Palestine Journal of Mathematics

Vol 14(4)(2025) , 913–922

THEORY OF GENERALIZED P-REDUCIBLE FINSLER SPACES

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Communicated by Zafar Ahsan

MSC 2010 Classifications: Primary 53C60; Secondary 53B40.

Keywords and phrases: Berwald manifold, Landsberg manifold, C-reducible, P-reducible, generalized P-reducible, scalar curvature; stretch manifold.

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Abstract. In this work, we investigate a class of Finsler spaces, that involves the class of P-reducible spaces, intrinsically. This class is referred to as P-reducible spaces generalized. We study the reduction of generalized P-reducible spaces with scalar flag curvature to C-reducible spaces under certain conditions. We establish the C-reducibility of a generalized P-reducible Finsler space with vanishing stretch curvature. We give an example of a manifold that is P-reducible but not C-reducible. In the end, we demonstrate that a stretch manifold is a generalized Landsberg manifold.

1 Introduction

Special Finsler manifolds can be constructed by assuming some restrictions on the curvature and torsion tensors, as well as by considering specific formulations of the Finsler structure. While the most ideal case of Finsler manifold is that the metric tensor is positive definite across the slit tangent bundle. Recent developments in Finsler geometry have shown the need to consider more flexible cases due to their applications in various scientific fields [1]. The strict requirement of positive definiteness can be overly restrictive, leading to the exploration of weaker cases such as pseudo or conic Finsler metrics.

Various interesting special Finsler spaces are defined by certain formulations of Cartan, Berwald and Landsberg tensors. For example, the C-reducible, P-reducible, and so on. In [10], Matsumoto has showed that every Randers metric is, in fact, C-reducible. In addition, Matsumoto-Shimada [12] introduced, locally, a Finsler metric which is more general than C-reducible metrics. This metric is called P-reducible metric. Moreover, this class of Finsler metrics have interesting physical meaning and includes the class of Randers metrics.

Despite the extensive research on special Finsler spaces, there is currently no comprehensive geometric theory that fully characterizes these spaces. Moreover, most investigations in this area have been conducted from a local perspective. Many authors have explored special Finsler spaces locally, for example, we refer to [4, 7, 8, 9, 11, 16, 19]. However, studies focusing on the global or intrinsic analysis of such spaces are few in the literature. Some contributions in this direction can be found in [18, 20, 21, 23].

In this paper, we investigate a coordinate-free study of P-reducible and generalized P-reducible Finsler spaces. The class of generalized P-reducible Finsler spaces includes the class of P-reducible spaces. We focus on generalized P-reducible spaces with scalar flag curvature and discover a condition under which they reduce to C-reducible spaces. We demonstrate that generalized P-reducible Finsler space with vanishing stretch curvature are also C-reducible. Finally, we study the generalized Landsberg and stretch manifolds and prove that every generalized Landsberg manifold is a stretch manifold. Also, we provide an example of P-reducible manifold which is not C-reducible.

2 Notation and Preliminaries

We cover some of the essential principles of the pullback approach to Finsler geometry in this section, which are necessary for our investigation. We can consult [14, 17, 24, 25], for example, for further information regarding the aforementioned approach.

Consider the tangent bundle $\pi : TM \longrightarrow M$ and its differential $d\pi : TTM \longrightarrow TM$, given an n-imensional smooth manifold M. The vertical bundle $V(TM)$ of TM is denoted by ker$(d\pi)$. The pullback bundle can be represented as $\pi^{-1}(TM)$. The subbundle of nonzero vectors is represented as $\pi : \mathcal{T}M \longrightarrow M$. Additionally, $\mathfrak{F}(TM)$ signifies the C^∞ functions algebra on TM, and $\mathfrak{X}(\pi(M))$ the $\mathfrak{F}(TM)$-module of differentiable sections of the $\pi^{-1}(TM)$ pullback bundle. The sections of $\mathfrak{X}(\pi(M))$ will be called π-vector fields, which are indicated by barred letters $\overline{X}$.

Consider the short exact sequence of vector bundle morphisms [6]

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(\mathcal{T}M) \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

where $\mathcal{T}M$ is the slit tangent bundle, γ is the natural injection and $\rho := (\pi_{TM}, \pi)$.

