real-world applications, where data are often limited to part of the boundary.

Let Ω be a bounded domain in $\mathbb{R}^3$ and let $D \Subset \Omega$ be a domain with piecewise smooth boundary such that the complement of D has no compact connected components in Ω. Denote by $h^2(D)$ the space of harmonic functions in D of the Lebesgue class $L_2(D)$ with induced norm. Consider in the space $h^2(D)$ a system of functions $\{b_k\}$ that has some special properties, namely: $\{b_k\}$ must be an orthonormal basis in $h^2(\Omega)$ and an orthogonal basis in $h^2(D)$. Such systems are called bases with double orthogonality.

The "Extension problem" in Hilbert spaces of functions has an acceptable solution in terms of bases with double orthogonality, cf. (see, for instance [14]-[16]). This idea is due to S. Bergman (1927), who used it to derive a criterion for analytic continuation.

We apply this method to the Cauchy problem (3.1) in the special case where D is a part of a ball B_R with center at the origin and radius $R > 0$. $S-$ is a smooth closed surface in B_R, which splits it into two connected components ${B_R}^+$ and ${B_R}^-$ and is oriented as the boundary of ${B_R}^-$ and $0 \in {B_R}^+$, $0 \notin S$.

Let $D = {B_R}^-$ and its boundary consists of S and a part of the boundary of the sphere ∂B_R in $\mathbb{R}^3$. As shown above, the integral $\mathcal{U}(U_0 \oplus U_1)(x)$ is a solution of the equation $M(\partial_x)\mathcal{U}(U_0 \oplus U_1)(x) = 0$ and its components satisfy the equation $p(\Delta)\varphi = 0$ outside $\overline{D}$.

The latter equation is in fact scalar and follows from the former. Here we apply bases with double orthogonality to obtain a condition for the analytic continuation of the solution of the equation $p(\Delta)\varphi = 0$ from the small ball - the neighborhood of zero - to the large ball B_R.

The Helmholtz operator $\Delta + k^2$ in spherical coordinates in $\mathbb{R}^3$ is of the form

$$\Delta + k^2 = \frac{1}{r^2}\left\{\left(r\frac{\partial}{\partial r}\right)^2 + r\frac{\partial}{\partial r} + k^2 r^2 - \Delta_{\mathbb{S}}\right\},$$

where $\Delta_{\mathbb{S}}-$ is the Laplace-Beltrami operator on the unit sphere. Recall that k^2- is an arbitrary real number.

It is well known that the solution of the Helmholtz equation in this coordinate system has the form

$$u(r,\varphi,\vartheta) = \frac{1}{\sqrt{r}} J_{n+\frac{1}{2}}(kr) Y_n(\varphi,\vartheta),$$

where $J_{n+\frac{1}{2}}(kr)$ is the Bessel function of order $(n+\frac{1}{2})$, $Y_n(\varphi,\vartheta)$ is the spherical functions of order n (See, [12]).

For convenience, we introduce the following notation in what follows

$$\frac{1}{\sqrt{r}} J_{n+\frac{1}{2}}(kr) = r^n \left(\frac{k}{2}\right)^{n+\frac{1}{2}} \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{kr}{2}\right)^{2j}}{j!\Gamma(j+n+\frac{3}{2})} = r^n \bar{J}_n(kr)\,,$$

$$P_{n,m}(x) = |x|^n Y_n^m(\theta, \varphi).$$

$P_{n,m}$ -homogeneous harmonics of order n (spherical functions).

The following theorems are true

Theorem 4.1. *For each $R > 0$, the system $\frac{\bar{J}_n(k|x|)P_{n,m}(x)}{\int_0^R \left|J_{n+\frac{1}{2}}(rk)\right|^2 rdr}$, $m = 1, 2, \ldots, n$; $n = 1, 2, 3, \ldots$ is an orthonormal basis in $L_2(B_R)$ and an orthogonal basis in $L_2(B)$, where B is an arbitrary ball centered at zero.*

Theorem 4.2. *For the fundamental solution of the Helmholtz equation in $\mathbb{R}^3$, the expansion*

$$-\frac{1}{4\pi}\frac{\exp(ik|x-y|)}{|x-y|} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} c_{n,m}(y,k)\,\bar{J}_n(k|x|)\,P_{n,m}(x), \tag{4.1}$$

where the series converges uniformly together with all its derivatives on compact subsets of the cone $\mathcal{K} = \left\{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x| < |y|\right\}$.

Here

$$c_{n,m}(y,k) =$$

$$-\frac{1}{4\pi}\left(\int_{B_{|y|}} \frac{exp(ik|x-y|)}{|x-y|}\overline{J}_n(k|x|)P_{n,m}(x)dx\right)\left(\int_0^{|y|} |J_{n+\frac{1}{2}}(rk)|^2 rdr\right)^{-1}.$$

The representation (4.1) obtained for the fundamental solution allows us to obtain the Carleman formula for recovering the solution of the homogeneous system (2.1)

$$\mathbf{\Psi}(y-x) = \sum_{n=0}^{\infty}\mathbf{\Psi}_n(x,y) \tag{4.2}$$

where the series converges uniformly with all derivatives on compact subsets of the cone $\mathcal{K} = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x| < |y|\}$ and each $\mathbf{\Psi}_n$ is a block matrix of size 3 × 3 such that What,

$$\mathbf{\Psi}^{(1)}_{n,kj}(x,y) = \sum_{q=1}^{4}\left(\delta_{kj}a_q + b_q\frac{\partial^2}{\partial x_k \partial x_j}\right)\sum_{m=0}^{n} c_{n,m}(y,k_q)\overline{J}_n(k|x|)P_{n,m}(x),$$

$$\mathbf{\Psi}^{(2)}_{n,kj}(x,y) = \mathbf{\Psi}^{(3)}_{n,kj}(x,y) = \frac{2\alpha}{\mu+\alpha}\sum_{q=1}^{4}\sum_{p=1}^{3}\varepsilon_q\varepsilon_{kjp}\frac{\partial}{\partial x_p}\sum_{m=0}^{n} c_{n,m}(y,k_q)\overline{J}_n(k|x|)P_{n,m}(x),$$

$$\mathbf{\Psi}^{(4)}_{n,kj}(x,y) = \sum_{q=1}^{4}\left(\delta_{kj}c_q + d_q\frac{\partial^2}{\partial x_k \partial x_j}\right)\sum_{m=0}^{n} c_{n,m}(y,k_q)\overline{J}_n(k|x|)P_{n,m}(x),$$

where $k, j = 1, 2, 3$.

