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On Graded and Augmented Graded Semirings

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Abstract This paper investigates certain properties of graded semirings in relation to their homogeneous components. We establish necessary and sufficient conditions under which a semiring R, graded by a semigroup G in which all elements are idempotent, exhibits several important properties: zero-sum free, additively regular, inverse, E-inversive, yoked, k-regular,c-semiring, or MV-semiring in terms of its homogeneous components. Furthermore, we derive necessary and sufficient conditions for an augmented graded semiring R, graded by a monoid G with identity in which all elements are idempotent, to be zero-sum free, additively regular, inversive, E-inversive, or yoked, in terms of the homogeneous component with respect to the identity element.

1 Introduction

The concept of semirings was first introduced by H. S. Vandiver in 1935 [20], and it has since attracted the attention of many researchers. In 2008, Sharma and Joseph [19] studied semirings graded by a finite group. The notion of augmented graduation of rings was introduced by Refai in 1997 [15], and was later extended to semirings by Darwish, Alnader, and Qrewi in 2021 [7], who investigated several results related to this concept.

This paper explores various properties of graded semirings through an examination of their homogeneous components. Specifically, we provide necessary and sufficient conditions under which a semiring R that is graded by a semigroup G whose elements are all idempotent, satisfies key properties such as being zero-sum free, additively regular, inverse, E-inversive, yoked, k-regular, c-semiring, or MV-semiring in terms of its homogeneous components. Moreover, we extend our analysis to the setting of augmented graded semirings. For a semiring R augmented graded by a monoid G with identity in which all elements are idempotent, we characterize the conditions under which R is zero-sum free, additively regular, inversive, E-inversive, or yoked, based on the structure of the homogeneous component corresponding to the identity element. The main objective of this study is to explore the behavior of graded semirings with respect to one or more of their homogeneous components. We present several results that contribute to the development of semiring theory and its applications. Note that, by a semigroup (a monoid), we mean a multiplicative semigroup (a multiplicative monoid) unless otherwise specified.

2 Definitions

This section introduces key definitions concerning semirings, which will be frequently used throughout the text [1, 3, 4, 5, 6, 7, 8, 10, 14, 16, 18, 19].

Definition 2.1 ([1, 3]). A semiring is a nonempty set R with associative operations (+) and (.) for which (+) is commutative and there exists $0 \in R$ such that for each $\alpha \in R$, we get $\alpha 0 = 0\alpha = 0$ and $\alpha + 0 = \alpha$, and the multiplication operation is distributive over addition on both sides.

If (.) is commutative, then R is called commutative. A subsemiring S of a semiring R with zero 0 is a subset of the semiring R in which $\alpha+\beta, \alpha\beta \in S$ for all $\alpha, \beta \in S$ and $0 \in S$.

If there exists an element $1 \in R$ such that $1x = x1 = x$ for all $x \in R$, then $1 \in R$ is said to be the unity in R and R is called a semiring with unity.

Definition 2.2 ([1]). Denote $E^+(R)$ the set of all additive idempotent elements in a semiring R where an additive idempotent element in R is such an element ζ for which $\zeta = \zeta + \zeta$. If all elements in R are additive idempotent, then R is called additive idempotent. Now, if for each element $\kappa \in R$, there exists an element $\psi \in R$ for which $\kappa + \psi \in E^+(R)$, then R is called E-inversive.

Definition 2.3 ([10, 19]). In a semiring R if for every $\zeta, \kappa \in R$, there exists $\psi \in R$ for which $\zeta + \psi = \kappa$ or $\kappa + \psi = \zeta$, then R is called yoked.

Definition 2.4 ([8, 16]). A semiring R is said to be zero-sum free if for all $\zeta, \kappa \in R$, $\zeta + \kappa = 0$ implies that $\zeta = \kappa = 0$.

Definition 2.5 ([5]). An element $\zeta \neq 0$ in a semiring R is called multiplicatively right cancellative if for every $\psi, \kappa \in R$, we have $\psi\zeta = \kappa\zeta$ follows that $\psi = \kappa$. If every non-zero element in R is multiplicatively right cancellative, then R is called multiplicatively right cancellative.

Definition 2.6 ([5, 8]). An infinite element in a semiring R is such an element ζ for which $\zeta + \kappa = \zeta$ for all $\kappa \in R$.

Definition 2.7 ([14]). An element ζ in a semiring R is called additively regular if there exists unique element $\kappa \in R$ such that $\zeta = \zeta + \kappa + \zeta$. If every element in R is additively regular, then R is called additively regular.

Definition 2.8 ([14]). An element ζ in a semiring R is called inverse if there exists unique element $\kappa \in R$ such that $\zeta = \zeta + \kappa + \zeta$ and $\kappa = \kappa + \zeta + \kappa$. If every element in R is inverse, then R is called inverse.

Definition 2.9 ([6, 18]). An element ζ in a semiring R is called k-regular if there exist $\eta, \kappa \in R$ for which $\zeta + \zeta\eta\zeta = \zeta\kappa\zeta$. If every element in R is k-regular, then R is called k-regular.

Definition 2.10. [13] In a semiring R if $\zeta\kappa = 0$ follows that $\zeta = \kappa = 0$ for all $\zeta, \kappa \in R$, then R is called zero-divisor free.

Definition 2.11 ([17]). A commutative additively idempotent semiring R with unity 1 is called c-semiring if $\zeta + 1 = 1$ for all $\zeta \in R$.

Definition 2.12 ([6, 18]). An additively idempotent semiring R is called k-semifield if for every $\zeta \in R$ and for every $0 \neq \eta \in R$, there exist $\psi_1, \psi_2, \theta_1, \theta_2 \in R$ for which $\zeta + \eta\psi_1 = \eta\psi_2$ and $\zeta + \theta_1\eta = \theta_2\eta$.

Definition 2.13 ([4]). On every additively idempotent semiring R there is a natural order $\leq$ given by $\zeta \leq \eta$ iff $\zeta + \eta = \eta$ for every $\zeta, \eta \in R$. Now, A commutative additively idempotent semiring R with unity 1 is called MV-semiring if for each element $\zeta \in R$, there exists a greatest element ζ^* for which $\zeta\zeta^* = 0$, and for every $\eta, \kappa \in R$, we have $\eta + \kappa = (\eta^*(\eta^*\kappa)^*)^*$.

Note that when R is MV-semiring with unity 1, we obtain the following properties [4]

- $0^* = 1, 1^* = 0$;
- $\eta \leq 1$ for all $\eta \in R$.

