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T-Hopfian Modules and Related Properties

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Abstract In this paper, we explore T-Hopfian modules, which are modules where each surjective endomorphism possesses a T-small kernel. We show that any Hopfian is T-Hopfian; however, the reverse implication does not necessarily hold. Furthermore, we examine the relationships between T-Hopfian modules, Dedekind finite modules, and quasi-projective modules. We also investigate the behavior of T-Hopfian modules in the context of direct sums and projective T-covers. Several important properties, such as the preservation of T-Hopfian modules under Morita equivalence, are demonstrated. These results contribute to the understanding of the classification and structure of modules in modern algebra.

1 Introduction

Exploring modules through the lens of their endomorphism properties has been a longstanding area of interest in algebraic research. In [14] the concept of Hopfian modules was first introduced by Hiremath, which provided a new way to classify modules according to their endomorphisms. Following this, the concept of co-Hopfian modules was first proposed by Varadarajan. The concept of generalized Hopfian (GH) modules, which extends the idea of Hopfian modules, was presented in [13]. A right R-module M is called GH if each surjective endomorphism of M has a kernel that is small. Small kernels and the generalization of Hopfian modules have been widely studied and extended by various authors, such as in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 17].

In this article, we explore several properties of Hopfian, co-Hopfian, and T-Hopfian module. We introduce the properties of T-Hopfian modules, and we explore the relationships between these classes of modules. Specifically, we show the T-Hopfian property is invariant under Morita equivalence (Theorem 3.5). A ring R is a Dedekind finite (DF) ring if for any elements a and b in R, the equation $ab = 1$ then it necessarily follows that $ba = 1$. Alternatively, R is DF if any element a that is right or left invertible is also invertible. This leads to the conclusion that if $ab = 1$, then ba must be a non-zero idempotent. Thus, a ring R is classified as DF iff R cannot be isomorphic to any proper left or right ideal that serves as a direct summand.

For a unital ring R and a unitary R-module M, the module M is DF if its endomorphism ring, $\text{End}(M)$, is DF. We also explore the implications of the T-Hopfian property for Dedekind finite modules, and we discuss the sufficient and necessary conditions for certain types of modules, such as quasi-projective modules, to exhibit T-Hopfian behavior.

The paper first presents an overview of Hopfian, co-Hopfian, and T-small modules, along with some foundational definitions. It then proves that the T-Hopfian property is invariant under Morita equivalence (Theorem 3.5). The paper demonstrates that each Hopfian R-module is T-Hopfian, Nonetheless, the reverse implication does not always hold, and shows that T-Hopfian implies DF (Proposition 3.10). Furthermore, it establishes that the concepts Hopfian, T-Hopfian, and DF modules coincide for quasi-projective modules (Corollary 3.11).

2 Definitions and preliminaries

The definitions and key facts listed below are recalled:

Definition 2.1. [14] An R-module is defined as *Hopfian* (or *co-Hopfian*) if each of its surjective (or injective) endomorphism is an R-automorphism.

Remark 2.2.
- Each Noetherian R-module M, meaning that M has the ACC on its submodules, is Hopfian. [1].
- Each Artinian module M, meaning that M satisfies the DCC on its submodules, is co-Hopfian [1].
- The group $\mathbb{Q}$ is a $\mathbb{Z}$-module that is neither Noetherian nor Artinian, yet it is both Hopfian and co-Hopfian. [15].

A submodule S of a module M ($S \leq M$) is referred to as T-small in M, written as $S \ll_T M$, where T is a submodule of M. The condition for T-smallness is as follows: for each submodule P of M such that $T \subseteq S + P$, it must be the case that $T \subseteq P$.

Lemma 2.3. *[2] Let L be a module, and suppose $P \leq T \leq L$ and $S \leq L$. Then, the following are true:*

(i) If $S \ll_T L$, then $S \cap T \ll L$.

(ii) $P \ll_T L$ iff $P \ll T$.

Lemma 2.4. *[[1], 5.1]. Suppose $h : L \to N$ and $k : N \to L$ are homomorphisms satisfying $hk = 1_N$. Then:*

$$L = \ker(h) \oplus \operatorname{Im}(k).$$

Additionally, h is surjective, and k is injective.

Lemma 2.5. *[2]. Let L be a module with $S \leq N \leq L$ and $S \leq T$. Then, we have*

$$N \ll_T L \quad \textit{iff} \quad S \ll_T L \quad \textit{and} \quad \frac{N}{S} \ll_{\frac{T}{S}} \frac{L}{S}.$$

Lemma 2.6. *[2].*

Consider L and N as R-modules, with $h : L \to N$ being an R-module homomorphism. Let K and T be submodules of L. If $K \ll_T L$, then the image $h(K)$ is T-small in N relative to $h(T)$, i.e., $h(K) \ll_{h(T)} N$.

