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Modules Whose Endomorphism Rings are Centrally AIP

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Abstract This paper presents the concept of centrally endo-AIP modules. For a ring R and an R-module A, A is termed as a centrally left endo-AIP module if the left annihilator of any fully invariant submodule B of A in the endomorphism ring $E = End_R(A)$ is a centrally s-unital ideal of E. We examine various characteristics of centrally endo-AIP modules and analyze their endomorphism ring. Furthermore, we investigate the characterization of quasi-Baer modules in relation to centrally endo-AIP modules.

1 Introduction

For clarity, we'll be working with rings that follow two properties throughout this paper: associativity and having a unity element. Similarly, all modules are assumed to be right unitary unless we state otherwise. In his work, I. Kaplansky [10] termed a ring R, *Baer (quasi-Baer)* if the right annihilator of any subset (ideal) of R is generated as a right ideal by an idempotent element of R. He introduced these concepts to investigate various properties of von Neumann regular algebras, AW^*-algebras, and C^*-algebras. Numerous researchers have directed their attention towards the Baer ring due to its origins in functional analysis and its significant connection to C^*-algebras and von Neumann algebras. A ring R is termed a right (left) Rickart ring or right (left) PP ring when the right (left) annihilator of any element of R in R forms a direct summand of R [15]. Birkenmeier et al. [3] introduced a further generalization of a Baer ring known as a principally quasi-Baer ring (PQ-Baer). They defined a ring R as principally quasi-Baer (PQ-Baer) if the right annihilator of every principal ideal of R is generated by an idempotent element of R.
An ideal I of a ring R is called right (left) *s-unital ideal* of R if for each $x \in I$, $xy = x$ (resp. $yx = x$) for some element $y \in I$ (see [12]). Additionally, an ideal I is identified as a centrally s-unital ideal of R if, for every $x \in I$, there exists a central element $z \in R$ such $xz = zx = x$ (see [14]). Furthermore, a submodule B of a right R-module A is called a *pure submodule* of A, if the sequences $0 \to B \to A$ and $0 \to B \otimes K \to A \otimes K$ remain exact for every left R-module K (see [5]). The condition for a right R-module A to be *flat* is that whenever $0 \to B_1 \to B_2$ is exact for left R-modules B_1 and B_2 then $0 \to A \otimes B_1 \to A \otimes B_2$ is also exact. By using the concept of s-unital ideal, Liu and Zhao [12] defined a generalized structure of PP rings and PQ-Baer rings. According to them, a ring R is said to be left APP if the left annihilator of every principal left ideal of R is pure as a left ideal of R, or equivalently the left annihilator of every principal left ideal of R is a right s-unital ideal of R. The class of left AIP rings encompasses both right p.q.-Baer rings and right PP rings (see [12]).
Majidinya et al. [14] define a ring R as a left (right) AIP-ring if the left (right) annihilator of each of its ideals is pure as a left (right) ideal of R, or alternatively the left (right) annihilator of each ideal in R is a right (left) s-unital ideal of R. They also introduced the centrally left AIP rings, and according to them, a ring is classified as a centrally left AIP-ring if the left annihilator of every ideal I of R is a centrally s-unital ideal of R. In [14], it is shown that for a ring, the centrally AIP condition is left-right symmetric property (see Proposition 2.10, [14]). The category of right AIP rings encompasses both right PQ-Baer rings and right PP rings.
P.A. Dana and A. Moussavi [6], introduced the module theoretical notion AIP and APP rings as endo-AIP and endo-APP modules. A module A is said to be an endo-AIP (endo-APP) if the left annihilator of every fully invariant (resp. cyclic) submodule of A is a right s-unital ideal

of E or a pure left ideal of E, where $E = End_R(A)$.