The endomorphism $J = \gamma \circ \rho$ is called the tangent structure of TM or the vertical endomorphism. Also, $\mathcal{C} := \gamma\,\overline{\eta}$ is the Liouville vector field $\mathcal{C}$; the fundamental π-vector field is $\overline{\eta}(u) = (u, u)$, $\forall\, u \in \mathcal{T}M$.

The corresponding connection map K for a linear connection D on $\pi^{-1}(TM)$ is described by $K : T\mathcal{T}M \longrightarrow \pi^{-1}(TM) : X \longmapsto D_X\overline{\eta}$, and the horizontal space $H_u(\mathcal{T}M)$ to M at u is $H_u(\mathcal{T}M) := \{X \in T_u(\mathcal{T}M) : K(X) = 0\}$. In the case when

$$T_u(\mathcal{T}M) = V_u(\mathcal{T}M) \oplus H_u(\mathcal{T}M) \;\; \forall\, u \in \mathcal{T}M,$$

the connection D is called regular.

For a regular connection D on M, the vector bundle maps γ, $\rho|_{H(\mathcal{T}M)}$ and $K|_{V(\mathcal{T}M)}$ are isomorphisms. The map $\beta := (\rho|_{H(\mathcal{T}M)})^{-1}$ is the horizontal map of D.

Definition 2.1. Suppose D is a regular connection on $\pi^{-1}(TM)$, where β is the horizontal map, $\mathbf{T}$ is the classical torsion tensor field, and $\mathbf{K}$ is the curvature tensor field. Next up, we have:

(i) For a π-tensor field A of type $(0, p)$, the h- and v-covariant derivatives $\overset{h}{D}$ and $\overset{v}{D}$ are given respectively by

$$(\overset{h}{D} A)(\overline{X}, \overline{X}_1, ..., \overline{X}_p) := (D_{\beta\overline{X}}A)(\overline{X}_1, ..., \overline{X}_p).$$
$$(\overset{v}{D} A)(\overline{X}, \overline{X}_1, ..., \overline{X}_p) := (D_{\gamma\overline{X}}A)(\overline{X}_1, ..., \overline{X}_p).$$

(ii) The (h)h-, (h)hv- and (h)v-torsion tensors of D are given respectively by

$$Q(\overline{X}, \overline{Y}) := \mathbf{T}(\beta\overline{X}, \beta\overline{Y}), \quad T(\overline{X}, \overline{Y}) := \mathbf{T}(\gamma\overline{X}, \beta\overline{Y}), \quad V(\overline{X}, \overline{Y}) := \mathbf{T}(\gamma\overline{X}, \gamma\overline{Y}),$$

(iii) The horizontal, mixed and vertical curvature tensors of D are given respectively by

$$R(\overline{X}, \overline{Y})\overline{Z} := \mathbf{K}(\beta\overline{X}, \beta\overline{Y})\overline{Z}, \quad P(\overline{X}, \overline{Y})\overline{Z} := \mathbf{K}(\beta\overline{X}, \gamma\overline{Y})\overline{Z},$$

$$S(\overline{X}, \overline{Y})\overline{Z} := \mathbf{K}(\gamma\overline{X}, \gamma\overline{Y})\overline{Z},$$

(iv) The (v)h-, (v)hv- and (v)v-torsion tensors of D are given respectively by

$$\widehat{R}(\overline{X}, \overline{Y}) := R(\overline{X}, \overline{Y})\overline{\eta}, \quad \widehat{P}(\overline{X}, \overline{Y}) := P(\overline{X}, \overline{Y})\overline{\eta}, \quad \widehat{S}(\overline{X}, \overline{Y}) := S(\overline{X}, \overline{Y})\overline{\eta}.$$

Throughout, for a Finsler manifold (M, F) of dimension n, we have the following geometric objects:

R°, P°, $\widehat{R^\circ}$	:	the h-, hv-curvature, $(v)h$-torsion tensors of Berwald connection.
$H := i_{\overline{\eta}}\,\widehat{R^\circ}$	:	the deviation tensor of Berwald connection.
$R(\bar{R})$, $P(\bar{P})$	:	the h-, hv-curvatures, tensors of Cartan(Chern) connection.
$\widehat{R}(=\widehat{\bar{R}})$	:	the $(v)h$-torsion tensors of Cartan(Chern) connection.
T	:	the $(h)hv$-torsion of Cartan connection.
C	:	the contracted torsion, where $C(\overline{X}) := Tr\{\overline{Y} \longmapsto T(\overline{X}, \overline{Y})\}$.
$R(\bar{R})$, $P(\bar{P})$	:	the h-, hv-curvatures, tensors of Cartan(Chern) connection.
$\widehat{R}(=\widehat{\bar{R}})$	:	the $(v)h$-torsion tensors of Cartan(Chern) connection.
$\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z})$	:	the Cartan torsion and defined by $\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) := g(T(\overline{X}, \overline{Y}), \overline{Z})$.
$\widehat{P}(=\widehat{\bar{P}})$	:	the $(v)hv$-torsion tensor of Cartan(Chern) connection.
$\widehat{\mathbf{P}}(\overline{X}, \overline{Y}, \overline{Z})$	:	the $(v)hv$-torsion and given by $\widehat{\mathbf{P}}(\overline{X}, \overline{Y}, \overline{Z}) := g(\widehat{P}(\overline{X}, \overline{Y}), \overline{Z})$.
$\mathbf{L} := D^\circ_{\beta\overline{\eta}}\mathbf{T} = \widehat{\mathbf{P}}$	:	the Landsberg curvature.
$\mathbf{J} := D^\circ_{\beta\overline{\eta}}C$	:	the mean Landsberg curvature.
$\sum$	:	the Stretch curvature.
$\mathbf{M}$	:	the Matsumoto torsion.

The relationship between the Berwald connection D°, the Chern connection $\bar{D}$, and the Cartan connection ∇ is given by the following result. Furthermore, the metricity of these connections is demonstrated.

Proposition 2.2. [22] *For a Finsler manifold (M, F), we have*

(i) $D^\circ_{\gamma\overline{X}}\overline{Y} = \nabla_{\gamma\overline{X}}\overline{Y} - T(\overline{X}, \overline{Y}) = \bar{D}_{\gamma\overline{X}}\overline{Y}$.

(ii) $D^\circ_{\beta\overline{X}}\overline{Y} = \nabla_{\beta\overline{X}}\overline{Y} + \widehat{P}(\overline{X}, \overline{Y}) = \bar{D}_{\beta\overline{X}}\overline{Y} + \widehat{P}(\overline{X}, \overline{Y})$,

where β is the horizontal map of ∇. Moreover, the metricity of these connections are given by

(iii) $(D^\circ_{\gamma\overline{X}}\, g)(\overline{Y}, \overline{Z}) = 2\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) = (\bar{D}_{\gamma\overline{X}}g)(\overline{Y}, \overline{Z})$, $\nabla_{\gamma\overline{X}}\, g = 0$.

(iv) $(D^\circ_{\beta\overline{X}}g)(\overline{Y}, \overline{Z}) = -2\widehat{\mathbf{P}}(\overline{X}, \overline{Y}, \overline{Z})$, $\nabla_{\beta\overline{X}}\, g = 0 = \bar{D}_{\beta\overline{X}}\, g$.

Remark 2.3. According to the above Proposition and the fact that $i_{\overline{\eta}}\,T = i_{\overline{\eta}}\,\widehat{P} = 0$, the horizontal (resp. vertical) covariant derivatives for every π-tensor field $\mathbf{A}$ along geodesics (resp. along $\mathcal{C}$) with respect to Cartan, Berwald and Chern connections coincide.

$$\text{i.e. } D^\circ_{\beta\overline{\eta}}\,\mathbf{A} = \nabla_{\beta\overline{\eta}}\,\mathbf{A} = \bar{D}_{\beta\overline{\eta}}\,\mathbf{A}\; (D^\circ_{\gamma\overline{\eta}}\,\mathbf{A} = \nabla_{\gamma\overline{\eta}}\,\mathbf{A} = \bar{D}_{\gamma\overline{\eta}}\,\mathbf{A}), \quad \mathcal{C} = \gamma\overline{\eta}.$$