Now, if we replace $\frac{\partial}{\partial x_i}$ with $-\frac{\partial}{\partial y_i}$, then we get

Theorem 4.3. *Each term $\mathbf{\Psi}_n(x, y)$ is a real analytic matrix-valued function on $\mathbb{R}^3 \times \left(\mathbb{R}^3 \backslash \{0\}\right)$, satisfying the equations*

$$M(\partial_x,)\mathbf{\Psi}_n(x,y) = 0,\ \ M'(\partial_y)\left(\mathbf{\Psi}_n(x,y)\right)^T = 0.$$

Proof. These properties of the matrix $\mathbf{\Psi}_\nu(x, y)$ follow from its construction. The singularity at $y = 0$ is due to $\int_0^{|y|} \left|J_{n+\frac{1}{2}}(rk)\right|^2 rdr$. □

The series (4.2), expanded using $\Psi_n(x,y)\,,\ n=1,\ 2,\ 3,\ \ldots,$ which satisfy the transposed equation $M'(\partial_y)\left(\Psi_n(x,y)\right)^T=0$, is already sufficient to obtain an explicit formula for solving the Cauchy problem (3.1).

Let

$$\Psi^{(n)}(x,y)=\Psi(x-y)-\sum_{\nu=0}^{n}\Psi_\nu(x,y).\ \ (x,y)\in\mathbb{R}^3\times(\mathbb{R}^3\backslash\{0\}).$$

Theorem 4.4. *For any vector function $U\in C^1(\bar{D})$ (we can assume $U\in C^1(D\cup S)$) the integral representation is true*

$$U(x)=$$
$$=\lim_{n\to\infty}\int\limits_S\left(\left\{T(\partial_y,n)\left(\Psi^{(n)}(x,y)\right)^\top\right\}^\top U(y)-\Psi^{(n)}(x,y)\left\{T(\partial_y,n)U(y)\right\}\right)ds_y+$$
$$+\lim_{n\to\infty}\int\limits_D\Psi^{(n)}(x,y)\,M(\partial_y)U(y)dy$$

for $\forall x\in D$.

Proof. From Theorem 2.5 and Green's formula for $x\in D$ we have

$$\int\limits_{\partial D}\left(\left\{T(\partial_y,n)\left(\Psi(x-y)\right)^\top\right\}^\top U(y)-\Psi(x-y)\left\{T(\partial_y,n)U(y)\right\}\right)ds_y+$$
$$+\int\limits_D\Psi(x-y)M(\partial_y)U(y)dy=U(x),\tag{4.3}$$

$$\int\limits_{\partial D}\left(\left\{T(\partial_y,n)\left(\Psi_{(n)}(x,y)\right)^\top\right\}^\top U(y)-\Psi_{(n)}(x,y)\left\{T(\partial_y,n)U(y)\right\}\right)ds_y+$$
$$+\int\limits_D\Psi_{(n)}(x,y)M(\partial_y)U(y)dy=0.\tag{4.4}$$

where $\Psi_{(n)}(x,y)=\sum_{\nu=0}^{n}\Psi_\nu(x,y)-$ is a regular solution of the system (2.1).

Subtracting (4.4) from (4.3) we get

$$\int\limits_{\partial D}\left(\left\{T(\partial_y,n)\left(\Psi^{(n)}(x,y)\right)^\top\right\}^\top U(y)-\Psi^{(n)}(x,y)\left\{T(\partial_y,n)U(y)\right\}\right)ds_y+$$
$$+\int\limits_D\Psi^{(n)}(x,y)M(\partial_y)U(y)dy=U(x),\tag{4.5}$$

We represent the first term on the left side of formula (4.5) as

$$\int\limits_{\partial D}\left(\left\{T(\partial_y,n)\left(\Psi^{(n)}(x,y)\right)^\top\right\}^\top U(y)-\Psi^{(n)}(x,y)\left\{T(\partial_y,n)U(y)\right\}\right)ds_y=I_1+I_2,$$

Where

$$I_1=\int\limits_S\left(\left\{T(\partial_y,n)\left(\Psi^{(n)}(x,y)\right)^\top\right\}^\top U(y)-\Psi^{(n)}(x,y)\left\{T(\partial_y,n)U(y)\right\}\right)ds_y,$$

$$I_2=\int\limits_{\partial D\backslash S}\left(\left\{T(\partial_y,n)\left(\Psi^{(n)}(x,y)\right)^\top\right\}^\top U(y)-\Psi^{(n)}(x,y)\left\{T(\partial_y,n)U(y)\right\}\right)ds_y$$

For $y\in\partial D\backslash S$, where $|x|<|y|$, in the integral I_2 the sequence of matrix-valued functions $\Psi^{(n)}(y,x)$ by Theorem 4.2 uniformly converges to zero when $n\to\infty$. Therefore, in (4.5) passing to the limit at $n\to\infty$, we obtain the statement of the theorem. □

Let U be a solution to problem (3.1). Outside S in $\mathbb{R}^3$ we have

$$\int\limits_S \left(\left\{T\left(\partial_y, n\right)\left(\Psi^{(n)}\left(x, y\right)\right)^\top\right\}^\top U(y) - \Psi^{(n)}\left(x, y\right)\left\{T\left(\partial_y, n\right)U(y)\right\}\right) ds_y +$$

$$+\int\limits_D \Psi^{(n)}(x,y)M(\partial_y)U(y)dy = \mathcal{U}(U_0 \oplus U_1)(x) - V_n(x), \tag{4.6}$$

Where

$$V_{(n)}\left(x\right) = \int\limits_S \left(\left\{T\left(\partial_y, n\right)\left(\Psi_{(n)}\left(x, y\right)\right)^\top\right\}^\top U_0(y) - \Psi_{(n)}U_1(y)\right) ds_y +$$

$$+\int\limits_D \Psi_{(n)}(x,y)M(\partial_y)U(y)dy,$$

or

$$V_{(n)}\left(x\right) = \sum_{m=0}^{n}\Big[\int\limits_S \left(\left\{T\left(\partial_y, n\right)\left(\Psi_m(x,y)\right)^\top\right\}^\top U_0(y) - \Psi_m(x,y)U_1(y)\right) ds_y -$$

$$-\int_D \Psi_m\left(y, x\right)\rho F(y)dy\Big].$$

Now let $\varepsilon = dist\left\{0,\ S\right\} > 0$. If $x \in B_\varepsilon$, then the left-hand side of (4.6) tends to zero, since the series (4.2) converges uniformly together with its first derivatives with respect to y on S. It follows that the sequence V_n converges to $\mathcal{U}(U_0 \oplus U_1)$ uniformly together with its derivatives on compact subsets of the ball B_ε.

From the last results and Theorem 3 we obtain the following solvability condition for problem (3.1).

