Palestine Journal of Mathematics

Vol 14(4)(2025) , 84–89

ON VIRTUALLY SEMISIMPLE MODULES AND RINGS

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Communicated by: Manoj Kumar Patel

MSC 2010 Classifications: Primary 16D69, 16S50; Secondary 16D70, 13F10.

Keywords and phrases: Virtually semisimple modules, Essentially iso-retractable modules, Iso-retractable modules, Semisimple modules.

The author would like to thank the reviewer and editor for their constructive comments and valuable suggestions that improved the quality of our paper

Abstract We find that the class of virtually semisimple modules is strictly between that of essentially iso-retractable modules and epi-retractable modules and we prove that a module is virtually semisimple module if and only if each of its essential submodule is isomorphic to a summand of it. Further, we show that a distributive module $M = \oplus_{i=1}^{n} M_i$ is virtually semisimple module if and only if every M_i is virtually semisimple. Also, we show that a module ${}_RM$ is co-semisimple if and only if all co-cyclic modules of $\sigma[{}_RM]$ are virtually semisimple.

1 Introduction

Unless otherwise specified, all rings in the work are associative and have identity and each module is a unital left module. Recall [12], a submodule E is called *large* or *essential* and symbolised by $E \leq_e M$ if whenever $E \cap K = 0$, then $K = 0$ for any submodule K. Recall [17], a non-zero module M is termed as *simple* if each of its non-zero submodule is equal to M; and a module M is *semisimple* if and only if each of its (essential) submodule is a summand if and only if all essential submodules are equal to M. Clearly, every simple module is semisimple. We refer readers to [12, 14, 17] for any undefined terminologies.

The idea of retractability is well-known in module theory. In the journey of retractability of modules, Ghorbani et al. [9] introduced the notion of epi-retractable modules in 2009. They called a module M *epi-retractable* if there is a surjective homomorphism from M to L for any non-zero submodule L. In 2016, Chaturvedi [4] introduced the notion of iso-retractable modules that is a proper subset of class of epi-retractable modules. He said a module M as *iso-retractable* if each of its non-zero submodule is isomorphic to M. A non-zero iso-retractable module was coined as *isosimple* by Facchini-Nazemian [8] and *virtually simple* by Behboodi et al. [1]. In 2018, Behboodi et al. [1] presented the idea of virtually semisimple modules as a common generalization of iso-retractable modules and semisimple modules. They called M as *virtually semisimple* if each of its submodule is isomorphic to a summand of itself. As a common generalization of semisimple modules and iso-retractable modules, we together with Chaturvedi et al. [6] presented the idea of essentially iso-retractable modules in 2022. They called M as *essentially iso-retractable* if its each essential submodule is isomorphic to M.

Here we share a connection of virtually semisimple modules with the theory of retractability of modules and discuss its some new properties. Also, we provide characterizations of some known classes of modules via virtually semi simple modules. Using Proposition 2.1 and 2.4, we have the following trivial implication diagram for a module M:

$$\begin{array}{ccccc}
\text{Iso-retractable} & \Longrightarrow & \text{essentially iso-retractable} & \Longrightarrow & \text{virtually semisimple} \\
\Uparrow & & \Uparrow & & \Downarrow \\
\text{simple} & \Longrightarrow & \text{semisimple} & \Longrightarrow & \text{epi-retractable}
\end{array}$$

2 Some properties of virtually semisimple modules

We begin by demonstrating that class of virtually semisimple modules contains class of essentially iso-retractable modules. Proposition 2.1 and 2.3 and their proofs are based on [1, Lemma 2.1].