This article presents the module theoretical notion of centrally AIP rings as centrally endo-AIP modules. An R-module A is labeled a centrally endo-AIP module if, within $E = End_R(A)$ the left annihilator of each fully invariant submodule of A becomes a centrally s-unital ideal of E. Every abelian Rickart module is a centrally endo-AIP module, and every centrally endo-AIP module is an endo-AIP module. We show that the centrally endo-AIP module is closed under direct summand. In general, the direct sum of centrally endo-AIP modules need not be centrally endo-AIP. We find the conditions for which the direct sum of centrally endo-AIP modules is centrally endo-AIP. We also prove that every projective R-module is centrally endo-AIP if and only if R is a centrally AIP ring.
In section 3, we study the endomorphism ring of centrally endo-AIP modules. The endomorphism ring of the centrally endo-AIP module is centrally AIP ring or semiprime ring. Further, we show that for a locally quasi-retractable module A, the ring of endomorphisms $E = End_R(A)$ is centrally AIP if and only if A is centrally endo-AIP module. Also, we prove that the endomorphism ring $E = End_R(A)$ of a centrally endo-AIP module A is a quasi-Baer if E has a finite left uniform dimension.
We denote the symbols $\subseteq$, $\leq$, $\leq^{\oplus}$, $\leq^{e}$, $\trianglelefteq$ and $\trianglelefteq^{p}$ to represent a variety of mathematical concepts: a subset, a submodule, a direct summand, an essential submodule, a fully invariant submodule (or an ideal), and a projection invariant submodule, respectively. For an R-module A with endomorphism ring $E = End_R(A)$, $r_A(T)$ (where T is a left ideal of E) and $\ell_E(B)$ (where $B \leq A$) will denote the right annihilator of T in A and left annihilator of B in E respectively. An idempotent element $x^2 = x$ of a ring R is said to be left (right) semi-central if for every $z \in R$, $xzx = zx$ ($xzx = xz$) (see [4]). By a regular ring, we always mean a von Neumann regular, and $T_n(R)$ stands for n by n upper triangular matrix ring over R. Before proceeding to the main section, we recall some definitions and results, which will be helpful to the clarity of further results.

Definition 1.1. Let A be an R-module with $E = End_R(A)$.

(i) A is said to be reduced if for each $\phi \in E$ and $a \in A$, $\phi(a) = 0$ implies $Im(\phi) \cap Ea = 0$. Equivalently, A is a reduced module if $\phi^2(a) = 0$ implies $\phi E(a) = 0$.

(ii) A is called a rigid module [1] if for every $\psi \in E$ and $a \in A$, $\psi^2(a) = 0$ implies $\psi(a) = 0$. Equivalently $Ker(\psi) \cap Im(\psi) = 0$ for every $\psi \in E$.

(iii) A ring R is said to be abelian if every idempotent element of R is central. Further, a module A is called abelian if its endomorphism ring E is abelian. In other words, A is an abelian module [1] if $\psi e(a) = e\psi(a)$ for every $a \in A$ where $\psi \in E$ and $e^2 = e \in E$.

(iv) A is said to be symmetric [1] if $\phi\psi(a) = 0$ implies $\psi\phi(a) = 0$ for every $\phi, \psi \in E$ and $a \in A$.

(v) A is known as semi-commutative [2] if $\psi(a) = 0$ implies $\psi E(a) = 0$ for every $\psi \in E$ and $a \in A$.

Reduced modules, rigid modules, symmetric modules, and semi-commutative modules are abelian; for details, see [1].

Lemma 1.2. *(Theorem 2.25, [1]), The following statements are equivalent for a Rickart module A:*

(i) A is an abelian module;

(ii) A is a reduced module;

(iii) A is a rigid module;

(iv) A is a semi-commutative module;

(v) A is a symmetric module.

Definition 1.3. Let A be an R-module and $E = End_R(A)$. Then

(i) A is said to be Baer (quasi-Baer) module [16], if for every submodule (fully invariant submodule) B of A, $\ell_E(B)$ is a direct summand of E. Further, A is called principally quasi-Baer module [13], if for every cyclic submodule C of A, $\ell_E(C)$ is a direct summand of E.

(ii) A module A is called Rickart if for every endomorphism $\phi \in E$, $Ker(\phi)$ is a direct summand of A.

(iii) A module A is called retractable if $Hom(A, B) \neq 0$, for all $0 \neq B \leq A$. Equivalently, A is retractable module if there exists $0 \neq \psi \in E$ with $Im(\psi) \subseteq B$ for every $B \leq A$.

It is clear that the following hierarchy is true,
Baer module $\Rightarrow$ Quasi-Baer module $\Rightarrow$ Principally Quasi-Baer module

2 Centrally endo-AIP Modules

In this segment, we present the module-theoretical concept of centrally AIP rings, terming it centrally endo-AIP modules. Centrally endo-AIP modules lies between Abelian Rickart modules and endo-AIP modules. We examine the conditions under which the direct sum of centrally endo-AIP modules retains its centrality in the endo-AIP property. To clarify our findings, we present illustrative examples that delineate our results.