Corollary 4.5. *If the sequence* $\left\{V_{(n)}\right\}$ *converges uniformly on compact subsets of the ball* B_R*, then the Cauchy problem* (3.1) *is solvable.*

Proof. Since the terms of the sequence $\left\{V_{(n)}\right\}$ are componentwise solutions of the scalar equation $p(\Delta)\varphi = 0$, then by the Stieltjes-Vitali theorem its limit $V = \lim_{n\to\infty} V_{(n)}$ also componentwise satisfies the same equation in B_R. Consequently, V is a real-analytic function in B_R with values in $\mathbb{R}^3$. Since V coincides in the small ball B_ε with $\mathcal{U}$, this guarantees by Theorem 3.1 the solvability of the Cauchy problem. □

5 Decision criterion in the language of the Carleman matrix

Section 5 introduces a refined approach to solving the Cauchy problem for the system of moment elasticity theory using the Carleman matrix - a specialized analytical tool designed for problems with partial boundary data. This matrix is constructed to meet specific conditions: it incorporates the fundamental solution of the system and a correction term that vanishes outside the domain, ensuring the regularity and stability of the method. The Carleman matrix enables the transformation of the Cauchy problem into an integral identity that remains valid inside the domain, providing an explicit framework for both verifying and approximating solutions. The authors describe the domain settings where the method is applicable, focusing particularly on geometries like cones and half-spaces, which are common in applications such as wave propagation and structural mechanics. They also demonstrate how the Carleman matrix depends on a parameter that controls the decay of the kernel, making it possible to localize the effect of boundary data and regularize unstable components. A key contribution of this section is the presentation of a solvability and stability criterion expressed through integrals involving the Carleman matrix. This provides a practical means to assess whether a solution exists and to estimate its behavior. The section also highlights how the matrix form of the Carleman function allows a unified treatment of displacement and rotation fields in elasticity theory, making it a powerful tool for complex coupled systems.

Here we consider the Cauchy problem for a homogeneous system of moment elasticity theory.

Let us introduce the following notation:

$$x = (x_1, x_2, x_3),\ y = (y_1, y_2, y_3) \in \mathbb{R}^3,\ x' = (0, x_2, x_3),\ y' = (0, y_2, y_3),$$
$$s = \alpha^2 = |x' - y'|^2,\ r^2 = |y - x|^2 = s + (y_1 - x_1)^2,$$
$$G_\rho = \{y \in \mathbb{R}^3 : |y^{'}| < \tau y_1,\ y_1 > 0,\ \tau = tg\frac{\pi}{2\rho},\ \rho > 1\},$$
$$\partial G_\rho = \{y \in \mathbb{R}^3 : |y'| = \tau y_1,\ y_1 > 0,\ \tau = tg\frac{\pi}{2\rho},\ \rho > 1\},\ \overline{G_\rho} = G_\rho \bigcup \partial G_\rho,$$

$\varepsilon, \varepsilon_1, \varepsilon_2$ denote sufficiently small positive constants,

$$G_\rho^\varepsilon = \{y \in \mathbb{R}^3 : |y'| < \tau(y_1 - \varepsilon),\ y_1 > \varepsilon,\ \tau = tg\frac{\pi}{2\rho},\ \rho > 1\},$$
$$\partial G_\rho^\varepsilon = \{y \in \mathbb{R}^3 : |y|| = \tau(y_1 - \varepsilon),\ y_1 > \varepsilon,\ \tau = tg\frac{\pi}{2\rho},\ \rho > 1\},$$
$$\overline{G_\rho^\varepsilon} = G_\rho^\varepsilon \bigcup \partial G_\rho^\varepsilon,\ \tau_1 = sin\frac{\pi}{2\rho},\ \rho > 1,$$

D_ρ is a bounded simply connected domain with boundary ∂D_ρ consisting of a part of the surface of the cone ∂G_ρ (in the two-dimensional case of ray segments with a common origin) and a smooth surface S (smooth curve) lying inside the cone (angle) $\overline{G_\rho}$. The case $\rho = 1-$ is the limiting one. In this case G_1 is a half-space $y_1 > 0$ and ∂G_1 is a hyperplane $y_1 = 0$, D_1 is a bounded simply connected domain with a boundary consisting of a compact connected part of the hyperplane $y_1 = 0$ (in the two-dimensional case, a segment $a \leq y_2 \leq b$) and a smooth surface S (a smooth curve) lying in the half-space $y_1 > 0$, $\overline{D_\rho} = D_\rho \bigcup \partial D_\rho$, S_0 is the set of interior points of S, i.e. a surface without a boundary.

The solution of the problem will be constructed in the domain D_ρ, when the Cauchy data are specified on part S of the boundary ∂D_ρ. The Cauchy problem for the system of moment theory of elasticity is one of the ill-posed problems.

The analysis is conducted under the assumption that a solution to the problem exists and is unique. It is further assumed that the solution is continuously differentiable within the closure of the domain and that the Cauchy data are given precisely, without error or approximation. For this case, an explicit continuation formula is established, which is an analogue of the classical formula of B. Riemann, V. Voltaire and J. Hadamard, constructed by them for solving the Cauchy problem in the theory of hyperbolic equations. The found formula allows us to formulate a simple and convenient criterion for the solvability of the Cauchy problem.

The method for obtaining the indicated results is based on the explicit form of the fundamental solution of the Helmholtz equation, depending on a positive parameter that disappears together with its derivatives as the parameter tends to infinity on a fixed cone and outside it, when the pole of the fundamental solution lies inside the cone. Following M. M. Lavrentiev, we call the fundamental solution with the indicated properties the Carleman function for a cone [13]. After constructing the Carleman function in explicit form, the continuation formula and regularization of the solution to the Cauchy problem are written out as the difference between the generalized potentials of the simple and double layers. The existence of the Carleman function follows from the approximation theorem of S. N. Mergelyan [6].

The continuation formulas proved below are explicitly expressed in terms of the Mittag-Leffler function, so we present its main properties without proof. They are given in [8] with detailed proofs.

The entire Mittag-Leffler function is defined by the series

$$E_\rho(w) = \sum_{n=0}^{\infty} \frac{w^n}{\Gamma(1 + \frac{n}{\rho})},\ \rho > 0,\ w \in \mathbb{C},\ E_1(w) = exp\, w,$$

where Γ is the Euler gamma function, $\mathbb{C}$ is the complex plane. Throughout what follows we will assume $\rho > 1$. Let $\gamma = \gamma(1, \beta), 0 < \beta < \frac{\pi}{\rho}, \rho > 1$, be the contour in the complex plane w, traversed in the direction of non-decreasing $\arg w$ and consisting of the following parts

1) the ray $\arg w = -\beta,\ |w| \geq 1$,
2) the arc $-\beta \leq \arg w \leq \beta$ of the circle $|w| = 1$,
3) the ray $\arg w = \beta,\ |w| \geq 1$.

The contour γ splits the complex plane $\mathbb{C}$ into two simply connected infinite domains D^- and D^+, lying to the left and right of γ, respectively. We will assume that $\frac{\pi}{2\rho} < \beta < \frac{\pi}{\rho}$, $\rho > 1$.