Also, the horizontal (resp. vertical) covariant derivatives for the Finsler metric g along geodesics (resp. along $\mathcal{C}$) with respect to Cartan, Berwald and Chern connections vanishes identically.

$$\text{i.e. } D^\circ_{\beta\overline{\eta}}\, g = \nabla_{\beta\overline{\eta}}\, g = \bar{D}_{\beta\overline{\eta}}\, g = 0, (D^\circ_{\gamma\overline{\eta}}\, g = \nabla_{\gamma\overline{\eta}}\, g = \bar{D}_{\gamma\overline{\eta}}\, g = 0).$$

Finsler geometry has various non-Riemannian π-tensor fields. For instance, the Cartan torsion $\mathbf{T}$, the Berwald mixed curvature P°, the deviation tensor $H := i_{\overline{\eta}}\,\widehat{R^\circ} = i_{\overline{\eta}}\,\widehat{R}$, the Landsberg curvature $\mathbf{L} := D^\circ_{\beta\overline{\eta}}\mathbf{T}$, the mean Landsberg curvature $\mathbf{J} := D^\circ_{\beta\overline{\eta}}C$, the Matsumoto torsion [1]

$$\mathbf{M}(\overline{X}, \overline{Y}, \overline{Z}) := \mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) - \frac{1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X}, \overline{Y})C(\overline{Z})\}\,, \tag{2.1}$$

[1] $\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}$ means the cyclic sum over $\overline{X}, \overline{Y}, \overline{Z}$.

the $\bar{\mathbf{M}}$-tensor field

$$\bar{\mathbf{M}}(\overline{X},\overline{Y},\overline{Z}) := \mathbf{L}(\overline{X},\overline{Y},\overline{Z}) - \frac{1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X},\overline{Y})\mathbf{J}(\overline{Z})\}, \tag{2.2}$$

and the stretch curvature

$$\sum(\overline{X},\overline{Y},\overline{Z},\overline{W}) := 2\{(D^{\circ}_{\beta\overline{W}}\mathbf{L})(\overline{X},\overline{Y},\overline{Z}) - (D^{\circ}_{\beta\overline{Z}}\mathbf{L})(\overline{X},\overline{Y},\overline{W})\}, \tag{2.3}$$

among others investigated locally in [15, 23], where $\hbar$ is the angular metric tensor and $\ell := F^{-1}i_{\overline{\eta}}\, g$ is the normalized supporting element. They all disappear for Riemannian spaces, so they are called non-Riemannian. The study of these π-tensor fields will help us to understand their distinction and the nature of Finsler geometry.

Definition 2.4. *[23]* We say that a Finsler manifold (M,F) with $n\geq 3$ has scalar curvature k if the deviation tensor H satisfies the following property:

$$H(\overline{X}) = kF^2\phi(\overline{X}),$$

where $i_{\phi(\overline{X})}\, g := i_{\overline{X}}\,\hbar$, ϕ is the vector π-form associated with $\hbar$, $k(x,y)$ is a scalar function on $\mathcal{T}M$, positively homogeneous of degree zero in y $(h^+(0))$[2]. Particularly, (M,F) is called a Finsler manifold of constant curvature if and only if the scalar curvature $k(x,y)$ is constant.

Definition 2.5. *[23]* A Finsler manifold (M,F) is said to be *:*

- Berwald if the Berwald mixed curvature P° vanishes identically
- Landsberg if the Landsberg curvature $\mathbf{L}$ vanishes identically.
- weakly Landsberg if the mean Landsberg curvature $\mathbf{J}$ vanishes identically.

3 Generalized P-reducible Spaces of Scalar curvature

The Cartan, Berwald, and Landsberg tensors have several interesting special forms that have been found by several Finslerians. These special forms of Finsler spaces have led to designations such as C-reducible and P-reducible. The concept of P-reducible metrics was introduced [12], locally, by Matsumoto-Shimada as a generalization of C-reducible metrics. Randers metrics are a specific instance of this class of Finsler metrics, which also has some intriguing physical means. Here, our research is fundamental. We will begin by defining C- and P-reducible Finsler manifolds.