Definition 2.14 ([7, 19]). A graded semiring R by a group (a semigroup, a monoid) G is a semiring in which $R = \bigoplus_{t \in G} R_t$ where R_t is an additive submonoid of R and $R_tR_s \subseteq R_{ts}$ for all $t, s \in G$. Denote (R, G) the grading of the semiring R by G. For all $t \in G$, R_t is called a homogeneous component of the grading (R, G). The support of the grading (R, G) is a subset of G denoted by $Supp(R, G)$ defined as $\{g \in G | R_g \neq \{0\}\}$.

It can be easily seen that if R is graded by a monoid G with identity e, then R_e is a subsemiring of R.

Definition 2.15 ([7]). An augmented graded semiring R by a monoid G with identity e is a semiring satisfies the following:

(i) Both $R = \underset{g\in G}{\oplus} R_g$ and $R_e = \underset{g\in G}{\oplus} R_{e-g} = \underset{g\in G}{\oplus} (R_e)_g$ are graded by G.

(ii) For each element $g \in G$, there exists an element ψ_g in R_g for which $R_g = \underset{h\in G}{\oplus} R_{e-h}\psi_g$.

(iii) For every $g, h \in G$ where $\psi_g \neq 0, \psi_h \neq 0$ are two elements in R_g, R_h respectively such that $R_g = \underset{t\in G}{\oplus} R_{e-t}\psi_g$ and $R_h = \underset{t\in G}{\oplus} R_{e-t}\psi_h$, we have $\psi_g\psi_h = \psi_{gh}$ such that ψ_{gh} is an element in R_{gh} where $R_{gh} = \underset{t\in G}{\oplus} R_{e-t}\psi_{gh}$ and $(x\psi_g)(y\psi_h) = xy\psi_{gh}$ for all $x, y \in R_e$.

3 On Graded Semirings

This section examines some properties of graded semirings (augmented graded semirings) and their homogeneous components. First, we derive necessary and sufficient conditions that if R is a graded semiring by a semigroup G in which all elements are idempotent, then R is zero- sum free, additively regular, inverse, E- inversive, yoked, k-regular, c- semiring, or MV-semiring in terms of its homogeneous components.

Theorem 3.1. *Let $R = \underset{g\in G}{\oplus} R_g$ be a graded semiring by a semigroup G in which all elements are idempotent. Then*

(i) R is zero-sum free iff R_g is zero sum free for all $g \in G$.

(ii) Let $\zeta = \sum_{g\in G} \zeta_g$, where $\zeta_g \in R_g$ for all $g \in G$, be an element in R. Then the element ζ is infinite in R iff ζ_g is infinite in R_g for all $g \in G$.

(iii) R is additively regular iff R_g is additively regular for all $g \in G$.

(iv) R is inverse iff R_g is inverse for all $g \in G$.

(v) R is E-inversive iff R_g is E-inversive for all $g \in G$.

(vi) R is yoked implies R_g is yoked for all $g \in G$.

(vii) Suppose $R_gR_h = \{0\}$ for all $g, h \in G$ with $g \neq h$. Then

a. R is k-regular iff R_g is k-regular for all $g \in G$.

b. Suppose $Supp(R, G)$ is finite. Then R is c-semiring iff R_g is c-semiring for all $g \in G$.

c. Suppose $Supp(R, G)$ is finite. Then R is MV-semiring iff R_g is MV-semiring for all $g \in G$.

d. R is k-semifield implies R_g is k-semifield for all $g \in G$.

Proof. Given that all elements of G are idempotent, we get R_g is a subsemiring of R for all $g \in G$.

(i) Assume that R is zero-sum free. Let g be an element in G and ζ_g, κ_g be two elements in R_g such that $\zeta_g + \kappa_g = 0$. Therefore $\zeta_g = \kappa_g = 0$ and hence R_g is zero-sum free. Conversely, assume that R_g is zero-sum free for all $g \in G$. Let ζ, κ be two elements in R. By hypothesis R is graded, we can write ζ, κ as follows $\zeta = \sum_{h\in G} \zeta_h, \kappa = \sum_{h\in G} \kappa_h$ such that $\zeta_h, \kappa_h \in R_h$ for all $h \in G$. Suppose $\zeta + \kappa = 0$. Therefore $\sum_{h\in G} \zeta_h + \sum_{h\in G} \kappa_h = \sum_{h\in G} (\zeta_h + \kappa_h) = 0$. Now, R being graded, it follows that $\zeta_g + \kappa_g = 0$ for all $g \in G$. Hence $\zeta_g = \kappa_g = 0$ for all $g \in G$. Therefore $\zeta = \kappa = 0$. Thus R is zero-sum free.

(ii) Assume that ζ is infinite. Therefore $\zeta + t = \zeta$ for all $t \in R$. Let h be an element in G. Suppose s_h is an element in R_h. Therefore $\zeta + s_h = \zeta$ and hence $\sum_{g\in G} \zeta_g + s_h = \sum_{g\in G} \zeta_g$. It follows that $\zeta_h + s_h = \zeta_h$. Thus ζ_h is infinite element in R_h. Conversely, assume that ζ_g infinite in R_g for all $g \in G$. Suppose s is an element in R. By hypothesis R is graded, we can write s as $s = \sum_{g\in G} s_g$ such that $s_g \in R_g$ for all $g \in G$. Therefore $\zeta_g + s_g = \zeta_g$ for all $g \in G$ and hence $\sum_{g\in G} \zeta_g + \sum_{g\in G} s_g = \sum_{g\in G} \zeta_g$. It follows that $\zeta + s = \zeta$. Thus ζ is infinite in R.

(iii) Assume that R is additively regular and g is an element in G. Let ζ_g be an element in R_g. Then there exists unique element $\kappa \in R$ such that $\zeta_g + \kappa + \zeta_g = \zeta_g$. As R is graded, we can write κ as $\kappa = \sum_{h\in G} \kappa_h$ such that $\kappa_h \in R_h$ for all $h \in G$. Therefore $\zeta_g + \sum_{h\in G} \kappa_h + \zeta_g = \zeta_g$. It follows that $\zeta_g + \kappa_g + \zeta_g = \zeta_g$. Now, R being graded and additively regular imply that κ_g is unique. Therefore, R_g is additively regular. Conversely, assume that R_g is additively regular for all $g \in G$. Let ζ be an element in R. As R is graded, we can write ζ as $\zeta = \sum_{g\in G} \zeta_g$ such that $\zeta_g \in R_g$ for all $g \in G$. Then for each $\zeta_g \in R_g$ and each $g \in G$ there exists a unique element $\kappa_g \in R_g$ such that $\zeta_g + \kappa_g + \zeta_g = \zeta_g$. Therefore $\sum_{g\in G} \zeta_g + \sum_{g\in G} \kappa_g + \sum_{g\in G} \zeta_g = \sum_{g\in G} \zeta_g$ and hence $\zeta + \sum_{g\in G} \kappa_g + \zeta = \zeta$. Now, put $\kappa = \sum_{g\in G} \kappa_g$. As κ_g is the unique element in R_g such that $\zeta_g + \kappa_g + \zeta_g = \zeta_g$ for each $g \in G$ and that R is graded, we get $\kappa = \sum_{g\in G} \kappa_g \in R$ is the unique element in R such that $\zeta + \kappa + \zeta = \zeta$. Therefore, R is additively regular.