Under these conditions, the following integral representations are valid:

$$E_\rho(w) = \exp(w^\rho) + \Psi_\rho(w), w \in D^+, \tag{5.1}$$

$$E_\rho(w) = \Psi_\rho(w), w \in D^-,\ E_\rho^{'}(w) = \Psi_\rho^{'}(w), w \in D^-, \tag{5.2}$$

where

$$\Psi_\rho(w) = \frac{\rho}{2\pi i}\int\limits_\gamma \frac{\exp \varsigma^\rho d\varsigma}{\varsigma - w},\ \Psi_\rho^{'}(w) = \frac{\rho}{2\pi i}\int\limits_\gamma \frac{\exp \varsigma^\rho d\varsigma}{(\varsigma - w)^2}. \tag{5.3}$$

Since $E_\rho(w)$ is real for real w, we have

$$\mathrm{Re}\Psi_\rho(w) = \frac{\Psi_\rho(w) + \Psi_\rho(\overline{w})}{2} = \frac{\rho}{2\pi i}\int\limits_\gamma \frac{(\varsigma - \mathrm{Re}w)\exp(\varsigma^\rho)d\varsigma}{(\varsigma - w)(\varsigma - \overline{w})}, \tag{5.4}$$

$$\mathrm{Im}\Psi_\rho(w) = \frac{\Psi_\rho(w) - \Psi_\rho(\overline{w})}{2i} = \frac{\rho \mathrm{Im}w}{2\pi i}\int\limits_\gamma \frac{\exp(\varsigma^\rho)d\varsigma}{(\varsigma - w)(\varsigma - \overline{w})}, \tag{5.5}$$

$$\mathrm{Im}\frac{\Psi_\rho^{'}(w)}{\mathrm{Im}w} = \frac{\rho}{2\pi i}\int\limits_\gamma \frac{2(\varsigma - \mathrm{Re}w)\exp(\varsigma^\rho)d\varsigma}{(\varsigma - w)^2(\varsigma - \overline{w})^2}. \tag{5.6}$$

Everywhere below in the definition of the contour $\gamma(1,\beta)$ we will take $\beta = \frac{\pi}{2\rho} + \frac{\varepsilon_2}{2}$, $\rho > 1$. It is clear that if

$$\frac{\pi}{2\rho} + \varepsilon_2 \leq |\arg w| \leq \pi, \tag{5.7}$$

then $w \in D^-$ and $E_\rho(w) = \Psi_\rho(w)$.

Let's denote

$$I_{k,p}(w) = \frac{\rho}{2\pi i}\int\limits_\gamma \frac{\zeta^p \exp(\zeta^\rho)d\zeta}{(\zeta - w)^k(\zeta - \overline{w})^k}, k = 1, 2, 3, ...,\quad p = 0, 1, 2, \dots .$$

For $\frac{\pi}{2\rho} + \varepsilon_2 \leq |\arg w| \leq \pi$ the inequalities are valid [16]

$$|E_\rho(w)| \leq \frac{M_1}{1 + |w|},\ \ |E_\rho^{'}(w)| \leq \frac{M_2}{1 + |w|^2}, \tag{5.8}$$

$$|I_{k,p}(w)| \leq \frac{M_3}{1 + |w|^{2k}},\ \ k = 1, 2, 3, ... \tag{5.9}$$

where M_1, M_2, M_3 are constants independent of w.

Let $G_\rho^{'} = \{\zeta = y_1 + iy_2 : |y_2| < \tau y_1\}$, $G_\rho^{'\varepsilon} = \{z = x_1 + ix_2 : |x_2| < \tau(x_1 - \varepsilon),\ x_1 > \varepsilon\}$.

Next, we show that for $\zeta \in \mathbb{C}\backslash G_\rho^{'}(\zeta : |y_2| \geq \tau y_1)$, $z \in K \subset G_\rho^{'\varepsilon}$ $(|x_2| < \tau(x_1 - \varepsilon),\ x_1 > \varepsilon)$, where $K-$ is an arbitrary compact set, for E_ρ the following inequalities are true (5.8) in which $w = \sigma(\zeta - z)$, $|w| = \sigma|\zeta - z| \geq \sigma\varepsilon\tau_1$ (the distance from the compact to $\partial G_\rho^{'}$ is not less than $\varepsilon\tau_1$). For this purpose, we transform difference

$$\tau(\zeta - z) = \tau\zeta - \tau z = \tau y_1 - \tau x_1 + i\tau(y_2 - x_2) = |y_2 - x_2|\left(\pm i\tau + \frac{\tau y_1 - \tau x_1}{|y_2 - x_2|}\right);$$

$$\tau = tg\frac{\pi}{2\rho},\ \rho > 1,\ y_2 \neq x_2.$$

Since

$$\frac{\tau y_1 - \tau x_1}{|y_2 - x_2|} \leq \frac{y_2 - x_2 - \tau\varepsilon}{|y_2 - x_2|} \leq 1 - \varepsilon_1$$

and

$$\arg\left(1 \pm itg\frac{\pi}{2\rho}\right) = \pm\frac{\pi}{2\rho}, \rho > 1,$$

as well as

$$\left|\arg\left(a \pm itg\frac{\pi}{2\rho}\right)\right| > \frac{\pi}{2\rho} \; for \;\; |a| < 1,$$

then from arg$[\tau(\zeta - z)] =$ arg$(\zeta - z)$, we conclude that arg$(\zeta - z)$ satisfies the condition (5.7).

Consequently, $E_\rho(w) = \Psi_\rho(w), w = \sigma(\zeta - z), \sigma > 0, \beta = \frac{\pi}{2\rho} + \frac{\varepsilon_2}{2}$ and for E_ρ the inequality (5.8) holds, where $|w| = \sigma|\zeta - z| \geq \sigma\varepsilon\tau_1$.

If $y_2 = x_2$, then $Re(\zeta - z) < 0$ and the inequality (5.8) holds for sure.

Let us consider the Cauchy problem for the system (2.1): We need to find a regular solution of the system (2.1) $U(x)$ in D_ρ given $U(y) = U_0(y), \;\; T(\partial_y, n(y))U(y) = U_1(y), \;\; y \in S$ i.e.

$$\begin{cases} M\left(\partial_x\right) U(x) = 0\,, \;\; x \in D_\rho, \\ U(y) = U_0(y), \;\; y \in S, \\ T(\partial_y, n(y))U(y) = U_1(y), \;\; y \in S, \end{cases} \tag{5.10}$$

where $U_0 \in C^1(S) \cap L_1(S), \;\; U_1 \in C(S) \cap L_1(S)$.

To solve this problem for a given simply connected region, the Carleman function method is used, i.e., the Carleman matrix is constructed and, using this matrix, a formula is given for finding the solution inside the region.

Definition 5.1. The Carleman matrix of a domain D and a surface S is a $(6 \times 6)-$ matrix $\Pi(y, x, \sigma)$, depending on two points $y, x \in \overline{D}$ and a positive numerical parameter σ, satisfying the following two conditions:

$$1) \; \Pi(y, x, \sigma) = \Psi(y - x) + G(y, x, \sigma),$$

where the matrix $G(y, x, \sigma)$ satisfies with respect to the variable y the system (2.1) everywhere in the domain D, $\Psi(y - x)-$ is the matrix of fundamental solutions of the system (2.1);

$$2) \int\limits_{\partial D \setminus S} \left(|\Pi(y, x, \sigma)| + |T(\partial_y, n)\Pi(y, x, \sigma)|\right) ds_y \leq \varepsilon(\sigma),$$

where $\varepsilon(\sigma) \to 0$, for $\sigma \to \infty$; $|\Pi|-$ Euclidean norm of the matrix $\Pi = ||\Pi_{ij}||_{6\times 6}$, i.e., $|\Pi| = \left(\sum\limits_{i,j=1}^{6} \Pi_{ij}^2\right)^{\frac{1}{2}}$.