Definition 2.1. An R-module A is said to be a centrally left endo-AIP module if the left annihilator of any fully invariant submodule of A in $E = End_R(A)$ is a centrally s-unital ideal of E. Equivalently, for every $B \trianglelefteq A$ and for each $\phi \in \ell_E(B)$ there exists a central element $\psi \in \ell_E(B)$ such that $\phi\psi = \phi = \psi\phi$. A right centrally endo-AIP module is defined similarly.

Consider R as an R-module, then we have $End_R(R) \cong R$. So above definition clearly gives a module theoretical notion of centrally AIP ring defined by Majidinya et al. [14].

The following proposition provides a rich source of examples of centrally endo-AIP modules.

Proposition 2.2. *Let A be an R-module and $E = End_R(A)$. Consider the following statements:*

(i) A is an abelian Rickart module;

(ii) A is a centrally endo-AIP module;

(iii) A is an endo-AIP module.

Then, $(i) \Rightarrow (ii) \Rightarrow (iii)$, while the converse of these implications need not be true.

Proof. $(i) \Rightarrow (ii)$ Let B be a fully invariant submodule of A and $f \in \ell_E(B)$ be arbitrary. Then, $f(B) = 0$ which implies $B \subseteq Ker(f)$. Since A is a Rickart module, $Ker(f)$ is a direct summand of A. So for some $e^2 = e \in E$, $B \subseteq Ker(f) = eA$. Thus, $(1-e)B = 0$ and $fe = 0$. Therefore, $(1-e) \in \ell_E(B)$ and $f(1-e) = f$. By hypothesis, A is an abelian module, so every idempotent of E is central. Therefore, $(1-e)$ is a central idempotent element of $\ell_E(B)$ such that $f(1-e) = f$. Hence, A is a centrally endo-AIP module.
$(ii) \Rightarrow (iii)$ Let B be a fully invariant submodule of A and $\psi \in \ell_E(B)$, then $\psi B = 0$ which implies that $B \subseteq r_A(\psi)$. Since A is a centrally endo-AIP module, so there exists a central element $\phi \in \ell_E(B)$ such that $\psi\phi = \psi$. Therefore, $\ell_E(B)$ is a right s-unital ideal of E. Hence, A is an endo-AIP module.
$(ii) \nRightarrow (i)$ Let L be a local prime ring which is not domain and J be the jacobson radical of L. Let $R = \{(x, \bar{y}) \ : \ x \in J, \ \bar{y} \in \bigoplus_{n=1}^{\infty} R_n\}$, where $R_n = L/J$ for each n, $\bar{y} = (\bar{y}_n)_{n=1}^{\infty}$ and $\bar{y}_n = y_n + J \in R_n$. It is clear from (Example 2.8, [14]) the ring R is centrally AIP-ring which is neither abelian ring nor the Rickart ring. Therefore, R_R is a centrally endo-AIP R-module while R_R is neither abelian R-module nor Rickart R-module.
$(iii) \nRightarrow (ii)$ Let $R = \begin{pmatrix} \Pi_{i=1}^{\infty}\mathbb{F}_i & \bigoplus_{i=1}^{\infty}\mathbb{F}_i \\ \bigoplus_{i=1}^{\infty}\mathbb{F}_i & \langle \bigoplus_{i=1}^{\infty}\mathbb{F}_i, 1\rangle \end{pmatrix}$ and $A = R$, where $\mathbb{F}$ is any field and $\mathbb{F}_i = \mathbb{F}$ for $i = 1, 2, 3, ...$ It is clear from (Example 1.6, [3]) that the ring R is semiprime left PP-ring, so it is semiprime left AIP-ring. Thus, A is an endo-AIP module. Now from (Example 2.11,

[14]), R is not a centrally AIP-ring. Therefore, A is not a centrally endo-AIP R-module.
$(iii) \nRightarrow (i)$ Let $R = T_n(F)$ be an upper triangular matrix ring over F, where F is a domain which is not a division ring. Then, by (Example 2.6, [6]) R_R is an endo-AIP R-module but not a Rickart R-module (see Example 2.9, [11]). □

Corollary 2.3. *A Rickart module A is a centrally endo-AIP module, if A satisfies any one of the following:*

(i) A is reduced.

(ii) A is rigid.

(iii) A is abelian.

(iv) A is semi-commutative.

(v) A is symmetric.