(iv) The proof of (iv) is done in the same way as the proof of (iii).

(v) Assume that R is E-inversive. Let g be an element in G and ζ_g be an element in R_g. Then there exists an element $x = \sum_{h\in G} x_h \in R$, where $x_h \in R_h$ for all $h \in G$ for which $x + \zeta_g \in E^+(R)$. Therefore $x_g + \zeta_g \in E^+(R_g)$. Thus R_g is E-inversive. Conversely, assume that R_g is E-inversive for all $g \in G$. Let ζ be an element in R. As R is graded, we can write ζ as $\zeta = \sum_{g\in G} \zeta_g$ such that $\zeta_g \in R_g$ for all $g \in G$. Therefore for each $g \in G$ and each $\zeta_g \in R_g$ there exists an element $x_g \in R_g$ such that $x_g + \zeta_g \in E^+(R_g)$ (we choose $x_g = 0$ when $\zeta_g = 0$ for all $g \in G$). Now, put $x = \sum_{g\in G} x_g$. Then $x \in R$ and $x + \zeta \in E^+(R)$. Thus R is E-inversive.

(vi) Assume that R is yoked. Let g be an element in G and ζ_g, κ_g be two elements in R_g. Then there exists an element $r = \sum_{g\in G} r_g$ in R such that $r_g \in R_g$ for all $g \in G$ for which $\zeta_g + r = \kappa_g$ or $\kappa_g + r = \zeta_g$. Therefore $\zeta_g + r_g = \kappa_g$ or $\kappa_g + r_g = \zeta_g$. Thus R_g is yoked.

(vii) a. Assume that R is k-regular. Let g be an element in G and ζ be an element in R_g. Then there exist $x, y \in R$ such that $\zeta + \zeta x \zeta = \zeta y \zeta$. By hypothesis R is graded, we can write x, y as follows $x = \sum_{h\in G} x_h$, $y = \sum_{h\in G} y_h$ such that $x_h, y_h \in R_h$ for all $h \in G$. Therefore $\zeta + \zeta \sum_{h\in G} x_h \zeta = \zeta \sum_{h\in G} y_h \zeta$. Since R is graded and $R_t R_s = \{0\}$ for all $t, s \in G$ with $s \neq t$, we get $\zeta + \zeta x_g \zeta = \zeta y_g \zeta$. Hence R_g is k-regular. Conversely, assume that R_g is k-regular for all $g \in G$. Let ζ be an element in R. Then we can write ζ as $\zeta = \sum_{g\in G} \zeta_g$ such that $\zeta_g \in R_g$ for all $g \in G$ and for each $\zeta_g \in R_g$ and each $g \in G$ there exist $x_g, y_g \in R_g$ such that $\zeta_g + \zeta_g x_g \zeta_g = \zeta_g y_g \zeta_g$ (we choose $x_g = y_g = 0$ when $\zeta_g = 0$ for all $g \in G$). Hence $\zeta + \sum_{g\in G} \zeta_g x_g \zeta_g = \sum_{g\in G} \zeta_g y_g \zeta_g$. As the condition $R_t R_s = \{0\}$ holds for all $t, s \in G$ with $s \neq t$, we get $\zeta + \zeta \sum_{g\in G} x_g \zeta = \zeta \sum_{g\in G} y_g \zeta$. Now, put $x = \sum_{g\in G} x_g$, $y = \sum_{g\in G} y_g$. Then $x, y \in R$ and $\zeta + \zeta x \zeta = \zeta y \zeta$. Thus R is k-regular.

b. Assume that R is c-semiring and the unity in R is $1 = \sum_{h\in G} \zeta_h$ such that $\zeta_h \in R_h$ for all $h \in G$. Let g be an element in G and let x be an element in R_g. Then $x1 = 1x = x$. Therefore $x \sum_{h\in G} \zeta_h = (\sum_{h\in G} \zeta_h)x = x$ and hence $\sum_{h\in G} x\zeta_h = \sum_{h\in G} \zeta_h x = x$. Now, by hypothesis R is graded and $R_t R_s = \{0\}$ for all $t, s \in G$ with $s \neq t$, we get $x\zeta_g = \zeta_g x = x$. It follows that R_g with unity ζ_g for all $g \in G$. R being c-semiring, it follows that R is commutative and $\zeta + \zeta = \zeta$ for all $\zeta \in R$. Therefore R_g is commutative and $\zeta + \zeta = \zeta$ for all $\zeta \in R_g$. Let y be an element in R_g. Then $y + 1 = 1$. Therefore

$y+\sum_{h\in G}\zeta_h = \sum_{h\in G}\zeta_h$ and hence $y+\zeta_g = \zeta_g$. Thus R_g is c-semiring. Conversely, assume that R_g is c-semiring for all $g \in G$. Suppose the unity in R_g is ζ_g for all $g \in G$, and put $1 = \sum_{g\in G}\zeta_g$. By hypothesis $Supp(R,G)$ is finite, we get $1 = \sum_{g\in G}\zeta_g \in R$. Let η be an element in R. Then we can write η as $\eta = \sum_{g\in G} x_g$ such that $x_g \in R_g$ for all $g \in G$. Now, R_g being c-semiring for all $g \in G$, it follows that $x_g + x_g = x_g$ and $x_g + \zeta_g = \zeta_g$ for all $g \in G$. Therefore $\eta + \eta = \eta$ and $\eta + 1 = 1$. Since the unity in R_g is ζ_g for all $g \in G$, we get $x_g\zeta_g = \zeta_g x_g = x_g$ for all $g \in G$. Therefore $\sum_{g\in G} x_g\zeta_g = \sum_{g\in G}\zeta_g x_g = \eta$. Hence by the condition $R_tR_s = \{0\}$ for all $t,s \in G$ with $s \neq t$, we get $\sum_{t\in G} x_t \sum_{g\in G}\zeta_g = \sum_{g\in G}\zeta_g\sum_{t\in G} x_t = \eta$. Thus $\eta 1 = 1\eta = \eta$. Let ψ, s be two elements in R. Then we can write ψ, s as follows $\psi = \sum_{g\in G}\psi_g, s = \sum_{g\in G} s_g$ such that $\psi_g, s_g \in R_g$ for all $g \in G$. By the condition $R_tR_s = \{0\}$ for all $t,s \in G$ with $s \neq t$ and since R_g is commutative for all $g \in G$, we get $\psi s = \sum_{g\in G}\psi_g\sum_{g\in G} s_g = \sum_{g\in G}\sum_{t\in G}\psi_g s_t = \sum_{g\in G}\psi_g s_g = \sum_{g\in G} s_g\psi_g = \sum_{t\in G}\sum_{g\in G} s_t\psi_g = \sum_{g\in G} s_g\sum_{g\in G}\psi_g = s\psi$. Therefore, R is commutative. Thus R is c-semiring.