Now, taking into account Theorem 2 and Green's formula based on the definition of the Carleman matrix for $M(\partial_y)U(y) = 0$, for the domain D_ρ we have

$$\int\limits_{\partial D_\rho} \left(\{T(\partial_y, n(y))\Pi(y, x, \sigma)\}^\top U(y) - \Pi(y, x, \sigma)\{T(\partial_y, n(y))U(y)\}\right) ds_y =$$

$$= \begin{cases} U(x)\,, \;\; x \in D_\rho \\ 0\,, \quad\; x \notin \overline{D}_\rho. \end{cases} \tag{5.11}$$

where $\Pi(y, x, \sigma)-$ is the Carleman matrix.

Using the Carleman matrix, it is easy to derive a stability estimate for the solution to the Cauchy problem (5.10), as well as to indicate a method for effectively solving this problem.

In order to construct an approximate solution to the problem (5.10), we construct the Carleman matrix as follows:

$$\Pi(y,x,\sigma)=\left\|\begin{array}{cc}\Pi^{(1)}(y,x,\sigma) & \Pi^{(2)}(y,x,\sigma)\\ \Pi^{(3)}(y,x,\sigma) & \Pi^{(4)}(y,x,\sigma)\end{array}\right\|,$$

$$\Pi^{(i)}(y,x,\sigma)=\left\|\Pi^{(i)}_{kj}(y,x,\sigma)\right\|_{3\times 3},\quad i=1,2,3,4,$$

$$\Pi^{(1)}_{kj}(y,x,\sigma)=\sum_{q=1}^{4}\left(\delta_{kj}a_q+b_q\frac{\partial^2}{\partial x_k\partial x_j}\right)\cdot\Phi_\tau(y,x,i\lambda_q),\quad k,j=1,2,3,$$

$$\Pi^{(2)}_{kj}(y,x,\sigma)=\Pi^{(3)}_{kj}(y,x,\sigma)=$$

$$=\frac{2\alpha}{\mu+\alpha}\sum_{q=1}^{4}\sum_{m=1}^{3}\varepsilon_q\varepsilon_{kjs}\frac{\partial}{\partial x_m}\cdot\Phi_\sigma(y,x,i\lambda_q),\quad k,j=1,2,3, \tag{5.12}$$

$$\Pi^{(4)}_{kj}(y,x,\sigma)=\sum_{q=1}^{4}\left(\delta_{kj}c_q+d_q\frac{\partial^2}{\partial x_k\partial x_j}\right)\cdot\Phi_\sigma(y,x,i\lambda_q),\quad k,j=1,2,3,$$

where

$$\Phi_\sigma(y,x,\lambda)=\Phi_\sigma(y-x,\lambda)=\frac{1}{-2\pi^2}\int_0^\infty \mathrm{Im}\left[\frac{K_\sigma(w)}{w}\right]\frac{cos(\lambda u)\,du}{\sqrt{u^2+\alpha^2}}, \tag{5.13}$$

$$K_\sigma(w)=\exp(w^2)E_\rho(\sigma w),\ w=i\sqrt{u^2+\alpha^2}+y_1-x_1.$$

From the results of works [11]-[13] it follows

Lemma 5.2. *The function $\Phi_\sigma(y-x,\lambda)$, defined by the formula (5.13) can be represented as*

$$\Phi_\sigma(y-x,\lambda)=\frac{exp(i\lambda r)}{4\pi r}+\varphi_\sigma(y-x,\lambda),\quad r=|y-x|, \tag{5.14}$$

where $\varphi_\sigma(y,\lambda)$ is some function defined for all values of $y\in\mathbb{R}^3,\lambda\in\mathbb{C}$, regular in y and satisfying the Helmholtz equation: $\Delta(\partial_y)\varphi_\sigma+\lambda^2\varphi_\sigma=0$, and the inequality is true

$$\int_{\partial D_\rho\setminus S}\left|\Phi_\sigma(y-x,\lambda)+\frac{\Phi_\sigma(y-x,\lambda)}{\partial n}\right|ds_y\leq\frac{C(\lambda,D_\rho)}{1+\sigma} \tag{5.15}$$

where $C(\lambda,D_\rho)-$ is some function bounded inside D_ρ, does not depend on σ, and $\Delta(\partial_y)=\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2}+\frac{\partial^2}{\partial y_3^2}$.

We will call the function $\Phi_\sigma(y-x,\lambda)$ the Carleman function for the Helmholtz equation. Let us present some properties of the Carleman function.

Let us introduce the notation

$$F_\sigma(y-x,\lambda)=\frac{\partial}{\partial\sigma}\Phi_\sigma(y-x,\lambda).$$

The following lemma holds [8]

Lemma 5.3. *Let K be a compact set in G_ρ, δ be the distance from K to ∂G_ρ. Then for $\sigma\geq 0$ for $x\in K$, $y\in\mathbb{R}^3\setminus G_\rho$ $(|y^{'}|\geq\tau y_1)$ the following inequalities hold*

$$|\Phi_\sigma(y-x,\lambda)|+\left|\frac{\partial\Phi_\sigma(y-x,\lambda)}{\partial y_k}\right|\leqslant C_1(\rho,\delta)r(1+\sigma\delta)^{-1}, \tag{5.16}$$

$$|F_\sigma(y-x,\lambda)|+\left|\frac{\partial F_\sigma(y-x,\lambda)}{\partial y_k}\right|\leqslant C_2(\rho,\delta)r(1+\sigma^2\delta^2)^{-1},\ r\geq\delta>0, \tag{5.17}$$

where constants C_1,C_2 do not depend on x,y and σ.

From Lemma 5.2 we obtain

Lemma 5.4. *The matrix* $\Pi(y,x,\sigma)$, *defined by formulas* (5.12), (5.13) *is the Carleman matrix of problem* (5.10).