Proof. It follows from Lemma 1.2 and from Proposition 2.2. □

According to Liu and Ouyang (Definition 3.2, [13]), a right R-module A is said to have insertion of factor property (IFP) if $r_A(\psi) \trianglelefteq A$ for all $\psi \in E = End_R(A)$. Equivalently, $\ell_E(a)$ is an ideal of E for every $a \in A$.

Proposition 2.4. *Let A be a module with insertion of factor property and $E = End_R(A)$. Then, the following statements are equivalent:*

(i) A is a centrally endo-AIP module;

(ii) A is an endo-AIP module;

(iii) A is an endo-APP module.

Proof. $(i) \Rightarrow (ii)$ It is clear from Proposition 2.2.
$(ii) \Rightarrow (i)$ Suppose that A has IFP and $B \trianglelefteq A$. Let $\phi \in \ell_E(B)$ be arbitrary. Then, $\phi(B) = 0$ which implies $B \subseteq r_A(\phi)$. By assumption, A is an endo-AIP module with IFP, so from (Proposition 2.10, [5]), A is a Rickart module. Therefore, $B \subseteq r_A(\phi) = e(A)$ for some idempotent element $e^2 = e \in E$. Thus, $(1-e)B = 0$ and $\phi e = 0$. So, we have $(1-e) \in \ell_E(B)$ and $\phi(1-e) = \phi$. Further, A has IFP property, so by (Proposition 3.4, [13]) E is an abelian ring. Therefore, the idempotent element $(1-e)$ is central. Hence, A is a centrally endo-AIP module.
$(ii) \Leftrightarrow (iii)$ It follows from (Proposition 4.3, [6]). □

In Proposition 2.4, the insertion of factor property (IFP) is not superfluous. We justify it by the following example.

Example 2.5. Let $R = T_n(\mathbb{F})$ and $A = R_R$, where $\mathbb{F}$ is a domain that is not a division ring. Then, by (Theorem 3.5, [14]) R is a centrally AIP ring. Therefore, A is an endo-AIP module but not a Rickart module (see Example 2.9, [11]). Thus, A is an endo-AIP module. Let $T_{ij} \in T_2(\mathbb{F})$, where T_{ij} with 1 at (i,j)-position and 0 elsewhere for every $i,j = 1,2$. Then, $T_{11}T_{22} = 0$ but $T_{11}T_{12}T_{22} \neq 0$. So, R does not have IFP. Therefore, A does not satisfy the insertion of factor property.

Proposition 2.6. *Direct summand of the centrally endo-AIP module is a centrally endo-AIP.*

Proof. Let A be a centrally endo-AIP module with $E = End_R(A)$ and $B \leq^{\oplus} A$. Then, for some idempotent $e^2 = e \in E$, $B = eA$ and $F = End_R(B) = eEe$. Let K be a fully invariant submodule of B. Clearly, EK is also a fully invariant submodule of A. Suppose $\psi \in \ell_F(K)$, then there exists some $\phi \in E$ such that $\psi = e\phi e$. Now, $e\phi e(EK) = e\phi(eE(eK)) = e\phi(eEe(K)) = e\phi(K) = e\phi e(K) = \psi(K) = 0$, implies that $e\phi e \in \ell_E(EK)$. As A is a centrally endo-AIP module, there exists a central element $\eta \in \ell_E(EK)$ such that $e\phi e\eta = e\phi e$. It is easy to see that $e\eta e \in \ell_F(K)$ and $\psi(e\eta e) = e\phi e(e\eta e) = (e\phi e\eta)e = e\phi e = \psi$. Now, it only remains to show that $e\eta e$ is a central element of $\ell_F(K)$. For it, let $\zeta \in \ell_F(K)$ be arbitrary. Then $\zeta(K) = 0$ and for some $\theta \in E$, $\zeta = e\theta e$. Now $e\eta e\zeta = e\eta e(e\theta e) = e\eta(e\theta e) = e(e\theta e)\eta = e\theta e(e\eta) = e\theta e(e\eta e) = \zeta e\eta e$. Thus, $e\eta e$ is a central element of $\ell_F(K)$ such that for $\psi \in \ell_F(K)$, $\psi e\eta e = \psi$. Therefore, $\ell_F(K)$ is a centrally s-unital ideal of F. Hence, B is a centrally endo-AIP module. □

Submodules of a centrally endo-AIP module need not be centrally endo-AIP. The following example illustrates it.