c. Assume that R is MV-semiring. Let g be an element in G. Then R is c-semiring and hence R_g is c-semiring. It follows that R_g is a commutative additively idempotent semiring. Now, let $a = \sum_{g\in G} a_g$, where $a_g \in R_g$ for all $g \in G$, be an element in R. Then there exists a greatest element $a^* \in R$ such that $aa^* = 0$. As R is graded, we can write a^* as $a^* = \sum_{g\in G} b_g$ such that $b_g \in R_g$ for all $g \in G$. Therefore, by the condition $R_tR_s = \{0\}$ for all $t,s \in G$ with $s \neq t$, we get $\sum_{g\in G}(a_g b_g) = 0$. Hence $a_g b_g = 0$ for all $g \in G$. Suppose for each $g \in G$ there exists $c_g \in R_g$ such that $a_g c_g = 0$. Now, put $c = \sum_{g\in G} c_g$. $Supp(R,G)$ being finite set, it follows that $c = \sum_{g\in G} c_g \in R$. We have $a_g c_g = 0$ for all $g \in G$. Hence $\sum_{g\in G} a_g c_g = 0$. By the condition $R_tR_s = \{0\}$ for all $t,s \in G$ with $s \neq t$, we get $ac = 0$. Thus $c \leq a^*$. It follows that $c_g \leq b_g$ for all $g \in G$. Therefore for each $g \in G$ the element b_g is the greatest element in R_g for which $a_g b_g = 0$. Denote $a_g^{*_g}$ the element b_g for all $g \in G$. Then $(\sum_{g\in G} a_g)^* = \sum_{g\in G} a_g^{*_g}$. Now, let ψ_g be an element in R_g. Then we can write ψ_g as $\psi_g = \sum_{h\in G}\psi_h$ such that $\psi_h = 0$ for all $h \in G-\{g\}$ and $\psi_g^* = (\sum_{h\in G}\psi_h)^* = \sum_{h\in G}\psi_h^{*_h}$ such that $\psi_h^{*_h}$ is the greatest element in R_h for which $\psi_h\psi_h^{*_h} = 0$ for all $h \in G$. Let ζ_g, κ_g be two elements in R_g. By hypothesis R is MV-semiring, we get $\zeta_g+\kappa_g = (\zeta_g^*(\zeta_g^*\kappa_g)^*)^*$. Therefore, by the condition $R_tR_s = \{0\}$ for all $t,s \in G$ with $s \neq t$ and R being graded, we get $\zeta_g + \kappa_g = (\zeta_g^{*_g}(\zeta_g^{*_g}\kappa_g)^{*_g})^{*_g}$. Thus R_g is MV-semiring. Conversely, assume that R_g is MV-semiring for all $g \in G$. Then R_g is c-semiring for all $g \in G$. Therefore R is c-semiring (see viib). Let ψ be an element in R. Then we can write ψ as $\psi = \sum_{g\in G}\psi_g$ such that $\psi_g \in R_g$ for all $g \in G$. Since for each $g \in G$ and each $\psi_g \in R_g$ there exists a greatest element $\psi_g^{*_g} \in R_g$ such that $\psi_g\psi_g^{*_g} = 0$, we get $\sum_{g\in G}\psi_g\psi_g^{*_g} = 0$. Therefore, by the condition $R_tR_s = \{0\}$ for all $t,s \in G$ with $t \neq s$, we get $\sum_{g\in G}\sum_{t\in G}\psi_g\psi_t^{*_t} = 0$. Hence $\sum_{g\in G}\psi_g\sum_{t\in G}\psi_t^{*_t} = 0$. Now, put $\psi^* = \sum_{t\in G}\psi_t^{*_t}$. Then $\psi^* \in R$ (since $Supp(R,G)$ is finite) and $\psi\psi^* = 0$. Assume that $s = \sum_{g\in G} s_g$ where $s_g \in R_g$ for all $g \in G$ is an element in R such that $\psi s = 0$. Therefore $\sum_{g\in G}\psi_g\sum_{g\in G} s_g = 0$ and hence $\sum_{g\in G}\sum_{t\in G}\psi_g s_t = 0$. Given that $R_sR_t = \{0\}$ for all $t,s \in G$ with $s \neq t$, we get $\sum_{g\in G}\psi_g s_g = 0$. It follows that $\psi_g s_g = 0$ for all $g \in G$. Now, $\psi_g^{*_g}$ being the greatest element in R_g such that $\psi_g\psi_g^{*_g} = 0$ for each $g \in G$, it follows that $s_g \leq \psi_g^{*_g}$ for all $g \in G$. Therefore $s_g + \psi_g^{*_g} = \psi_g^{*_g}$ for all $g \in G$ and hence $\sum_{g\in G} s_g + \sum_{g\in G} \psi_g^{*_g} = \sum_{g\in G} \psi_g^{*_g}$. It follows that $s + \psi^* = \psi^*$. Thus $s \leq \psi^*$. Therefore ψ^* is the greatest element in R for which $\psi\psi^* = 0$. Note that $(\sum_{g\in G} \psi_g)^* = (\sum_{g\in G} \psi_g^{*_g})$. Now, let $\zeta = \sum_{g\in G} \zeta_g, \kappa = \sum_{g\in G} \kappa_g$, where $\zeta_g, \kappa_g \in R_g$ for all $g \in G$, be two elements in R. By hypothesis R_g is MV-semiring for all $g \in G$, we get $\zeta_g + \kappa_g = (\zeta_g^{*_g}(\zeta_g^{*_g}\kappa_g)^{*_g})^{*_g}$ for all $g \in G$. Therefore $\zeta + \kappa = \sum_{g\in G} \zeta_g + \sum_{g\in G} \kappa_g = \sum_{g\in G} (\zeta_g^{*_g}(\zeta_g^{*_g}\kappa_g)^{*_g})^{*_g}$. We shall prove that $\sum_{g\in G} (\zeta_g^{*_g}(\zeta_g^{*_g}\kappa_g)^{*_g})^{*_g} = (\zeta^*(\zeta^*\kappa)^*)^*$. By the condition $R_sR_t = \{0\}$ for all $t, s \in G$ with $s \neq t$, we get $\zeta^*(\zeta^*\kappa)^* = \sum_{g\in G} \zeta_g^{*_g}(\sum_{h\in G} \zeta_h^{*_h} \sum_{t\in G} \kappa_t)^* = \sum_{g\in G} \zeta_g^{*_g}(\sum_{h\in G}\sum_{t\in G} \zeta_h^{*_h}\kappa_t)^* = \sum_{g\in G} \zeta_g^{*_g}(\sum_{h\in G} \zeta_h^{*_h}\kappa_h)^* = \sum_{g\in G} \zeta_g^{*_g} \sum_{h\in G} (\zeta_h^{*_h}\kappa_h)^{*_h} = \sum_{g\in G} \zeta_g^{*_g}(\zeta_g^{*_g}\kappa_g)^{*_g}$. Therefore $(\zeta^*(\zeta^*\kappa)^*)^* = (\sum_{g\in G} \zeta_g^{*_g}(\zeta_g^{*_g}\kappa_g)^{*_g})^*$ and hence $(\zeta^*(\zeta^*\kappa)^*)^* = \sum_{g\in G} (\zeta_g^{*_g}(\zeta_g^{*_g}\kappa_g)^{*_g})^{*_g}$. Thus R is MV-semiring.