Proposition 2.1. *Every essentially iso-retractable module is virtually semi simple.*

Proof. Let $L \leq M$. Then $L \oplus K \leq_e M$ for some submodule K. So there is an isomorphism $\alpha : L \oplus K \to M$ by assumption. It follows that $L \cong \alpha(L)$ and $M = \alpha(L) \oplus \alpha(K)$. Hence L is isomorphic to a direct summand $\alpha(L)$. □

Remark 2.2. Virtually semisimple modules are not necessarily essentially iso-retractable. For instant, let $M = \mathbb{Z}_4 \bigoplus_{i=1}^{\infty} \mathbb{Z}_2$ and $R = \mathbb{Z}_4$. Then M is a virtually semisimple R-module by [2, Example 2.7] and $Soc({}_RM) = 2\mathbb{Z}_4 \bigoplus_{i=1}^{\infty} \mathbb{Z}_2 \leq_e M$. If $Soc({}_RM) \cong M$, then M is semisimple which is a contradiction as $Soc({}_RM) \neq M$. Hence, it follows that ${}_RM$ is not essentially iso-retractable. But, we observe the following for it:

Proposition 2.3. *Every virtually semisimple module M which is finitely generated is essentially iso-retractable.*

Proof. As M is finitely generated, M is Noetherian by [2, Proposition 2.8]. Let $N \leq_e M$. Then by [10, Corollary 5.21], $u.dim(N) = u.dim(M)$ as M is Noetherian. Also by assumption, $N \cong K$ for some decomposition $M = K \oplus K'$. Hence $u.dim(N) = u.dim(K) = u.dim(M)$ as K is summand and $N \cong K$. So, $K \leq_e M$ by [10, Corollary 5.21]. Thus $M = K \cong N$ and so M is essentially iso-retractable. □

Proposition 2.4. *Every virtually semisimple module M is epi-retractable.*

Proof. Let $N \leq M$. Then $N \oplus L \leq_e M$ for some submodule L. So by assumption, $D \cong N \oplus L$ for a direct summand $D \leq M$. Let $\theta : D \to N \oplus L$ be the isomorphism. Then for canonical projection maps $\pi_D : M \to D$ and $\pi_N : N \oplus L \to N$, homomorphism $\pi_N \theta \pi_D : M \to N$ is surjective. Thus M is epi-retractable. □

Remark 2.5. Epi-retractable modules are not surely virtually semisimple. Consider example of a module M constructed in [1, Example 3.2]. Let $M = RX \oplus RY$, where $R = F[[x, y]]/ < xy >$ with field F, $X = x+ < xy >$ and $Y = y+ < xy >$. Then by [1, Example 3.2], ${}_RM$ is not virtually semisimple. Let $0 \neq N \leq M$. Then $N = RX^i$ or RY^j or $RX^i \oplus RY^j$ or $R(X^i + Y^j)$. Since we will get an epimorphism form M to N for each case, so M is epi-retractable.

Proposition 2.6. *Every quasi-projective virtually semisimple module M is semi simple.*

Proof. Let $K \leq M$. Then $K \cong L$ for some summand L by assumption. Since L is summand of a quasi-projective module M, hence $L \cong K$ is M-projective. Since M is virtually semisimple, so by Proposition 2.4, M is epi-retractable. Hence we get a surjective homomorphism $\alpha : M \to K$. So by M-projectivity of K, there is a homomorphism $\beta : K \to M$ so that $\alpha o \beta = I_K$. It follows that $K \leq^{\oplus} M$. So M is semisimple. □

Corollary 2.7. *Let M be virtually semisimple. Then M is projective if and only if its all submodules are projective.*

Proof. By Proposition 2.6, M is semisimple if M is projective. Hence its each submodule is a summand and so a projective submodule. The converse follows from the fact that M is also a submodule of M. □

Recall [17], R is termed as left hereditary if each of its left ideal is projective. As consequence of Proposition 2.6 and Corollary 2.7, we derive that if R is left virtually semisimple, then R is left hereditary, semisimple.

Recall [3], M is said *distributive* if for any three submodules N_1, N_2, N_3, we have $N_1 \cap (N_2 + N_3) = (N_1 \cap N_2) + (N_1 \cap N_3)$. Recall [15], M is termed as *square free* if $L \cap K = 0$ and $L \cong K$ implies that $L = K = 0$ for any two submodules L and K. Clearly distributive modules are square free.