Definition 3.1. *[23]* A Finsler manifold (M,F) is said to be

(i) C-reducible if the Matsumoto torsion $\mathbf{M}$, defined in (2.1), vanishes i.e.

$$\mathbf{T}(\overline{X},\overline{Y},\overline{Z}) = \frac{1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X},\overline{Y})\,C(\overline{Z})\},$$

(ii) P-reducible if the $\bar{\mathbf{M}}$-tensor field, defined in (2.2), vanishes i.e.

$$\mathbf{L}(\overline{X},\overline{Y},\overline{Z}) = \frac{1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X},\overline{Y})\,\mathbf{J}(\overline{Z})\}.$$

It is clear that every C-reducible manifold is P-reducible, but the converse is not true. As a counter-example, we have the following example.

[2] ω is $h^+(k)$ in y if and only if $D^{\circ}_{\gamma\overline{\eta}}\,\omega = k\,\omega$.

Example 3.2. [5] For $n \geq 3$, the class

$$F = \left(a\beta + \sqrt{\alpha^2 - \beta^2}\right) e^{\frac{a\beta}{a\beta + \sqrt{\alpha^2-\beta^2}}}, \quad a \neq 0$$

on a manifold M of dimension n represents a family of non-Berwaldian Landsberg metrics, where $\beta = f(x^1)y^1$, $\alpha = f(x^1)\sqrt{(y^1)^2 + \varphi(\hat{y})}$, $f(x^1)$ is a positive function on $\mathbb{R}$ and φ is a quadratic function in $\hat{y}$ ($\hat{y}$ stands for the variables $y^2, ..., y^n$) and it should be chosen so that α's metric tensor is non-degenerate.

By [13], a non-Riemannian Finsler space (M, F) is C-reducible if and only if it is of Randers or Kropina types. The function $\phi(s)$, $s = \frac{\beta}{\alpha}$, of a Randers metric and Kropina metric are given, respectively, by

$$\phi(s) = 1 + s, \quad \phi(s) = \frac{1}{s}.$$

Now, in this example the function $\phi(s)$ takes the form

$$\phi(s) = \left(as + \sqrt{1 - s^2}\right) e^{\frac{as}{as + \sqrt{1-s^2}}},$$

which is not of Randers or Kropina types, that is, F is not C-reducible. Now, since the metric F is Landsbergian, then the Landsberg tensor $\mathbf{L}$ and mean Landsberg tensor $\mathbf{J}$ vanish and hence the metric is P-reducible and not C-reducible.

As a specific case, we are currently providing an intrinsically new class of Finsler spaces that includes the notion of P-reducible:

Definition 3.3. A Finsler manifold (M, F) is said to be generalized P-reducible if the Landsberg curvature $\mathbf{L}$ has the form:

$$\mathbf{L}(\overline{X}, \overline{Y}, \overline{Z}) = \lambda(x, y)\, \mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) + \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X}, \overline{Y})\, \mathbf{A}(\overline{Z})\}, \tag{3.1}$$

where $\lambda(x, y)$ is a scalar function on TM which homogeneous of degree 1 in y , $\mathbf{A}$ is a π-form which homogeneous of degree 0 in y and $\hbar$ is the angular metric.

Remark 3.4. In view of the above Definition and the fact that $i_{\overline{\eta}}\mathbf{L} = i_{\overline{\eta}}\hbar = i_{\overline{\eta}}\mathbf{T} = 0$, we conclude that $\mathbf{A}(\overline{\eta}) = 0$. If $\mathbf{A}$ vanishes, then (M, F) reduces to a general isotropic Landsberg manifold and if $\lambda(x, y)$ vanishes then (M, F) is a P-reducible manifold. As a result, investigating this class of Finsler spaces helps to understand the geometric meaning of Randers metrics.

Matsumoto has shown (locally) in [10] that a C-reducible Finsler space is reducible to any P-reducible Finsler spaces with non-zero scalar flag curvature. In order to find out under what circumstances these spaces reduce to C-reducible spaces, we now study intrinsically the class of generalized P-reducible Finsler spaces with scalar flag curvature. More precisely, we exhibit the following.