d. Assume that R is k-semifield. Therefore, R is additively idempotent and hence R_t is additively idempotent for all $t \in G$. Let g be an element in G. If $g \notin supp(R, G)$, $R_g = \{0\}$. Therefore R_g is k-semifield. Again, if $g \in supp(R, G)$, let ζ, κ be two elements in R_g such that $\kappa \neq 0$. Then there exist $x_1 = \sum_{h\in G} (x_1)_h, x_2 = \sum_{h\in G} (x_2)_h, y_1 = \sum_{h\in G} (y_1)_h$, and $y_2 = \sum_{h\in G} (y_2)_h$ in R where $(x_1)_h, (x_2)_h, (y_1)_h, (y_2)_h \in R_h$ for all $h \in G$ such that $\zeta + x_1\kappa = x_2\kappa$ and $\zeta + \kappa y_1 = \kappa y_2$. Therefore $\zeta + (\sum_{h\in G} (x_1)_h)\kappa = (\sum_{h\in G} (x_2)_h)\kappa$ and $\zeta + \kappa(\sum_{h\in G} (y_1)_h) = \kappa(\sum_{h\in G} (y_2)_h)$. Now, given that R is graded and that $R_tR_s = \{0\}$ for all $t, s \in G$ with $t \neq s$, we get $\zeta + (x_1)_g\kappa = (x_2)_g\kappa$ and $\zeta + \kappa(y_1)_g = \kappa(y_2)_g$. Thus R_g is K-semifield.

□

We have to mention that the conditions from (i) to (vi) are satisfied when R is graded by a monoid G in which all elements are idempotent. By (vii) of Theorem 3.1, we have the following corollaries.

Corollary 3.2. *Let $R = \bigoplus_{g\in G} R_g$ be a graded semiring by a semigroup G in which all elements are idempotent. Assume that $R_gR_h = \{0\}$ for all $g, h \in G$ with $g \neq h$. Suppose R with unity $1 = \sum_{g\in G} \zeta_g$ where $\zeta_g \in R_g$ for all $g \in G$. Then R_g with unity ζ_g for all $g \in G$ (see the proof of (viib)-Theorem 3.1).*

Corollary 3.3. *Let $R = \bigoplus_{g\in G} R_g$ be a graded semiring by a monoid G with identity e in which all elements are idempotent. Then*

(i) Suppose R with unity $1 = \sum_{g\in G} \zeta_g$ where $\zeta_g \in R_g$ for all $g \in G$. Then R_e with unity ζ_e.

(ii) R is k-regular implies R_e is k-regular.

(iii) R is c-semiring implies R_e is c-semiring.

(iv) R is k-semifield implies R_e is k-semifield.

Next, we shall prove that if R is an augmented graded semiring by a monoid G with identity in which all elements are idempotent, then each homogeneous component is a graded subsemiring of R.

Theorem 3.4. *Let $R = \bigoplus_{g\in G} R_g$ be an augmented graded semiring by a monoid G with identity e in which all elements are idempotent. Since R is augmented graded, we get $R_e = \bigoplus_{h\in G} R_{e-h}$*

is a graded subsemiring of R by G and for each $g \in G$ there exists $\psi_g \in R_g$ such that $R_g = \bigoplus_{h \in G} R_{e-h}\psi_g$. Assume that ψ_g is unique. Then R_g is a graded subsemiring of R by G for all $g \in G$.

Proof. Let g be an element in G. We have g is idempotent and R_g is an additive submonoid in R. Therefore, R_g is a subsemiring of R and hence R_g is a semiring. Let h be an element in G and $\zeta_{e-h}, \kappa_{e-h}$ be two elements in R_{e-h}. As $0 = 0\psi_g \in R_{e-h}\psi_g$ and $\zeta_{e-h}\psi_g + \kappa_{e-h}\psi_g = (\zeta_{e-h} + \kappa_{e-h})\psi_g \in R_{e-h}\psi_g$, we get $R_{e-h}\psi_g$ is an additive submonoid of R. Let t, h be two elements in G. By hypothesis R is augmented graded, we get $(R_{e-t}\psi_g)(R_{e-h}\psi_g) \subseteq R_{e-t}R_{e-h}\psi_{gg} = R_{e-t}R_{e-h}\psi_g \subseteq R_{e-th}\psi_g$. □

By Theorem 3.1 and Theorem 3.4, we have the following corollary.