Theorem 2.8. *Let M be distributive and $M = \oplus_{i=1}^n M_i$. Then M is virtually semisimple if and only if each M_i is virtually semisimple.*

Proof. ($\Longleftarrow$) Let $N \leq M$. Then $N \cap M_i \leq M_i$ and $N = N \cap (\oplus_{i=1}^n M_i) = \oplus_{i=1}^n (N \cap M_i)$ as M is distributive. By assumption, $N \cap M_i \cong K_i$ for some decomposition $M_i = K_i \oplus K_i'$. Let $\phi_i : K_i \to N \cap M_i$ be the isomorphism . Then, $\oplus_{i=1}^n K_i$ is a summand of M and the map $\phi : \oplus_{i=1}^n K_i \to N (= \oplus_{i=1}^n (N \cap M_i))$ given by $\phi(\sum_{i=1}^n k_i) = \sum_{i=1}^n \phi_i(k_i)$ is an isomorphism.

($\Longrightarrow$) Let $N \leq M_i$. Then $N \leq M$ and so there is an isomorphism $\phi : D \to N$ for some decomposition $M = D \oplus D'$. Let $\bigoplus_{j=1, j \neq i}^n M_j = L_i$. Then $M = M_i \oplus L_i$. So, by distributive property, we have $D = (D \cap M_i) \oplus (D \cap L_i)$. If $D \cap L_i$ is non-zero, then $D \cap L_i \cong \phi(D \cap L_i)$ and $\phi(D \cap L_i) \cap (D \cap L_i) = 0$. But then $D \cap L_i = \phi(D \cap L_i) = 0$ as M is square free. This is a contradiction. Hence $D \cap L_i = 0$ and so $D = D \cap M_i$. Hence $D \leq M_i$ and so $M_i = M_i \cap M = M_i \cap (D \oplus D') = (M_i \cap D) \oplus (M_i \cap D') = D \oplus (M_i \cap D')$. Thus $N \cong D \leq^{\oplus} M_i$. Hence M_i is virtually semisimple. □

Proposition 2.9. *Let L and K be virtually semisimple R-modules with the property $ann_R(K) + ann_R(L) = R$. Then $M = K \oplus L$ is virtually semisimple.*

Proof. Let $N \leq M$. Then by [11, Proposition 3.9], $N = K_1 \oplus L_1$ where $K_1 \leq K$ and $L_1 \leq L$. By assumption, there is decomposition $K = K_2 \oplus K_2'$ and $L = L_2 \oplus L_2'$ such that $K_1 \cong K_2$ and $L_1 \cong L_2$. Hence $N = K_1 \oplus L_1 \cong K_2 \oplus L_2$ and $M = K \oplus L = K_2 \oplus K_2' \oplus L_2 \oplus L_2' = (K_2 \oplus L_2) \oplus (K_2' \oplus L_2')$. Thus M is virtually semisimple. □

Corollary 2.10. *Let $\{M_i\}_{i=1}^n$ be virtually semisimple R-modules with the property $\sum_{i=1}^n ann_R (M_i) = R$. Then $M = \bigoplus_{i=1}^n M_i$ is virtually semisimple.*

3 Characterizations via virtually semisimple module

By observing virtually semisimple module definition and semisimple module characterization, we get the following trivial but fruitful characterization of virtually semisimple modules.