Theorem 3.5. *Let (M, F) be a generalized P-reducible Finsler manifold with nonzero scalar curvature $k(x, y)$. Assume that F fulfills $D^{\circ}_{\beta\overline{\eta}}\lambda + \lambda^2 + kF^2 \neq 0$. Then (M, F) is a C-reducible manifold.*

To prove the above result, we need the following lemmas.

Lemma 3.6. *For a generalized P-reducible Finsler manifold, the horizontal covariant derivatives of Matsumoto torsion $\mathbf{M}$ along geodesics has the form :*

$$(D^{\circ}{}_{\beta\overline{\eta}}\mathbf{M})(\overline{X}, \overline{Y}, \overline{Z}) = \lambda(x, y)\, \mathbf{M}(\overline{X}, \overline{Y}, \overline{Z})$$

Proof. Suppose that (M, F) is generalized P-reducible, then by Definition 3.3 we have

$$\mathbf{L}(\overline{X}, \overline{Y}, \overline{Z}) = \lambda(x, y)\, \mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) + \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X}, \overline{Y})\, \mathbf{A}(\overline{Z})\}. \tag{3.2}$$

Contracting $\overline{Y}$ with $\overline{Z}$, taking into account the facts that $\mathbf{L} := D^{\circ}{}_{\beta\overline{\eta}}\mathbf{T}$ and $\mathbf{J} := D^{\circ}{}_{\beta\overline{\eta}}C$, the above relation takes the form

$$\mathbf{J}(\overline{X}) = \lambda(x, y)\, C(\overline{X}) + (n + 1)\, \mathbf{A}(\overline{X}).$$

From which together with (3.2), we obtain

$$\begin{aligned}\mathbf{L}(\overline{X},\overline{Y},\overline{Z}) &= \lambda(x,y)\,\mathbf{T}(\overline{X},\overline{Y},\overline{Z})\\ &\quad -\frac{\lambda(x,y)}{n+1}\{\hbar(\overline{X},\overline{Y})C(\overline{Z})+\hbar(\overline{Y},\overline{Z})C(\overline{X})+\hbar(\overline{Z},\overline{X})C(\overline{Y})\}\\ &\quad +\frac{1}{n+1}\{\hbar(\overline{X},\overline{Y})\mathbf{J}(\overline{Z})+\hbar(\overline{Y},\overline{Z})\mathbf{J}(\overline{X})+\hbar(\overline{Z},\overline{X})\mathbf{J}(\overline{Y})\}.\end{aligned}\tag{3.3}$$

Applying the horizontal covariant derivative on both sides of (2.1) along geodesics, taking into account the facts that $D^\circ{}_{\beta\overline{\eta}}\,\hbar=0$, we have

$$(D^\circ{}_{\beta\overline{\eta}}\,\mathbf{M})(\overline{X},\overline{Y},\overline{Z})=\mathbf{L}(\overline{X},\overline{Y},\overline{Z})-\frac{1}{n+1}\{\hbar(\overline{X},\overline{Y})\mathbf{J}(\overline{Z})+\hbar(\overline{Y},\overline{Z})\mathbf{J}(\overline{X})+\hbar(\overline{Z},\overline{X})\mathbf{J}(\overline{Y})\}.$$

Hence, the result follows by plugging (2.1) and (3.3) into the above equation. □

Lemma 3.7. *Landsberg curvature* $\mathbf{L}$ *and the deviation tensor* H *are related by the following relation*

$$\begin{aligned}(D^\circ{}_{\beta\overline{\eta}}\,\mathbf{L})(\overline{X},\overline{Y},\overline{Z})+\mathbf{T}(H(\overline{X}),\overline{Y},\overline{Z}) &= -\frac{1}{3}g((D^\circ{}_{\gamma\overline{Y}}\,H)(\overline{X}),\overline{Z})-\frac{1}{3}g((D^\circ{}_{\gamma\overline{Z}}\,H)(\overline{X}),\overline{Y})\\ &\quad -\frac{1}{6}g((D^\circ{}_{\gamma\overline{X}}\,H)(\overline{Y}),\overline{Z})-\frac{1}{6}g((D^\circ{}_{\gamma\overline{X}}\,H)(\overline{Z}),\overline{Y}).\end{aligned}$$