Corollary 3.5. *Let $R = \bigoplus_{g \in G} R_g$ be an augmented graded semiring by a monoid G with identity e in which all elements are idempotent. Since R is augmented graded, we get $R_e = \bigoplus_{h \in G} R_{e-h}$ is a graded subsemiring of R by G and for each $g \in G$ there exists $\psi_g \in R_g$ such that $R_g = \bigoplus_{h \in G} R_{e-h}\psi_g$. Assume that ψ_g is unique. Then*

(i) Let $\zeta = \sum_{g \in G} \zeta_g = \sum_{g \in G}\sum_{h \in G} (\zeta_g)_{e-h}\psi_g$, where $\zeta_g \in R_g$ and $(\zeta_g)_{e-h} \in R_{e-h}$, be an element in R. Then ζ is infinite iff $(\zeta_g)_{e-h}\psi_g$ is infinite for all $g, h \in G$.

(ii) R is zero-sum free iff $R_{e-h}\psi_g$ is zero sum free for all $g, h \in G$.

(iii) R is additively regular iff $R_{e-h}\psi_g$ is additively regular for all $g, h \in G$.

(iv) R is inverse iff $R_{e-h}\psi_g$ is inverse for all $g, h \in G$.

(v) R is E-inversive iff $R_{e-h}\psi_g$ is E-inversive for all $g, h \in G$.

(vi) R is yoked implies $R_{e-h}\psi_g$ is yoked for all $g, h \in G$.

Finally, we established necessary and sufficient conditions that if R is an augmented graded semiring by a monoid G with identity e in which all elements are idempotent, then R is zero divisor free, additively regular, inversive, E- inversive, and yoked in terms of homogeneous components with respect to the identity element.

Theorem 3.6. *Let $R = \bigoplus_{g \in G} R_g$ be an augmented graded semiring by a monoid G with identity e in which all elements are idempotent. Since R is augmented graded, we get $R_e = \bigoplus_{h \in G} R_{e-h}$ is a graded subsemiring of R by G and for each $g \in G$ there exists $\psi_g \in R_g$ such that $R_g = \bigoplus_{h \in G} R_{e-h}\psi_g$. Assume that ψ_g is unique. Then*

(i) Suppose R is zero divisor free. Then R is zero-sum free iff R_e is zero-sum free.

(ii) Suppose R is right multiplicatively cancellative. Then

a. R is additively regular iff R_e is additively regular.

b. R is inverse iff R_e is inverse.

(iii) R is E-inversive iff R_e is E-inversive.

(iv) R is yoked implies R_e is yoked.

Proof. Given that all elements of G are idempotent, we get R_g is a subsemiring of R and $R_{e-h}\psi_g$ is a subsemiring of R_g for all $g, h \in G$.

(i) Assume that R_e is zero-sum free. Therefore, R_{e-t} is zero-sum free for all $t \in G$. Let g, h be two elements in G. If $\psi_g \neq 0$, let ζ, κ be two elements in $R_{e-h}\psi_g$ such that $\zeta + \kappa = 0$. Then there exist two elements $\zeta_{e-h}, \kappa_{e-h} \in R_{e-h}$ such that $\zeta = \zeta_{e-h}\psi_g$, $\kappa = \kappa_{e-h}\psi_g$, and $\zeta_{e-h}\psi_g + \kappa_{e-h}\psi_g = 0$. It follows that $(\zeta_{e-h} + \kappa_{e-h})\psi_g = 0$. Therefore $\zeta_{e-h} + \kappa_{e-h} = 0$. Hence $\zeta_{e-h} = 0$ and $\kappa_{e-h} = 0$. Thus $\zeta = 0$ and $\kappa = 0$. Again, if $\psi_g = 0$, it can be easily seen that $R_{e-h}\psi_g$ is zero-sum free. Now, by Corollary 3.5, we get R is zero-sum free. Conversely, assume that R is zero-sum free. By Theorem 3.1, we get R_e is zero-sum free.

(ii) a. Assume that R_e is additively regular. Then R_{e-t} is additively regular for all $t \in G$. Let g, h be two elements in G. If $\psi_g = 0$, it can be easily seen that $R_{e-h}\psi_g$ is additively regular. Again, if $\psi_g \neq 0$, let ζ be an element in $R_{e-h}\psi_g$. Then there exists an element $\zeta_{e-h} \in R_{e-h}$ such that $\zeta = \zeta_{e-h}\psi_g$. Since $\zeta_{e-h} \in R_{e-h}$ and R_{e-h} is additively regular, there exists a unique element $\kappa_{e-h} \in R_{e-h}$ such that $\zeta_{e-h} + \kappa_{e-h} + \zeta_{e-h} = \zeta_{e-h}$. Hence $\zeta_{e-h}\psi_g + \kappa_{e-h}\psi_g + \zeta_{e-h}\psi_g = \zeta_{e-h}\psi_g$. Assume that $\theta_{e-h}\psi_g$ is an element of $R_{e-h}\psi_g$ such that $\zeta_{e-h}\psi_g + \theta_{e-h}\psi_g + \zeta_{e-h}\psi_g = \zeta_{e-h}\psi_g$. Therefore $(\zeta_{e-h} + \theta_{e-h} + \zeta_{e-h})\psi_g = \zeta_{e-h}\psi_g$. Now, R being multiplicatively right cancellative, it follows that $\zeta_{e-h} + \theta_{e-h} + \zeta_{e-h} = \zeta_{e-h}$. It follows that $\theta_{e-h} = \kappa_{e-h}$. Therefore $\theta_{e-h}\psi_g = \kappa_{e-h}\psi_g$ and hence $\kappa_{e-h}\psi_g$ is unique. By Corollary 3.5, we find that R is additively regular. Conversely, assume that R is additively regular. By Theorem 3.1, we get R_e is additively regular.

b. The proof of (b) is done in the same way as the proof of (a).

(iii) Assume that R_e is E-inversive. Then R_{e-t} is E-inversive for all $t \in G$. Let g, h be two elements in G and let ζ be an element in $R_{e-h}\psi_g$. Then there exists an element $\zeta_{e-h} \in R_{e-h}$ such that $\zeta = \zeta_{e-h}\psi_g$. Since R_{e-h} is E-inversive and $\zeta_{e-h} \in R_{e-h}$, there exists an element $x_{e-h} \in R_{e-h}$ such that $\zeta_{e-h} + x_{e-h} \in E^+(R_{e-h})$. Therefore $\zeta_{e-h} + x_{e-h} + \zeta_{e-h} + x_{e-h} = \zeta_{e-h} + x_{e-h}$ and hence $\zeta + x_{e-h}\psi_g + \zeta + x_{e-h}\psi_g = \zeta + x_{e-h}\psi_g$. Thus $\zeta + x_{e-h}\psi_g \in E^+(R_{e-h}\psi_g)$. By Corollary 3.5, we get R is E-inversive. Conversely, assume that R is E-inversive. By Theorem 3.1, we get R_e is E-inversive.