In particular, when $K \ll_T L$ and $L \leq N$, the submodule K is T-small in N, denoted as $K \ll_T N$.

Lemma 2.7. *[2].*

Let P be a module, and suppose $S_1 \leq P_1 \leq P$, $S_2 \leq P_2 \leq P$, with $T \subseteq P_1 \cap P_2$.

Then the following equivalence holds:

$$S_1 + S_2 \ll_T P_1 + P_2 \quad \iff \quad (S_1 \ll_T P_1 \text{ and } S_2 \ll_T P_2).$$

Definition 2.8. [9]. An R-module L is referred to as a Fitting module if

$$\forall f \in \operatorname{End}(L),\ \exists n \geq 1 : L = \ker(f^n) \oplus \operatorname{Im}(f^n).$$

Remark 2.9.
- Each Noetherian Artinian module is Fitting. [1]
- Each Fitting module is co-Hopfian. [1]
- Each Fitting module is Hopfian. [1]

3 T-HOPFIAN MODULES AND RELATED PROPERTIES

Definition 3.1. [11] A module L is referred to as a *T-Hopfian* (TH) module if for every surjective endomorphism $f : M \to M$, the kernel of f is T-small.

This example illustrates that Hopfian modules constitute a proper subset of TH modules.

Example 3.2. [11]
In the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$. For each $n \in \mathbb{N}$, define the submodule $H_n = \langle \frac{1}{p^n} + \mathbb{Z} \rangle$ of $\mathbb{Z}_{p^\infty}$, and for $m > n$, define $H_m = \langle \frac{1}{p^m} + \mathbb{Z} \rangle$. By [2, Example 2.2(e)], we have the inclusion $H_n \ll_{H_m} \mathbb{Z}_{p^\infty}$, meaning that H_n is H_m-small in $\mathbb{Z}_{p^\infty}$. As a consequence, each submodule of H_n is H_m-small, which implies that H_n is H_m-Hopfian as a $\mathbb{Z}$-module. However, the multiplication by p is a surjective endomorphism on H_n, but it is not an automorphism.

This outcome can be viewed as a variant of Fitting's result.

Proposition 3.3. *Let L be an R-module and $T \leq L$ such that $\forall h : L \to L$ (an endomorphism of L), there exists $n \in \mathbb{Z}$ such that $n \geq 1$ and*

$$\ker(h^n) \cap \operatorname{Im}(h^n) \ll_T L.$$

Then L is TH.

Proof. Let $h : L \to L$ be an R-Endomorphism. Then there exists $n \in \mathbb{Z}$ such that $n \geq 1$ and

$$\ker(h^n) \cap \operatorname{Im}(h^n) \ll_T L.$$

If h is surjective, then h^n is also surjective, i.e., $\operatorname{Im}(h^n) = M$, which implies $\ker(h^n) \ll_T L$. As $\ker(h) \subseteq \ker(h^n)$, it follows that $\ker(h) \ll_T L$, and therefore, L is TH.

□

Lemma 3.4. *Let N be a TH R-module, with $T \subseteq N$. For all R-modules X, if there exists a surjective endomorphism $h : N \to N \oplus X$, then $X = 0$.*

Proof. Consider $h : N \to N \oplus X$ be surjective, and let $\pi : N \oplus X \to N$ be the natural projection. It follows that

$$\ker(\pi h) = h^{-1}(i(X)),$$

with $i : X \to X \oplus N$ is the natural injection. As N is TH, hence $\ker(\pi h) \ll_T N$. By Lemma 2.6,

$$i(X) = h[h^{-1}(i(X))] = h(\ker(\pi h)) \ll_{h(T)} N \oplus X.$$

Therefore, it must be the case that $X = 0$.

□

Theorem 3.5. *The property of being T-Hopfian is preserved under Morita equivalence.*

Proof. Let Y and Z are Morita equivalent rings, and let the inverse category equivalences γ: Mod-Y $\to$ Mod-Z and μ: Mod-Z $\to$ Mod-Y. Suppose $M \in$ Mod-Y is TH. To prove that $\gamma(M)$ is TH in Mod-Z, consider an epimorphism $f : \gamma(M) \to \gamma(M) \oplus X$ in Mod-Z, where X is an Z-module.

As category equivalences preserve both direct sums and epimorphisms, the map $\mu(f) : \mu\gamma(M) \to \mu\gamma(M) \oplus \mu(X)$ is an epimorphism in Mod-R. As $\mu\gamma(M) \simeq M$, we obtain an epimorphism $M \to M \oplus \mu(X)$ in Mod-R. By Lemma 3.4, this implies that $\mu(X) = 0$, and therefore $X = 0$. Thus, $\gamma(M)$ is TH in Mod-Z.