Proof. For Chern connection $\bar{D}$ [25], we have

$$\begin{aligned}&(\bar{D}_{\gamma\overline{X}}\bar{R})(\overline{Y},\overline{Z},\overline{W})+(\bar{D}_{\beta\overline{Y}}\bar{P})(\overline{Z},\overline{X},\overline{W})-(\bar{D}_{\beta\overline{Z}}\bar{P})(\overline{Y},\overline{X},\overline{W})\\ &-\bar{P}(\overline{Z},\widehat{P}(\overline{Y},\overline{X}))\overline{W}+\bar{P}(\overline{Y},\widehat{P}(\overline{Z},\overline{X}))\overline{W}=0.\end{aligned}$$

Setting $\overline{Y}=\overline{W}=\overline{\eta}$, taking into account the facts that $K\circ\gamma=id_{\pi^{-1}(TM)}$, $K\circ\beta=0$, $i_\eta\,\widehat{P}=0$, $\widehat{\bar{P}}=\widehat{P}$, $\bar{D}_{\gamma\overline{X}}\overline{Y}=D^\circ{}_{\gamma\overline{X}}\overline{Y}$, $\bar{D}_{\beta\overline{\eta}}\overline{Y}=D^\circ{}_{\beta\overline{\eta}}\overline{Y}$ and $H:=i_\eta\widehat{\bar{R}}=i_\eta\widehat{R}$, the above equation becomes

$$(D^\circ{}_{\gamma\overline{X}}H)(\overline{Z})-\widehat{R}(\overline{X},\overline{Z})-\bar{R}(\overline{\eta},\overline{Z})\overline{X}+(D^\circ{}_{\beta\overline{\eta}}\widehat{P})(\overline{Z},\overline{X})=0$$

From which together with the facts that $D^\circ{}_{\beta\overline{\eta}}\,g=0$ and $\mathbf{L}=\widehat{\mathbf{P}}$, we obtain

$$g((D^\circ{}_{\gamma\overline{X}}H)(\overline{Z}),\overline{W})+g(\widehat{R}(\overline{Z},\overline{X}),\overline{W})-\bar{R}(\overline{\eta},\overline{Z},\overline{X},\overline{W})+(D^\circ{}_{\beta\overline{\eta}}\mathbf{L})(\overline{Z},\overline{X},\overline{W})=0\tag{3.4}$$

On the other hand, we have[25]

$$\begin{aligned}\bar{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) &= -\bar{R}(\overline{X},\overline{Y},\overline{W},\overline{Z})-2\mathbf{T}(\widehat{R}(\overline{X},\overline{Y}),\overline{Z},\overline{W}).\\ \widehat{R}(\overline{X},\overline{Y}) &= \frac{1}{3}\{(D^\circ{}_{\gamma\overline{X}}H)(\overline{Y})-(D^\circ{}_{\gamma\overline{Y}}H)(\overline{X})\}\end{aligned}$$

Hence, by (3.4) the result follows. □

Proof of Theorem 3.5: Suppose that (M,F) be a generalized P-reducible Finsler manifold of scalar curvature $k(x,y)$. Then, by Definition 2.4, we have

$$H(\overline{Y})=k(x,y)F^2\phi(\overline{Y}).\tag{3.5}$$

Applying the Berwald vertical covariant derivative on both sides of (3.5) and taking into account the facts $D^\circ{}_{\gamma\overline{X}}\,F=\ell(\overline{X})$ and $(D^\circ{}_{\gamma\overline{X}}\phi)(\overline{Y})=-F^{-2}\hbar(\overline{X},\overline{Y})\,\overline{\eta}-F^{-1}\,\phi(\overline{X})\ell(\overline{Y})$, we obtain

$$(D^\circ{}_{\gamma\overline{X}}H)(\overline{Y})=F^2\phi(\overline{Y})D^\circ{}_{\gamma\overline{X}}\,k+2kF\ell(\overline{X})\phi(\overline{Y})-k\hbar(\overline{X},\overline{Y})\,\overline{\eta}-kF\phi(\overline{X})\ell(\overline{Y}).\tag{3.6}$$