Example 2.7. Let $A = \mathbb{Z}_p \oplus \mathbb{Q}$ (p is any prime) be a $\mathbb{Z}$-module. By (Example 2.9, [1]) A is a reduced module. Since A is also a Rickart module, from Corollary 2.3 A is a centrally endo-AIP module. While, the submodule $B = \mathbb{Z}_p \oplus \mathbb{Z}$ of A is not centrally endo-AIP. In fact, the submodule B of A is not endo-APP (see Example 4.4, [6])

Now, we discuss in the following proposition when a submodule of a centrally endo-AIP module is a centrally endo-AIP.

Proposition 2.8. *Let B be a fully invariant submodule of a centrally endo-AIP module A with $E = End_R(A)$ and $F = End_R(B)$. If every $\psi \in F$ can be extended to $\bar{\psi} \in E$, then B is a centrally endo-AIP submodule.*

Proof. Let X be a fully invariant submodule of B. As $X \trianglelefteq B$ and $B \trianglelefteq A$ implies $X \trianglelefteq A$. Suppose that $\phi \in \ell_F(X)$, then $\bar{\phi}(X) = \phi(X) = 0$ which implies that $\bar{\phi} \in \ell_E(X)$. Since A is a centrally endo-AIP module, there is a central element $\eta \in \ell_E(X)$ such that $\bar{\phi}\eta = \bar{\phi}$. Therefore, $\phi\eta|_B = \phi$ and $\eta|_B(X) = 0$, which gives $\eta|_B \in \ell_F(X)$ as $B \trianglelefteq A$. Thus, $\ell_F(X)$ is a centrally s-unital ideal of F. Hence, B is a centrally endo-AIP module. □

Example 2.9. If a finitely generated $\mathbb{Z}$-module A is a centrally endo-AIP module, then A is a torsion-free or semisimple module.

Proof. Let A be a finitely generated centrally endo-AIP $\mathbb{Z}$-module then, A is an endo-APP $\mathbb{Z}$-module. So from (Proposition 4.8, [6]), A is a semisimple or a torsion-free module. □

The following example shows that the direct sum of centrally endo-AIP modules need not be a centrally endo-AIP.

Example 2.10. The $\mathbb{Z}$-modules $\mathbb{Z}$ and $\mathbb{Z}_p$ (where p is prime) both are centrally endo-AIP modules, while the direct sum $A = \mathbb{Z} \oplus \mathbb{Z}_p$ is not a centrally endo-AIP $\mathbb{Z}$-module. In fact, A is neither semisimple nor torsion-free, so by Proposition 2.9, A is not a centrally endo-AIP module.

In the following proposition, we discuss when the direct sum of centrally endo-AIP modules is centrally endo-AIP.

Proposition 2.11. *Let $A = A_1 \oplus A_2$, where A_1 and A_2 are centrally endo-AIP modules. If every $\phi \in Hom_R(A_i, A_j)$ (where $i \neq j \in \{1, 2\}$) is a monomorphism, then A is a centrally endo-AIP module.*

Proof. Assume that $B \trianglelefteq A_1 \oplus A_2$, then by (Lemma 1.10, [16]), $B = B_1 \oplus B_2$, where $B_1 \trianglelefteq A_1$ and $B_2 \trianglelefteq A_2$. Now, let $E = End_R(A_1 \oplus A_2) \cong \begin{pmatrix} F_1 & F_{12} \\ F_{21} & F_2 \end{pmatrix}$, where $F_i = End_R(A_i)$ for $i = 1, 2$ and $F_{ij} = Hom_R(A_j, A_i)$, for $i \neq j \in \{1, 2\}$. Let $\alpha = \begin{pmatrix} \alpha_1 & \alpha_{12} \\ \alpha_{21} & \alpha_2 \end{pmatrix}$ and $\alpha \in \ell_E(B)$ then $\alpha(B) = 0$ and for every $\beta_{ij} \in Hom_R(A_j, A_i)$, $\beta_{ij}\alpha_{ji} \in \ell_{F_i}(B_i)$, $i \neq j \in 1, 2$. Since A_1 and A_2 are centrally endo-AIP modules, there are some central elements $\phi_i \in \ell_{F_i}(B_i)$ for $i = 1, 2$ such that $\beta_{ij}\alpha_{ji}\phi_i = \beta_{ij}\alpha_{ji}$, $i \neq j \in \{1, 2\}$. Thus, for every $a \in A_i$, $(\beta_{ij}\alpha_{ji}\phi_i)(a) = (\beta_{ij}\alpha_{ji})(a)$, and so $\beta_{ij}((\alpha_{ji}\phi_i)(a) - (\alpha_{ji})(a)) = 0$. Therefore, by assumption $(\alpha_{ji}\phi_i)(a) - (\alpha_{ji})(a) = 0$ for every $i \neq j \in \{1, 2\}$ and $a \in A_i$, which implies that $\alpha_{ji}\phi_i = \alpha_{ji}$. Thus, for $\phi_1 \in F_1$ and $\phi_2 \in F_2$, $\phi = \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix} \in \ell_E(B)$ and $\alpha\phi = \alpha$. Therefore, $\ell_E(B)$ is a centrally s-unital ideal of E. Hence, A is a centrally endo-AIP module. □