□

The concept of projective T-covers is now introduced.

Definition 3.6. Let G be a module and $T \subseteq G$. A pair (L, p) is referred to as a projective T-cover of G if L is projective and $p : L \to G$ is an R-epimorphism such that the kernel of p, $\ker(p)$, is T-small in L, i.e., $\ker(p) \ll_T L$.

Proposition 3.7. *Let G be a quasi-projective R-module with a projective T-cover $p : L \to G$ and $T \subseteq G$. If G is TH, then L is TH.*

Proof. Since p is an epimorphism, its kernel is T-small and a fully invariant submodule of L due to the quasi-projectivity of G. If G is TH, it follows that $L/\ker(p)$ is TH as well, implying that L is TH.

□

Proposition 3.8. *If G is a quasi-projective module and T is a submodule of G, then G is TH whenever M is co-Hopfian.*

Proof. Let $h : G \to G$ be an R-epimorphism. As G is quasi-projective, there exists an endomorphism $k : G \to G$ with $hk = \mathrm{id}_G$. Therefore, k is injective. As G is co-Hopfian, it follows that k is an R-automorphism. Hence, h is also an R-automorphism, which implies that G is TH.

□

Lemma 3.9. *Let G be a TH module with a submodule $T \subseteq G$. If N is a proper submodule of G and $h : G \to G$ is a surjective homomorphism, then $h(N)$ is a proper submodule of G, i.e., $h(N) \neq G$.*

Proof. Let N be a proper submodule of G, and let $h : G \to G$ be an R-epimorphism such that $h(N) = G$. It follows that $G = \ker(h) + N$. Now, let $k : G/\ker(h) \to G$ be an R-epimorphism. The composition $k\pi : G \to G$, where $\pi : G \to G/\ker(h)$ is the canonical R-epimorphism, is a surjective of G. As G is TH, we have $\ker(k\pi) \ll_T G$. Given that $\ker(h)$ is a submodule of $\ker(k\pi)$, we conclude that $\ker(h) \ll_T G$. By Lemma 2.3, this implies $\ker(h) \ll T$ and thus $\ker(h) \ll G$. As a result, we obtain $G = N$, This contradicts the assumption that N is a proper submodule of G. □

Proposition 3.10. *If G is a TH module with T is a submodule of G, then G must be DF.*

Proof. Let $h, k \in \mathrm{End}(G)$ such that $hk = 1$. It is clear that k is monomorphism and h is epimorphism. Since $h(\mathrm{Im}(k)) = G$, by Lemma 3.9, we conclude that $\mathrm{Im}(k) = G$, which implies that k is an automorphism. Consequently, we have $kh = 1$, and therefore, G is DF.

□

Corollary 3.11. *Let G be a module and let T be a submodule of M. We examine the assertions.*
(i) G is Hopfian.
(ii) G is TH.
(iii) G is DF.
We have $(i) \implies (ii) \implies (iii)$. Moreover, if M is quasi-projective, then (i), (ii), and (iii) are identical.

Proof. $(i) \implies (ii)$: This is clear.
$(ii) \implies (iii)$: Follows from Proposition 3.10.
$(iii) \implies (i)$: This is established by [[13], Corollary 1.4].

□

Definition 3.12. [10] An R-module G is termed as semi-Hopfian if for every epimorphism of G, the kernel is a direct summand. In other words, every surjective endomorphism of M splits.

Proposition 3.13. *For a semi-Hopfian module G, the statements are identical:*

(1) G is Hopfian.

(2) G is TH.

Proof. (1) ⇒ (2) This is straightforward.
(2) ⇒ (1) Assume $h : G \to G$ is an epimorphism. As G is semi-Hopfian, the map h splits, meaning there exists $k : G \to G$ such that $hk = 1$. As G is TH, Proposition 3.10 implies that G is DF. Consequently, $kh = 1$, making h a monomorphism. Being both surjective and injective, h is an automorphism. Hence, G is Hopfian.

□

Proposition 3.14. *If G is a TH R-module, then each direct summand of G is also TH.*

Proof. Let P be a direct summand of G, so there exists a submodule Q such that $G = P \oplus Q$. Consider a surjective endomorphism $f : P \to P$. This induces a surjective endomorphism $h \oplus 1_Q : G \to G$ defined by

$$(h \oplus 1_Q)(p + q) = h(p) + q,$$

where $p \in P$ and $q \in Q$. As G is TH, we have $\ker(h \oplus 1_Q) \ll_T G$, which implies that $\ker(h) \ll_T P$. Therefore, P is TH.