(iv) Assume that R is yoked. By Theorem 3.1, we get R_e is yoked.

□

Next, by Corollary 3.3, we have the following corollary.

Corollary 3.7. *Let $R = \bigoplus_{g\in G} R_g$ be an augmented graded semiring by a monoid G with identity e in which all elements are idempotent. Since R is augmented graded by G, we get $R_e = \bigoplus_{h\in G} R_{e-h}$ is a graded subsemiring of R by G and for each $g \in G$ there exists $\psi_g \in R_g$ such that $R_g = \bigoplus_{h\in G} R_{e-h}\psi_g$. Assume that ψ_g is unique. Then*

(i) Suppose R with unity $1 = \sum_{g\in G} \zeta_g$ such that $\zeta_g \in R_g$ for all $g \in G$. Assume that $\zeta_e = \sum_{g\in G} \zeta_{e-g}$ such that $\zeta_{e-g} \in R_{e-g}$ for all $g \in G$. Then R_{e-e} with unity ζ_{e-e}.

(ii) R is k-regular implies R_{e-e} is k-regular.

(iii) R is c-semiring implies R_{e-e} is c-semiring.

(iv) R is k-semifield implies R_{e-e} is k-semifield.

Theorem 3.8. *Let $R = \bigoplus_{g\in G} R_g$ be an augmented graded semiring by a monoid G with identity e in which all elements are idempotent. Since R is augmented graded, $R_e = \bigoplus_{h\in G} R_{e-h}$ is a graded subsemiring of R by G and for each $g \in G$ there exists $\psi_g \in R_g$ such that $R_g = \bigoplus_{h\in G} R_{e-h}\psi_g$. Assume that ψ_g is unique and assume that $R_gR_h = \{0\}$ for all $g, h \in G - \{e\}$ with $g \neq h$. Then*

(i) Suppose R with unity $1 = \sum_{g\in G} \zeta_g$ such that $\zeta_g \in R_g$ for all $g \in G$. Assume that $\zeta_g = \sum_{h\in G} (\zeta_g)_{e-h}\psi_g$ such that $(\zeta_g)_{e-h} \in R_{e-h}$ for all $g, h \in G$. Then $R_{e-e}\psi_g$ with unity $((\zeta_g)_{e-e} + (\zeta_e)_{e-e})\psi_g$ for all $g \in G - \{e\}$.

(ii) R is k-regular implies $R_{e-e}\psi_g$ is k-regular for all $g \in G - \{e\}$.

(iii) R is c-semiring implies $R_{e-e}\psi_g$ is c-semiring for all $g \in G - \{e\}$.

(iv) R is k-semifield implies $R_{e-e}\psi_g$ is k-semifield for all $g \in G - \{e\}$.

Proof. Given that all elements of G are idempotent, we get $R_{e-h}\psi_g$ is a subsemiring of the semiring R_g for all $g, h \in G$.

(i) Let g be an element in $G-\{e\}$ and x be an element of $R_{e-e}\psi_g$. Then there exists an element $x_{e-e}\in R_{e-e}$ such that $x=x_{e-e}\psi_g$. If $\psi_g\neq 0$, we have $x1=1x=x$. Therefore $x_{e-e}\psi_g\sum\limits_{t\in G}\zeta_t=\sum\limits_{t\in G}\zeta_t x_{e-e}\psi_g=x_{e-e}\psi_g$. Hence $x_{e-e}\psi_g\sum\limits_{t\in G}\sum\limits_{h\in G}(\zeta_t)_{e-h}\psi_t=\sum\limits_{t\in G}\sum\limits_{h\in G}(\zeta_t)_{e-h}\psi_t x_{e-e}\psi_g=x_{e-e}\psi_g$. By the condition $R_gR_h=\{0\}$ for all $g,h\in G-\{e\}$ with $g\neq h$ and R being augmented graded, we get $x_{e-e}((\zeta_g)_{e-e}+(\zeta_e)_{e-e})\psi_{gg}=((\zeta_g)_{e-e}+(\zeta_e)_{e-e})x_{e-e}\psi_{gg}=x_{e-e}\psi_g$. Therefore

$$x_{e-e}\psi_g((\zeta_g)_{e-e}+(\zeta_e)_{e-e})\psi_g=((\zeta_g)_{e-e}+(\zeta_e)_{e-e})\psi_g x_{e-e}\psi_g=x_{e-e}\psi_g$$

and hence

$$x((\zeta_g)_{e-e}+(\zeta_e)_{e-e})\psi_g=((\zeta_g)_{e-e}+(\zeta_e)_{e-e})\psi_g x=x.$$

Again, if $\psi_g=0$, we get $R_g=\{0\}$. Therefore $R_{e-e}\psi_g$ with unity $((\zeta_g)_{e-e}+(\zeta_e)_{e-e})\psi_g=0$.

(ii) Suppose R is k-regular. Let g be an element of $G-\{e\}$ and x be an element of $R_{e-e}\psi_g$. If $\psi_g\neq 0$, there exists an element $x_{e-e}\in R_{e-e}$ such that $x=x_{e-e}\psi_g$ and there exist $\zeta=\sum\limits_{g\in G}\zeta_g$, $\kappa=\sum\limits_{g\in G}\kappa_g$ in R where $\zeta_g,\kappa_g\in R_g$ for all $g\in G$ such that $x+x\kappa x=x\zeta x$. Hence $x_{e-e}\psi_g+x_{e-e}\psi_g\sum\limits_{t\in G}\kappa_t x_{e-e}\psi_g=x_{e-e}\psi_g\sum\limits_{t\in G}\zeta_t x_{e-e}\psi_g$. Since $\zeta_t,\kappa_t\in R_t$ for all $t\in G$, we can write for each $t\in G$ the elements ζ_t,κ_t as follows $\zeta_t=\sum\limits_{h\in G}(\zeta_t)_{e-h}\psi_t$ and $\kappa_t=\sum\limits_{h\in G}(\kappa_t)_{e-h}\psi_t$ such that $(\zeta_t)_{e-h},(\kappa_t)_{e-h}\in R_{e-h}$ for all $h\in G$. Therefore $x_{e-e}\psi_g+x_{e-e}\psi_g\sum\limits_{t\in G}\sum\limits_{h\in G}(\kappa_t)_{e-h}\psi_t x_{e-e}\psi_g=x_{e-e}\psi_g\sum\limits_{t\in G}\sum\limits_{h\in G}(\zeta_t)_{e-h}\psi_t x_{e-e}\psi_g$. By the condition $R_gR_h=\{0\}$ for all $g,h\in G-\{e\}$ with $g\neq h$ and R being augmented graded, we get $x_{e-e}\psi_g+x_{e-e}\psi_g((\kappa_g)_{e-e}+(\kappa_e)_{e-e})\psi_g x_{e-e}\psi_g=x_{e-e}\psi_g((\zeta_g)_{e-e}+(\zeta_e)_{e-e})\psi_g x_{e-e}\psi_g$. Therefore $x+x((\kappa_g)_{e-e}+(\kappa_e)_{e-e})\psi_g x=x((\zeta_g)_{e-e}+(\zeta_e)_{e-e})\psi_g x$. Thus $R_{e-e}\psi_g$ is k-regular. Again, if $\psi_g=0$, then $R_{e-e}\psi_g=\{0\}$. Therefore $R_{e-e}\psi_g$ is k-regular.