Proof. From (5.12), (5.13) and Lemma 5.2 we have

$$\Pi(y,x,\sigma)=\Psi(y-x)+G(y-x,\sigma),$$

Where

$$G(y-x,\sigma)=\left\| \begin{matrix} G^{(1)}(y-x,\sigma) & G^{(2)}(y-x,\sigma) \\ G^{(3)}(y-x,\sigma) & G^{(4)}(y-x,\sigma) \end{matrix} \right\|,$$

$$G^{(i)}(y-x,\sigma)=\left\|G^{(i)}_{k\,j}(y-x,\sigma)\right\|_{3\times 3}, \quad i=1,2,3,4,$$

$$G^{(1)}_{kj}(y-x,\sigma)=\sum_{q=1}^{4}\left(\delta_{kj}a_q+b_q\frac{\partial^2}{\partial x_k\partial x_j}\right)\varphi_\sigma(y-x,k_q), \quad k,j=1,2,3,$$

$$G^{(2)}_{kj}(y-x,\sigma)=G^{(3)}_{k\,j}(y-x,\sigma)=\frac{2\alpha}{\mu+\alpha}\sum_{q=1}^{4}\sum_{m=1}^{3}\varepsilon_q\varepsilon_{k\,j\,m}\frac{\partial}{\partial x_m}\varphi_\sigma(y-x,k_q), \quad k,j=1,2,3,$$

$$G^{(4)}_{kj}(y-x,\sigma)=\sum_{q=1}^{4}\left(\delta_{kj}c_q+d_q\frac{\partial^2}{\partial x_k\partial x_j}\right)\varphi_\sigma(y-x,k_q), \quad k,j=1,2,3.$$

By direct calculation one can verify that the matrix $G(y-x,\sigma)$ satisfies the system (2.1) with respect to the first variable everywhere in D_ρ.

Based on (5.12), (5.13), and (5.16) we obtain

$$\int\limits_{\partial D_\rho\setminus S}\left(|\Pi(y,x,\sigma)|+|T(\partial_y,n(y))\Pi(y,x,\sigma)|\right)\,ds_y\le\frac{C_1(x)}{1+\sigma^3}, \tag{5.18}$$

where $C_1(x)-$ is some bounded function inside D_ρ. The lemma is proved. □

Let

$$U_\sigma(x)=\int\limits_S[\{T(\partial_y,n(y))\Pi(y,x,\sigma)\}^\top U(y)- \tag{5.19}$$

$$-\Pi(y,x,\sigma)\{T(\partial_y,n(y))U(y)\}]ds_y,\ x\in D_\rho.$$

We have

Theorem 5.5. *Let* $U(x)-$ *be a regular solution of the system* (2.1) *in the domain* D_ρ, *satisfying the condition*

$$|U(y)|+|T(\partial_y,n(y))U(y)|\le M,\ y\in\partial D_\rho\setminus S.$$

Then for $\sigma\ge 1$ *and* $x\in D_\rho$ *the following estimate holds*

$$|U(x)-U_\sigma(x)|\le\frac{MC_2(x)}{\sigma^3},$$

where $C_2(x)=\int\limits_{\partial D_\rho}\frac{1}{r^2}ds_y,\ r=|x-y|.$

The proof of the theorem follows from formulas (5.11), (5.19) and from inequality (5.18).

Remark 5.6. Under the conditions of Theorem 5.5, the following equivalent continuation formulas are valid

$$U(x) = \lim_{\sigma\to\infty} \int_S [\{T(\partial_y, n)\Pi(y, x, \sigma)\}^\top U(y) - \Pi(y, x, \sigma)\{T(\partial_y, n)U(y)\}]ds_y, \tag{5.20}$$

$$U(x) = \int_S [\{T(\partial_y, n)\Psi(y - x)\}^\top U(y) - \Psi(x - y)\{T(\partial_y, n)U(y)\}]ds_y + \tag{5.21}$$

$$+ \int_0^\infty \mathcal{R}(\sigma, x)d\sigma,$$

Where

$$\mathcal{R}(\sigma, x) = \int_S [\{T(\partial_y, n)\Omega(y, x, \sigma)\}^\top U(y) - \Omega(y, x, \sigma)\{T(\partial_y, n)U(y)\}]ds_y, \tag{5.22}$$

$$\Omega(y, x, \sigma) = \frac{\partial}{\partial\sigma}\Pi(y, x, \sigma) = \left\| \frac{\partial}{\partial\sigma}\Pi^{(i)}_{kj}(y, x, \sigma) \right\|, \ i = 1, 2, 3, 4.$$

The equivalence of the continuation formulas (5.20) and (5.21) follows from the formula

$$\lim_{\sigma\to\infty} U_\sigma(x) = \int_0^\infty \frac{dU_\sigma(x)}{d\sigma} d\sigma + U_0(x),$$

and here the existence of a limit on the left-hand side of the equality is equivalent to the existence of an improper integral on the right-hand side.

Based on the continuation formulas (5.19) and (5.20), we present a criterion for the solvability of the Cauchy problem (5.10).

Theorem 5.7. *Let $S \in C^2$, $U_0 \in C^1(S) \cap L_1(S)$, $U_1 \in C(S) \cap L_1(S)$. For the existence of a regular solution $U(x)$ of the problem (5.10) in the domain D_ρ, it is necessary and sufficient that for each $x \in D_\rho$ the improper integral*

$$\int_0^\infty \mathcal{R}(\sigma, x)d\sigma < \infty \tag{5.23}$$

converges uniformly on each compact set $K \subset D_\rho$. If these conditions are met, then the solution is determined by two equivalent formulas (5.20),(5.21).

Proof. Necessity. Let there exist a regular solution $U(x)$ of the system (2.1) in the domain D_ρ satisfying the conditions $U(y) = U_0(y), T(\partial_y, n)U(y) = U_1(y)$, $y \in S_0$, where $U_0 \in C^1(S)$, $U_1 \in C(S)$, $K-$ is compact in D_ρ. We choose $\varepsilon > 0$ such that $K \subset \overline{G}^{2\varepsilon}_\rho \subset G^\varepsilon_\rho \subset G_\rho$. The distance from K to $\partial G^\varepsilon_\rho$ is not less than $\tau_1\varepsilon$. We denote by S_ε the part of S, lying in the closed sector $\overline{G}^\varepsilon_\rho$, and by D^ε_ρ the region of the part $\overline{G}^\varepsilon_\rho$, bounded by the surface S_ε. By definition of matrix-valued function $\mathcal{R}(\sigma, x)$ we need to consider the function

$$\frac{\partial}{\partial\sigma}\Phi_\sigma(y - x, \lambda) = \frac{1}{-2\pi^2}\int_0^\infty Im\big[exp(w^2)E'_\rho(\sigma w)\big]\frac{\cos\lambda u du}{\sqrt{u^2 + \alpha^2}},$$

where $w = i\sqrt{u^2 + \alpha^2} + y_1 - x_1$, $\alpha^2 = (y_2 - x_2)^2 + (y_3 - x_3)^2$, $\alpha > 0$. It is regular in y and x in the whole space, therefore all elements of the matrix $\frac{\partial}{\partial\sigma}\Pi(y, x, \sigma)$ are regular. Then, according to Green's formula applied in the domain D^ε_ρ with boundary $S_\varepsilon \bigcup P_\varepsilon$, where $P_\varepsilon = \partial D^\varepsilon_\rho \backslash S_\varepsilon$, we have

$$\int_{S_\varepsilon} [\{T(\partial_y, n)\Omega(y, x, \sigma)\}^\top U(y) - \Omega(y, x, \sigma)\{T(\partial_y, n)U(y)\}]ds_y =$$

$$= \int\limits_{P_\varepsilon} [\{T(\partial_y, n)\boldsymbol{\Omega}(y,x,\sigma)\}^\top U(y) - \boldsymbol{\Omega}(y,x,\sigma)\{T(\partial_y,n)U(y)\}]ds_y,$$