Proposition 3.1. *M is virtually semi simple if and only if its every essential submodule is isomorphic to a summand of M.*

Proof. Suppose each essential submodule is isomorphic to some summand of M. Let $L \leq M$. Then $L \oplus K \leq_e M$ for some submodule K. Hence, there is an isomorphism $\alpha : L \oplus K \to P$ for some decomposition $M = P \oplus Q$. So, we get $L \cong \alpha(L)$ and $M = P \oplus Q = \alpha(L) \oplus \alpha(K) \oplus Q$. Thus M is virtually semisimple. The converse is clear. □

Recall [12], a submodule Q is termed as *complement* of K into a module M if Q be maximal with the property $Q \cap K = 0$. If Q is complement of certain submodule of M, Q is referred as *complement submodule* of it and denoted by $Q \leq_c M$. Recall [5], M is referred as *iso-c-retractable* if its each non-zero complement submodule is isomorphic to it. We will now characterize iso-retractable modules below.

Proposition 3.2. *M is iso-retractable if and only if M is virtually semisimple and iso-c-retractable.*

Proof. Suppose M is virtually semisimple and iso-c-retractable. If $0 \neq L \leq M$, then $L \cong Q$ for some decomposition $M = Q \oplus Q'$, as M is virtually semisimple. This implies that $0 \neq Q \leq_c M$. So $M \cong Q$ as M is iso-c-retractable. Hence $M \cong Q \cong L$. Thus M is iso-retractable. The converse is obvious. □

Recall [13], M is called *dual Rickart* if image of all its endomorphisms are summands. The next result is a characterization of semisimple modules.

Theorem 3.3. *The following statements are equivalent for a module M:*

(i) M is semisimple;

(ii) M is satisfies C_2-condition and virtually semisimple;

(iii) M dual Rickart and is virtually semisimple.

Proof. 1) $\Longrightarrow$ 2). Clear.

2) $\Longrightarrow$ 3). If $f \in End(M)$, then $f(M) \cong D$ for some direct summand D as M is virtually semisimple. So $f(M)$ is a summand as M is with C_2-condition. Thus M is dual Rickart.

3) $\Longrightarrow$ 1). Proposition 2.4 indicates that M will be epi-retractable. So for $N \leq M$, we get a surjective homomorphism $\phi : M \to N$. So for the inclusion map $i : N \to M$, the map $\phi' = io\phi : M \to M$ will be a homomorphism with property that $\phi'(M) = N$. Since M is dual Rickart, $\phi'(M) = N$ a summand of M. Therefore M is semisimple. □

We present a characterization of semisimple rings below.

Theorem 3.4. *The assertions below are identical for a ring R:*

(i) R is semisimple;

(ii) All left (right) R-modules are essentially iso-retractable;

(iii) All injective left (right) R-modules are essentially iso-retractable;

(iv) All projective left (right) R-modules are essentially iso-retractable;

(v) All injective left (right) R-modules are direct sum of essentially iso-retractable modules.

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (5) and (1) $\Longrightarrow$ (2) $\Longrightarrow$ (4) are evident. (4) $\Longrightarrow$ (1) This may be deduced from the projectivity of $_RR$ and [6, Proposition 13]. (5) $\Longrightarrow$ (1) Since injective envelope $E(_RR)$ is injective, so by assumption, $E(_RR) = \oplus_{i\in I} M_i$ where each M_i is essentially iso-retractable. Since M_i is a direct summand of an injective module, M_i is also injective. Hence M_i is semisimple by [6, Proposition 13] and so $E(_RR)$ is semisimple. Now since $_RR \leq_e E(_RR)$, $_RR = E(_RR)$ is semisimple. □

Recall [7], $_RM$ is referred as *co-semisimple* (or *V-module*) if each simple module of $\sigma[_RM]$ is M-injective. If $_RR$ is co-semisimple, it is said that R is a left *V-ring*. A module that contains an essential simple submodule is called *co-cyclic*.

Theorem 3.5. *$_RM$ is co-semisimple if and only if all co-cyclic modules of $\sigma[_RM]$ are virtually semisimple.*

Proof. ($\Longrightarrow$) Let $N \in \sigma[_RM]$ and it is co-cyclic. Then there is $K \leq N$ with the property that K is essential and simple. Since M is co-semisimple, K is M-injective and so $K \leq^{\oplus} M$. But since K is essential, $K = M$. So M is simple and hence virtually semisimple.