Theorem 2.12. *Let $A = \bigoplus_{\lambda \in \Lambda} A_\lambda$, where A_λ is a centrally endo-AIP module for every $\lambda \in \Lambda$ and $A_\lambda \cong A_\nu$ for every $\lambda, \nu \in \Lambda$. Then A is a centrally endo-AIP module.*

Proof. First, we prove the theorem for $\Lambda = \{1, 2, ..., n\}$. Now, we assume $(A_\lambda)_R \cong X_R$ for every $\lambda \in \Lambda$. Let B be a fully invariant submodule of A, then by (Lemma 1.10, [16]), $B = \bigoplus_{\lambda=1}^{n} B_\lambda$, where $B_\lambda = B \cap A_\lambda \subseteq X$ and each B_λ is fully invariant in X. It is observe that, if $S' = End_R(X)$ then $S \cong Mat_n(S')$ and $\ell_S(B) = \begin{pmatrix} \ell_{S'}(B_1) & \ell_{S'}(B_2) & ... & \ell_{S'}(B_n) \\ \ell_{S'}(B_1) & \ell_{S'}(B_2) & ... & \ell_{S'}(B_n) \\ ... & ... & ... & ... \\ \ell_{S'}(B_1) & \ell_{S'}(B_2) & ... & \ell_{S'}(B_n) \end{pmatrix}$.

So, if $\phi \in \ell_S(B)$ then $\phi = \begin{pmatrix} \phi_1 & \phi_2 & ... & \phi_n \\ \phi_1 & \phi_2 & ... & \phi_n \\ ... & ... & ... & \\ \phi_1 & \phi_2 & ... & \phi_n \end{pmatrix}$, where $\phi_\lambda \in \ell_{S'}(B_\lambda)$ for $\lambda \in \Lambda$. Since X is a centrally endo-AIP module, so for each $\phi_\lambda \in \ell_{S'}(B_\lambda)$, there exist some central elements $\psi_\lambda \in \ell_{S'}(B_\lambda)$ such that $\phi_\lambda \psi_\lambda = \phi_\lambda$.

Hence, $\phi = \begin{pmatrix} \phi_1 & \phi_2 & ... & \phi_n \\ \phi_1 & \phi_2 & ... & \phi_n \\ ... & ... & ... & ... \\ \phi_1 & \phi_2 & ... & \phi_n \end{pmatrix} = \begin{pmatrix} \phi_1\psi_1 & \phi_2\psi_2 & ... & \phi_n\psi_n \\ \phi_1\psi_1 & \phi_2\psi_2 & ... & \phi_n\psi_n \\ ... & ... & ... & ... \\ \phi_1\psi_1 & \phi_2\psi_2 & ... & \phi_n\psi_n \end{pmatrix}$ which implies that $\phi = \begin{pmatrix} \phi_1 & \phi_2 & ... & \phi_n \\ \phi_1 & \phi_2 & ... & \phi_n \\ ... & ... & ... & ... \\ \phi_1 & \phi_2 & ... & \phi_n \end{pmatrix} \begin{pmatrix} \psi_1 & 0 & ... & 0 \\ 0 & \psi_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \psi_n \end{pmatrix}$. Since, for each $\lambda \in \Lambda$, $\psi_\lambda \in \ell_{S'}(B_\lambda)$ is a central element, therefore $\psi = \begin{pmatrix} \psi_1 & 0 & ... & 0 \\ 0 & \psi_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \psi_n \end{pmatrix}$ is a central element of $\ell_S(B)$. Now $\psi(B) = 0$ because $\psi_\lambda \in \ell_{S'}(B_\lambda)$ for every $\lambda \in \Lambda$. Therefore, $\ell_S(B)$ is a centrally s-unital ideal of S. By assuming Λ an infinite set, the proof can be extended to a column finite matrix ring. Hence, A is a centrally endo-AIP module. □

Theorem 2.13. *The following statements are equivalent:*

(i) Every projective right R-module is centrally endo-AIP module;

(ii) Every free R-module is centrally endo-AIP module;

(iii) R is a centrally AIP ring.