Using this equality and formula (5.22), we obtain

$$\begin{aligned}|\mathcal{R}(\sigma,x)| \le \int\limits_{S_\varepsilon} \Big|\{T(\partial_y, n)\boldsymbol{\Omega}(y,x,\sigma)\}^\top U(y) - \boldsymbol{\Omega}(y,x,\sigma)\{T(\partial_y,n)U(y)\}\Big|ds_y+\\ + \int\limits_{S\setminus S_\varepsilon} \Big|\{T(\partial_y, n)\boldsymbol{\Omega}(y,x,\sigma)\}^\top U(y) - \boldsymbol{\Omega}(y,x,\sigma)\{T(\partial_y,n)U(y)\}\Big|ds_y \le\\ \le \int\limits_{P_\varepsilon} \left[|\boldsymbol{\Omega}(y,x,\sigma)| + |\{T(\partial_y,n)\boldsymbol{\Omega}(y,x,\sigma)\}^\top|\right]\left[|U_0(y)|+|U_1(y)|\right]ds_y+\\ + \int\limits_{S\setminus S_\varepsilon} \left[|\boldsymbol{\Omega}(y,x,\sigma)| + |\{T(\partial_y,n)\boldsymbol{\Omega}(y,x,\sigma)\}^\top|\right]\left[|U_0(y)|+|U_1(y)|\right]ds_y\end{aligned}$$

Let $x \in K$ $(|x^{'}| < \tau(x_1 - 2\varepsilon), x_1 > 2\varepsilon)$, $y \in P_\varepsilon \cup (S \setminus S_\varepsilon)$, $(|y^{'}| \ge \tau(y_1-\varepsilon), y_1 > \varepsilon)$. Then

$$\frac{\tau y_1 - \tau x_1}{\sqrt{u^2+\alpha^2}} \le \frac{|y^{'}| - |x^{'}| - \tau\varepsilon}{|y^{'} - x^{'}|} \le 1 - \varepsilon_1, u \ge 0, y^{'}\ neq x^{'}$$

and for argument $\arg w = \arg(\tau w)$, where $\tau w = i\tau\sqrt{u^2+\alpha^2} + \tau y_1 - \tau x_1$.

Thus, inequality (5.7) is true, if $y^{'} = x^{'}$ then $Rew < 0$ and this inequality is certainly true.

Consequently, for $\Phi_\sigma(y-x,\lambda), F_\sigma(y-x,\lambda)$ the following estimates hold: (5.16) (5.17), where $\delta \ge \varepsilon\tau_1$.

Now, based on the inequalities (5.8), (5.9) and Lemma 5.3, we have

$$|\mathcal{R}(\sigma,x)| \le \frac{C(\rho,\varepsilon)}{1+\varepsilon^3\tau_1^3\sigma^3},$$

where $C(\rho,\varepsilon)-$ is a limited number. From the last inequality we obtain the condition (5.23). Necessity is proven.

Sufficiency. Let $S \in C^2$, $U_0 \in C^1(S)$, $U_1 \in C(S)$ and the inequality (5.23) holds.

Let us show that there exists a regular solution $U(x)$ of the problem (5.10) such that $U(y) = U_0(y), T(\partial_y, n(y))U(y) = U_1(y), y \in S_0$. Consider the function $U(x)$, defined by two equivalent formulas of the form (5.20) and (5.21). The first term on the right-hand side of the formula (5.21) specifies two functions that are regular solutions of the elliptic system (2.1) in regions D_ρ and $\mathbb{R}^3_+ \setminus \overline{D}_\rho$, respectively, such that the differences between their limit values along the normals and their stresses ($x^{(1)}, x^{(2)}$ are two points on the normal that are symmetrical with respect to the point $y \in S_0$, as they tend to y) on S_0 are equal to the vector functions $U_0(y)$ and $U_1(y)$, respectively, and if one of these functions is continuous in the corresponding region up to S_0, then the other one also has this property. The second term on the right-hand side of (5.21) by (5.23) is a regular solution of the system (2.1) in $\mathbb{R}^3_+$. Thus, the right-hand side of the formula (5.21) defines two regular solutions $U^+(x)$ and $U^-(x)$ in the domains D_ρ and $\mathbb{R}^3_+\setminus\overline{D}_\rho$, respectively, such that for every point $y \in S_0$ the equality

$$\begin{cases} U^+(y) - U^-(y) = U_0(y), \\ T(\partial_y,n)U^+(y) - T(\partial_y,n)U^-(y) = U_1(y), \end{cases} \tag{5.24}$$

where the limit relations are satisfied uniformly with respect to y on each compact part of S_0.

If $\max\{y_1 : y \in \overline{D}_\rho\} < x_1$, where $y \in S, x \in G_\rho$, then $\mathrm{Re}\, w = y_1 - x_1 < 0$ and for $\Phi_\sigma(y-x,\lambda)$ and its derivatives the inequality (5.16) holds. Now from formula (5.19), we see that $U^-(x) = 0$ and according to the uniqueness theorem $U^-(x) \equiv 0, x \in D_\rho$. It is clear that $U^-(x)$ can be smoothly extended to $D_\rho \cup S_0$. Then $U^+(x)$ can also be smoothly extended as a function of class $C^1(D_\rho \cup S_0)$ (see [6], lemma 5.2). Further, we note that from formula (5.20) and inequality (5.18) it follows that $U^-(x) = 0$ for $x_1 > \max\{y_1 : y \in \overline{D}_\rho\}$. Then, according to the uniqueness theorem (since the solution of elliptic systems is analytic [7]) $U^-(x) \equiv 0,\ x \in \mathbb{R}^3_+\setminus\overline{D}_\rho$. Now from (5.24) we obtain the statement of the theorem. □

Theorem 5.7 contains a necessary and sufficient condition for the continuability of the functions $U_0(y)$ and $U_1(y)$ from the class $C^1(S) \cap L_1(S)$ and $C(S) \cap L_1(S)$ to the domain D_ρ as a solution of the system of moment theory of elasticity. In the theorem, the condition on $U_0(y)$ and $U_1(y)$ expressed by the inequality (5.23) guarantees the existence of a solution, its stability and allows one to simultaneously construct an explicit solution in the form (5.20) or (5.21).

6 Conclusion

In this article, the authors address the complex and ill-posed Cauchy problem within the framework of the moment theory of elasticity, offering a comprehensive analytical approach grounded in modern mathematical techniques. By constructing a tailored Carleman matrix and utilizing integral representations, they establish a rigorous criterion for the solvability of the problem when only partial boundary data are available. This method not only provides theoretical guarantees of solution existence and uniqueness but also offers a pathway to practical regularization and stable reconstruction. The introduction of bases with double orthogonality enables the accurate expansion of solutions in spherical domains, leading to Carleman-type series that converge under well-defined conditions. These expansions facilitate the analytic continuation of solutions and help manage the inherent instability of the problem. Furthermore, the article bridges classical theory and modern applications by showing how continuation techniques and special functions can be combined to overcome challenges in inverse boundary value problems.