($\Longleftarrow$)Let $N \in \sigma[_RM]$ and it is simple. Let $\widehat{N}$ be its M-injective envelope. Then $N \leq_e \widehat{N}$ and $N = Rn$ for some $n \in N$. Let $0 \neq a \in \widehat{N}$. Then $ra \neq 0$ and $ra \in N = Rn$ for some $r \in R$. Hence $ra = r'n \neq 0$ for some $r' \in R$. This implies that $Rra = Rr'n \leq Rn = N$. But since $ra \neq 0$ and N is simple, so $Rra = Rn$ and hence $N = Rn = Rra \leq Ra$. Thus $N \leq_e Ra$ and N is simple. Hence by assumption, Ra is virtually semisimple and so Ra is essentially iso-retractable by Proposition 2.3. Hence $Ra \cong N$ and so Ra is simple. This implies that $Ra = N$ and hence $a \in N$. Therefore, $\widehat{N} = N$ is M-injective. □

The characterization of left V-rings afterwards is because of Theorem 3.5.

Corollary 3.6. *R is a left V-ring if and only if each co-cyclic all left R-modules are virtually semisimple.*

Recalling from [16] that if every non-zero endomorphism of M is surjective homomorphism, then M is referred to satisfy the $(**)$-*property*. Now, we provide a characterization of simple modules.

Proposition 3.7. *A non-zero module M is simple if and only if M is virtually semisimple and satisfies $(**)$-property.*

Proof. Let M satisfies $(**)$-property and be virtually semisimple. Proposition 2.4 indicates that M will be epi-retractable. So for $L \leq M$, we get a surjective homomorphism $f : M \to L$. So for the inclusion map $i : L \to M$, $f' = iof : M \to M$ is homomorphism and $f'(M) = L$. Hence $L = f'(M) = M$ as M satisfies $(**)$-property. Therefore M is simple. □

Lemma 3.8. *A nonzero module M is not-singular and iso-retractable if and only if M is uniform, non-singular and virtually semisimple.*

Proof. Let M be not-singular and iso-retractable. Then $Z(M) \neq M$ and so $Z(M) = 0$ because $Z(M) \neq 0$ implies $Z(M) \cong M$ and hence $Z(M) = M$ which will be a contradiction. Thus M is non-singular; and M is uniform by [4, Theorem 1.12].

On the other hand, if M is uniform, non-singular and virtually semisimple, then M is not-singular. Also, Proposition 2.4 indicates to M be epi-retractable. Let $0 \neq N \leq M$. Then there is an epimorphism $\alpha : M \to N$. Since M is non-singular, $ker(\alpha) \nleq_e M$ and so $ker(\alpha) = 0$ as M is uniform. So $N \cong M$. Thus M is iso-retractable. □

Proposition 3.9. *A ring R is a left principal ideal domain if and only if $_RR$ is uniform and there exists a non-zero uniform, non-singular and virtually semisimple left R-module.*

Proof. Assume that $_RR$ is uniform and there is a left R-module M which is non-zero uniform, non-singular and virtually semisimple. Then M is iso-retractable by Lemma 3.8. Let $0 \neq x \in M$. Then the map $\theta : R \to Rx$ given by $\theta(r) = rx$ is a one-one homomorphism as $ann(x) = 0$. Hence $R \cong \theta(R) \leq Rx \leq M$ and so $R \cong \theta(R) \cong M$ as M is iso-retractable. This implies that $_RR$ is iso-retractable and so by [1, Proposition 2.2], R is left principal ideal domain.

On the other hand, if R is left principal ideal domain, the R is non-singular and $_RR$ is iso-retractable by [1, Proposition 2.2]. So result follows from Lemma 3.8. □

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Received: 2024-10-01
Accepted: 2025-11-18