Proof. $(i) \Rightarrow (ii)$ It is clear.
$(ii) \Rightarrow (i)$ It is well known that a projective module is a direct summand of a free module. Since, from (ii) every free module is a centrally endo-AIP module. Therefore, from Proposition 2.6 every projective module is a centrally endo-AIP module.
$(ii) \Rightarrow (iii)$ It is clear that R_R is a free right R-module. So, by (ii) R_R is a centrally endo-AIP R-module. Therefore, R is a centrally AIP ring.
$(iii) \Rightarrow (ii)$ Let $A = R^{(\Lambda)}$ be a free R-module and Λ be an arbitrary index set. Since R is a centrally AIP ring, so from Theorem 2.12 A is a centrally endo-AIP module. □

Remark 2.14. From Theorem 2.13 it is clear that, if R is a centrally AIP ring then polynomial ring $R[x]$ and matrix ring $M_n(R)$ are centrally AIP rings, see also (Lemma 3.4, [14]) and (Proposition 3.14, [14]).

3 Endomorphism rings of centrally endo-AIP modules

In this segment, we delve into the examination of the endomorphism ring concerning centrally endo-AIP modules, exploring their equivalence with quasi-Baer modules and endo-AIP modules.

Proposition 3.1. *The endomorphism ring of the centrally endo-AIP module is centrally AIP ring.*

Proof. Let $E = End_R(A)$ be the endomorphism ring of A, F be an ideal of E and $\phi \in \ell_E(F)$. Then $\phi(F(A)) = 0$ which implies that $\phi \in \ell_E(F(A))$. It is clear that $F(A)$ is a fully invariant submodule of A. Since A is a centrally endo-AIP module, there exists a central element $\psi \in \ell_E(F(A))$ such that $\phi\psi = \phi$. Thus, $\psi F(A) = 0 \Rightarrow \psi F = 0$ which implies that $\psi \in \ell_E(F)$. Hence, E is a centrally AIP ring. □

Corollary 3.2. *The endomorphism ring of a centrally endo-AIP module is a semiprime ring.*

Proof. Since, from (Proposition 2.9, [14]) every centrally AIP ring is a semiprime ring. Therefore, the proof follows from Proposition 3.1. □

Remark 3.3. We observe that when we take the class of finitely generated projective module A over a centrally AIP ring R, then the endomorphism ring of A is a centrally AIP. In particular, the centrally AIP property is Morita invariant (Theorem 3.5, [14]).

The following example shows that the converse of the proposition 3.1 need not be true in general.

Example 3.4. Consider the $\mathbb{Z}$-module $A = \mathbb{Z}_{p^\infty}$, where p is a prime number. It is well known that $End_R(A) \cong \mathbb{Z}_{(\bar{p})}$ (ring of p-adic integers) (see Example 3, page 216 [7]), which is a commutative domain and endo-AIP ring (see Example 3.2, [6]). Therefore, it is a centrally AIP ring. Also, A is not an endo-AIP module (see Example 3.2, [6]). Hence, by Proposition 2.2 A is not a centrally endo-AIP module.

Recall that an R-module A is *locally principally quasi-retractable* module, if for every principal ideal P of $E = End_R(A)$ such that $r_A(P) \neq 0$, then there exists a non-zero endomorphism $\psi \in E$ such that $r_A(P) = \psi(A)$ (see Definition 3.3, [6]).

Proposition 3.5. *Let A be a locally principally quasi-retractable module. If $E = End_R(A)$ is a centrally AIP ring, then A is a centrally endo-AIP module.*

Proof. Let B be a fully invariant submodule of A. Then for every $f \in \ell_E(B)$, $EfE \subseteq \ell_E(B)$. Thus, $0 \neq B \subseteq r_A(EfE)$. Since A is a locally principally quasi-retractable module and $r_A(EfE) \neq 0$, so there exists $0 \neq g \in E$ such that $B \subseteq r_A(EfE) = g(A)$ and $f \in \ell_E(EgE)$. Since E is a centrally AIP ring, there is a central element $h \in \ell_E(EgE)$ such that $fh = f = hf$. Now as $B \subseteq g(A)$ so $h(B) \subseteq h(g(A)) = 0$. Therefore, $h \in \ell_E(B)$. Hence, A is a centrally endo-AIP module. □

Recall that a right R-module A has uniform dimension n (written as $u.dim(A_R) = n$) if there is an essential submodule B of A, which is a direct sum of n uniform submodules. If no such an integer exists, then $u.dim(A_R) = \infty$. For a left R- modules, the definition is simultaneous. Further, a ring R has finite right (left) uniform dimension, if $u.dim(R_R) = n$ ($u.dim(_RR) = n$) for a positive integer n.