(iii) Suppose R is c-semiring. Let g be an element in $G-\{e\}$. If $\psi_g=0$, it can be easily seen that $R_{e-e}\psi_g=\{0\}$ is c-semiring. Again, if $\psi_g\neq 0$, then R with unity 1 and we can write the unity as $1=\sum\limits_{t\in G}\zeta_t=\sum\limits_{t\in G}\sum\limits_{h\in G}(\zeta_t)_{e-h}\psi_t$ such that $\zeta_t\in R_t$ and $(\zeta_t)_{e-h}\in R_{e-h}$ for all $t,h\in G$. By (i)-Theorem 3.8, we get $R_{e-e}\psi_g$ with unity $((\zeta_g)_{e-e}+(\zeta_e)_{e-e})\psi_g$. R being additively idempotent, it follows that $R_{e-e}\psi_g$ is additivelly idempotent. Let x be an element in $R_{e-e}\psi_g$. Then there exists an element $x_{e-e}\in R_{e-e}$ such that $x=x_{e-e}\psi_g$. We have $x+1=1$. Therefore $x+\sum\limits_{t\in G}\sum\limits_{h\in G}(\zeta_t)_{e-h}\psi_t=\sum\limits_{t\in G}\sum\limits_{h\in G}(\zeta_t)_{e-h}\psi_t$ and hence $x_{e-e}\psi_g+\sum\limits_{t\in G}\sum\limits_{h\in G}(\zeta_t)_{e-h}\psi_t=\sum\limits_{t\in G}\sum\limits_{h\in G}(\zeta_t)_{e-h}\psi_t$. Now, by hypothesis R is augmented graded, we get $x_{e-e}\psi_g+(\zeta_g)_{e-e}\psi_g=(\zeta_g)_{e-e}\psi_g$. Therefore $x+((\zeta_g)_{e-e}+(\zeta_e)_{e-e})\psi_g=((\zeta_g)_{e-e}+(\zeta_e)_{e-e})\psi_g$. Finally, R being commutative, it follows that $R_{e-e}\psi_g$ is commutative. Thus $R_{e-e}\psi_g$ is c-semiring.

(iv) Assume that R is k-semifield. Since R is additively idempotent, $R_{e-e}\psi_t$ is additively idempotent for all $t\in G$. Let g be an element in $G-\{e\}$. If $R_{e-e}\psi_g=\{0\}$, it can be easily seen that $R_{e-e}\psi_g$ is k-semifield. Again, if $R_{e-e}\psi_g\neq\{0\}$, let $\zeta,\kappa\neq 0$ be two elements in $R_{e-e}\psi_g$. Then there exist $\zeta_{e-e},\kappa_{e-e}\in R_{e-e}$ such that $\zeta=\zeta_{e-e}\psi_g$, $\kappa=\kappa_{e-e}\psi_g$ and there exist $x_1=\sum\limits_{t\in G}(x_1)_t, x_2=\sum\limits_{t\in G}(x_2)_t, y_1=\sum\limits_{t\in G}(y_1)_t, y_2=\sum\limits_{t\in G}(y_2)_t\in R$ where $(x_1)_t,(x_2)_t,(y_1)_t,(y_2)_t\in R_t$ for all $t\in G$ such that $\zeta+x_1\kappa=x_2\kappa$ and $\zeta+\kappa y_1=\kappa y_2$. Given that R is augmented graded, we can write for each $t\in G$ the elements $(x_i)_t$, $(y_i)_t$ as follows $(x_i)_t=\sum\limits_{h\in G}((x_i)_t)_{e-h}\psi_t$, $(y_i)_t=\sum\limits_{h\in G}((y_i)_t)_{e-h}\psi_t$ such that $((x_i)_t)_{e-h},((y_i)_t)_{e-h}\in R_{e-h}$ for all $h\in G$, $i\in\{1,2\}$. Therefore

$$\zeta_{e-e}\psi_g+\sum_{t\in G}\sum_{h\in G}((x_1)_t)_{e-h}\psi_t\kappa_{e-e}\psi_g=\sum_{t\in G}\sum_{h\in G}((x_2)_t)_{e-h}\psi_t\kappa_{e-e}\psi_g$$

and

$$\zeta_{e-e}\psi_g + \sum_{t\in G}\sum_{h\in G}\kappa_{e-e}\psi_g((y_1)_t)_{e-h}\psi_t = \sum_{t\in G}\sum_{h\in G}\kappa_{e-e}\psi_g((y_2)_t)_{e-h}\psi_t.$$

By the condition $R_gR_h = \{0\}$ for all $g, h \in G - \{e\}$ with $g \neq h$ and R being augmented graded, we get

$$\zeta_{e-e}\psi_g + (((x_1)_g)_{e-e} + ((x_1)_e)_{e-e})\psi_g\kappa_{e-e}\psi_g = (((x_2)_g)_{e-e} + (x_2)_e)_{e-e})\psi_g\kappa_{e-e}\psi_g$$

$$and$$

$$\zeta_{e-e}\psi_g + \kappa_{e-e}\psi_g(((y_1)_g)_{e-e} + ((y_1)_e)_{e-e})\psi_g = \kappa_{e-e}\psi_g(((y_2)_g)_{e-e} + ((y_2)_e)_{e-e})\psi_g.$$

Therefore

$$\zeta + (((x_1)_g)_{e-e} + ((x_1)_e)_{e-e})\psi_g\kappa = (((x_2)_g)_{e-e} + ((x_2)_e)_{e-e})\psi_g\kappa$$

$$and$$

$$\zeta + \kappa(((y_1)_g)_{e-e} + ((y_1)_e)_{e-e})\psi_g = \kappa(((y_2)_g)_{e-e} + ((y_2)_e)_{e-e})\psi_g.$$

Thus $R_{e-e}\psi_g$ is K-semifield.

□

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