□

The direct sum of TH modules need not necessarily result in a T-Hopfian module, as demonstrated by this example.

Example 3.15. According to [[13], Remark 1.5] and [[16], Page 19], let F be a field and R be the F-algebra generated by the set $\{a, b, c, d, e, f, g, h\}$, subject to the relations:

$$af + ch = 1, \quad ag + ce = 0, \quad bf + dh = 0, \quad bg + de = 1.$$

This R serves as a classical example of a DF ring for which the matrix ring $Mat_{2\times 2}(R)$ is not DF. Moreover, the R-module R^2 is free but not DF because its endomorphism ring is not DF. As a result, R_R is T-Hopfian, but the module R^2_R is not T-Hopfian by Corollary 3.11.

The following result provides a condition under which the direct sum of two TH modules is TH.

Proposition 3.16. *Let $G = G_1 \oplus G_2$ and T be a submodule of G. If each G_i for $i \in \{1, 2\}$ is a fully invariant submodule of G, then G is TH iff both G_1 and G_2 are TH.*

Proof. The necessity is immediate from Proposition 3.14.

For the sufficiency, let $h = (h_{ij})$ be an epimorphism of G, where each $h_{ij} \in \mathrm{Hom}(G_i, G_j)$ for $i, j \in \{1, 2\}$. By assumption, $\mathrm{Hom}(G_i, G_j) = 0$ for all $i \neq j$. As h is an epimorphism, the maps h_{ii} must be surjective endomorphisms on G_i for each $i \in \{1, 2\}$. Given that each G_i is TH, we have $\ker(h_{ii}) \ll_T G_i$. Because $\ker(h) = \ker(h_{11}) \oplus \ker(h_{22})$, it follows that $\ker(h) \ll_T G$ by Lemma 2.7. Therefore, G is TH.

□

References

[1] Anderson, F. W., Fuller, K. R.: Rings and Categories of Modules, Grad. Texts in Math. Springer-Verlag, New York. **13**, (1992).

[2] Beyranvand, R. and Moradi, F.: Small submodules with respect to an arbitrary submodule. Journal of Algebra and Related Topics, **3**(2), 43-51, (2015).

[3] El Moussaouy, A.: Jacobson monoform modules. Journal of Algebraic Systems **12**(2), 379-390 (2025).

[4] El Moussaouy, A., Moniri Hamzekolaee, A. R., Ziane, M.: Jacobson Hopfian modules. Algebra and Discrete Mathem **33**(1), 116-127 (2022).

[5] El Moussaouy, A., Moniri Hamzekolaee, A. R., Khoramdel, M., Ziane, M. Weak Hopfcity and singular modules. Ann Univ Ferrara **68**, 69-78 (2022).

[6] El Moussaouy, A., Ziane, M. Modules in which every surjective endomorphism has a μ-small kernel. Ann Univ Ferrara **66**, 325-337 (2020).

[7] El Moussaouy, A., Hopfcity and Jacobson small submodules, Algebraic Structures and Their Applications, **10**(2), 31–40 (2023).

[8] El Moussaouy, A., Ziane, M.: A new generalization of Hopfian modules. Palestine Journal of Mathematics **12**(3), 309-319 (2023).

[9] El Moussaouy, A., Ziane, M.: Some properties of endomorphism of modules. Palestine Journal of Mathematics **11**(1), 122-129 (2022).

[10] El Moussaouy, A., Ziane, M. Notes on generalizations of Hopfian and co-Hopfian modules. Jordan Journal of Mathematics and Statistics **15**(1), 43-54 (2022).

[11] El Moussaouy, A., Ziane, M. On T-Hopfcity of modules. International Journal of Mathematics and Computer Sciences **19**(2), 307-310 (2024).

[12] El Moussaouy, A., Ziane, M.: Modules whose surjective endomorphisms have a γ-small kernels. Algebraic Structures and Their Applications, **9**(2), 121-133 (2022).

[13] Ghorbani, A., Haghany, A.: Generalized Hopfian modules. J. Algebra. **255**(2), 324-341 (2002).

[14] Hiremath, V. A.: Hopfian rings and Hopfian modules, Indian J. Pure Appl.Math. **17**(7), 895-900 (1986).

[15] Hmaimou, A., Kaidi, A., Sanchez Campos, E.: Generalized Fitting modules and rings. J. Algebra. **308**, 199-214 (2007).

[16] Lam, T. Y.: Lectures on modules and rings, New York: Springer-Verlag, 1998.

[17] Wang, Y.: Generalizations of Hopfian and co-Hopfian modules. Int. J. Math. Sci. **9**, 1455-1460 (2005).

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