Proposition 3.6. *Let A be a centrally endo-AIP module and $E = End_R(A)$. If E has a finite right uniform dimension, then E is a quasi-Baer ring.*

Proof. Let A be a centrally endo-AIP module. Then, from Proposition 3.1 E is centrally AIP ring. So, by assumption, E is a centrally AIP ring with a finite right uniform dimension. Hence, from (Theorem 5.1, [14]) E is a quasi-Baer ring. □

A right R-module A is called semi-projective [9], if for any cyclic right ideal F of $E = End_R(A)$, $F = Hom_R(A, FA)$.

Corollary 3.7. *The endomorphism ring of a centrally endo-AIP semi-projective retractable module with finite uniform dimension is quasi-Baer.*

Proof. Let A be a semi-projective retractable module with finite uniform dimension. Then, from (Theorem 2.6, [9]) $E = End_R(A)$ has finite right uniform dimension. Therefore, from Proposition 3.6, E is a quasi-Baer ring. □

Corollary 3.8. *Let A be an R-module with endomorphism ring $E = End_R(A)$. If A is a centrally endo-AIP module and $u.dim(E_E) = 1$, then E is a prime ring.*

Proof. Let A be a centrally endo-AIP module and $u.dim(E_E) = 1$. Then from Corollary 3.2 and Proposition 3.6, E is a semiprime quasi-Baer ring. If E is not a prime ring, then $\ell_E(E\phi E) = Ee$ for some $0 \neq \phi \in E$ and e is right semicentral. Since E is a semiprime quasi-Baer ring, $e \in E$ is central. Thus $E = Ee \oplus E(1 - e)$, a contradiction as $u.dim(E_E) = 1$. Hence, E is a prime ring. □

Proposition 3.9. *Let A be an endo-AIP module and $E = End_R(A)$. If E is a local ring then E is prime.*

Proof. It is clear from (Theorem 3.1, [6]) that the endomorphism ring of an endo-AIP module is AIP ring. Thus, E is a local AIP ring. Then by (Proposition 5.3, [14]), every local AIP ring is a prime ring. □

Recall from [17], a fully invariant submodule $B \trianglelefteq A$ is said to be a prime submodule of A (in this case B is said to be prime in A), if for any ideal F of $E = End_R(A)$, and for any fully invariant submodule $B' \trianglelefteq A$, $F(B') \subset B$ implies $F(A) \subset B$ or $B' \subset B$. Further, a fully invariant submodule B of A is called semiprime if it is an intersection of prime submodules of A. A right R-module A is called prime if $\{0\}$ is prime in A while A is called semiprime module if $\{0\}$ is semiprime submodule of A.

Proposition 3.10. *Let A be a semiprime (prime) module, and $E = End_R(A)$ satisfies ascending chain condition on its principal left ideals. Then, the following conditions are equivalent:*

(i) A is a quasi-Baer module;

(ii) A is an endo-AIP module;

(iii) A is a centrally endo-AIP module.

Proof. $(i) \Rightarrow (ii)$ It is clear.
$(ii) \Rightarrow (iii)$ Let A be an endo-AIP module, B be a fully invariant submodule of A and E satisfies ascending chain condition on its principal left ideals. Then, from (Proposition 3.9, [6]) M is a quasi-Baer module. So, $\ell_E(B) = Ee$ for some $e^2 = e \in E$. Since A is a semiprime module, so by (Theorem 2.9, [17]) E is a semiprime ring. Thus, from (Proposition 1.17, [3]), all left semicentral element of E is right semicentral. Hence, A is a centrally endo-AIP module.
$(iii) \Rightarrow (i)$ A is a centrally endo-AIP module and E satisfies ascending chain condition on principal left ideal. Since every centrally endo-AIP module is endo-AIP, by (Proposition 3.9, [6]) A is a quasi-Baer module. □

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