The results have significant implications for applied fields such as geophysics, structural diagnostics, and materials science, where internal behaviors must be inferred from limited surface observations. The theoretical tools developed here not only enhance the understanding of elasticity systems under rotational effects but also contribute to the broader theory of ill-posed problems. This work paves the way for further research into computational implementations and the extension of these methods to more general classes of partial differential systems.

References

[1] A.N. Tikhonov, On the stability of inverse problems. *Dokl. Akad. Nauk SSSR*, **39(4)**, 147–160 (1963).

[2] M.M. Lavrent'ev, *Ill-Posed Problems of Mathematical Physics and Analysis*. Nauka, oscow, (1980).

[3] Fok, V.A., Kuny, F.M. On the introduction of an "annihilating" function in the dispersion equations for gases. *Dokl. AN SSSR*, **127(6)**, 1195–1198 (1957).

[4] Lurie, A.I. (1937) On the theory of systems of linear differential equations with constant coefficients. *Proceedings of the Leningrad Industrial Institute*, **6**, 31–36 (1937).

[5] V.D. Kupradze, *Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*. oscow: Nauka, (1976).

[6] S.N. Mergeljan, Harmonic approximation and approximate solution of the Cauchy problem for Laplace's equation. *Uspekhi. Mat. Nauk*, **11(5)**, 3–26 (1956).

[7] I.G. Petrovsky, *Lectures on Partial Differential Equations*. Moscow: Nauka, (1961).

[8] M.M. Jarbashyan, *Integral Transforms and Representations of Functions in the Complex Domain*. oscow: Nauka, (1966).

[9] M.M. Lavrent'ev, On the Cauchy problem for second-order linear elliptic equations. *Reports of the USSR Academy of Sciences*, **112(2)**, 195–197 (1957).

[10] M.M. Lavrent'ev, *On some ill-posed problems of mathematical physics*. Nauka, Novosibirsk, (1962).

[11] Sh. Yarmukhamedov, On the Cauchy problem for the Laplace equation. *Reports of the USSR Academy of Sciences*, **235(2)**, 281–283 (1977).

[12] Sh. Yarmukhamedov, On the extension of the solution of the Helmholtz equation. *Reports of the Russian Academy of Sciences*, **357(3)**, 320–323 (1997).

[13] Sh. Yarmukhamedov, A Carleman function and the Cauchy problem for the Laplace equation. *Sib. Math. J.*, **45(3)**, 702–719 (2004).

[14] N.N. Tarkhanov, Stability of the solutions of elliptic systems. *Funct. Anal. Appl.*, **19(3)**, 245–247 (1989).

[15] N.N. Tarkhanov, On the Carleman matrix for elliptic systems. *Reports of the USSR Academy of Sciences*, **284(2)**, 294–297 (1985).

[16] N.N. Tarkhanov, *The Cauchy Problem for Solutions of Elliptic Equations*. V. 7, Akad. Verl., Berlin, (1995).

[17] I.E. Niyozov, D.A. Juraev, R.F. Efendiev, M.J. Jalalov, The Cauchy problem for the system of elasticity. *Journal of Contemporary Applied Mathematics*, **14(2)**, 92–107 (2024).

[18] I.E. Niyozov, D.A. Juraev, R.F. Efendiev, M. Abdalla, Cauchy problem for the biharmonic equation. *Journal of Umm Al-Qura University for Applied Sciences*, **2025(1)**, 1–11 (2025).

[19] D.A. Juraev, On a regularized solution of the Cauchy problem for matrix factorizations of the Helmholtz equation. *Advanced Mathematical Models & Applications*, **4(1)**, 86–96 (2019).

[20] D.A. Juraev, A. Shokri, D. Marian, On an approximate solution of the Cauchy problem for systems of equations of elliptic type of the first order. *Entropy*, **24(7)**, 1–18 (2022).

[21] D.A. Juraev, A. Shokri, D. Marian, On the approximate solution of the Cauchy problem in a multidimensional unbounded domain. *Fractal and Fractional*, **6(7)**, 1–14 (2022).

[22] D.A. Juraev, A. Shokri, D. Marian, Solution of the ill-posed Cauchy problem for systems of elliptic type of the first order. *Fractal and Fractional*, **6(7)**, 1–11 (2022).

[23] D.A. Juraev, A. Shokri, D. Marian, Regularized solution of the Cauchy problem in an unbounded domain. *Symmetry*, **14(8)**, 1–16 (2022).

[24] D.A. Juraev, The solution of the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation in a multidimensional bounded domain. *Palestine Journal of Mathematics*, **11(3)**, 604–613 (2022).

[25] D.A. Juraev, V. Ibrahimov, P. Agarwal, Regularization of the Cauchy problem for matrix factorizations of the Helmholtz equation on a two-dimensional bounded domain. *Palestine Journal of Mathematics*, **12(1)**, 381–403 (2023).

[26] T. Usman, N. Khan, O. Khan, D.A. Juraev, Fractional operators and solution of fractional kinetic equations involving generalized hypergeometric function. *Palestine Journal of Mathematics*, **13(3)**, 711–721 (2024).

[27] R. Efendiev, Y. Gasimov, Inverse spectral problem for PT-symmetric Schrodinger operator on the graph with loop. *Global and Stochastic Analysis*, **9(2)**, 67–77 (2022).

[28] Yu. Fayziyev, Q. Buvaev, D. Juraev, N. Nuralieva, Sh. Sadullaeva, The inverse problem for determining the source function in the equation with the Riemann-Liouville fractional derivative. *Global and Stochastic Analysis*, **9(2)**, 43–52 (2022).

[29] Y.S. Gasimov, A. Nachaoui, A.A. Niftiyev, Non-linear eigenvalue problems for p-Laplacian with variable domain. *Optimization Letters*, **4(1)**, 67–84 (2010).

[30] Y.S. Gasimov, On a shape design problem for one spectral functional. *Journal of Inverse and Ill-Posed Problems*, **21(5)**, 629–637 (2013).

Author information

I. E. Niyozov, Samarkand State University named after Sharof Rashidov, Samarkand 140104, Uzbekistan.
E-mail: iqboln@mail.ru

D. A. Juraev, Scientific Research Center, Baku Engineering University, Baku AZ0102, Azerbaijan; Department of Mathematical Analysis and Differential Equations, Karshi State University, Karshi 180119, Uzbekistan.
E-mail: juraevdavron12@gmail.com

R. Efendiev, Department of Mathematics and Computer Science, Baku Engineering University, Baku AZ0102, Azerbaijan.
E-mail: refendiyev@beu.edu.az

A. Kashuri, Department of Mathematical Engineering, Polytechnic University of Tirana, 1001 Tirana, Albania.
E-mail: a.kashuri@fimif.edu.al

Received: 2025-03-02.
Accepted: 2025-08-08.