On the other hand, from Lemma 3.7, we have

$$
\begin{aligned}
(D^{\circ}{}_{\beta\overline{\eta}}\,\mathbf{L})(\overline{X},\overline{Y},\overline{Z}) \;=\; & -\frac{1}{3}g((D^{\circ}{}_{\gamma\overline{Y}}\,H)(\overline{X}),\overline{Z}) - \frac{1}{3}g((D^{\circ}{}_{\gamma\overline{Z}}\,H)(\overline{X}),\overline{Y}) \\
& -\frac{1}{6}g((D^{\circ}{}_{\gamma\overline{X}}\,H)(\overline{Y}),\overline{Z}) - \frac{1}{6}g((D^{\circ}{}_{\gamma\overline{X}}\,H)(\overline{Z}),\overline{Y}) \\
& -\mathbf{T}(H(\overline{X}),\overline{Y},\overline{Z}). \qquad (3.7)
\end{aligned}
$$

Now, from (3.5), (3.6) and (3.5), after some computations, we get

$$
\begin{aligned}
(D^{\circ}{}_{\beta\overline{\eta}}\,\mathbf{L})(\overline{X},\overline{Y},\overline{Z}) \;=\; & -\frac{F^2}{3}\{\hbar(\overline{Y},\overline{Z})D^{\circ}{}_{\gamma\overline{X}}\,k + \hbar(\overline{X},\overline{Z})D^{\circ}{}_{\gamma\overline{Y}}\,k \\
& +\hbar(\overline{X},\overline{Y})D^{\circ}{}_{\gamma\overline{Z}}\,k + 3k\,\mathbf{T}(\overline{X},\overline{Y},\overline{Z})\}. \qquad (3.8)
\end{aligned}
$$

Contracting $\overline{Y}$ with $\overline{Z}$, we have

$$
(D^{\circ}{}_{\beta\overline{\eta}}\,\mathbf{J})(\overline{X}) = -\frac{F^2}{3}\{(n+1)D^{\circ}{}_{\gamma\overline{X}}\,k + 3k\,C(\overline{X})\}. \qquad (3.9)
$$

Taking twice horizontal covariant derivatives along geodesic for Matsumoto torsion and using the facts that $D^{\circ}{}_{\beta\overline{\eta}}\,\hbar = 0$, $D^{\circ}{}_{\beta\overline{\eta}}\,D^{\circ}{}_{\beta\overline{\eta}}\,\mathbf{T} = D^{\circ}{}_{\beta\overline{\eta}}\,\mathbf{L}$ and $D^{\circ}{}_{\beta\overline{\eta}}\,D^{\circ}{}_{\beta\overline{\eta}}\,C = D^{\circ}_{\beta\overline{\eta}}\,\mathbf{J}$, we obtain

$$
\begin{aligned}
(D^{\circ}{}_{\beta\overline{\eta}}\,D^{\circ}{}_{\beta\overline{\eta}}\,\mathbf{M})(\overline{X},\overline{Y},\overline{Z}) \;:=\; & (D^{\circ}{}_{\beta\overline{\eta}}\,D^{\circ}{}_{\beta\overline{\eta}}\,\mathbf{T})(\overline{X},\overline{Y},\overline{Z}) \\
& -\frac{1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X},\overline{Y})(D^{\circ}{}_{\beta\overline{\eta}}\,D^{\circ}{}_{\beta\overline{\eta}}\,C)(\overline{Z})\}, \\
=\; & (D^{\circ}{}_{\beta\overline{\eta}}\,\mathbf{L})(\overline{X},\overline{Y},\overline{Z}) \\
& -\frac{1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X},\overline{Y})(D^{\circ}{}_{\beta\overline{\eta}}\,\mathbf{J})(\overline{Z})\}
\end{aligned}
$$

In view of (3.8) and (3.9), the above relation becomes

$$
(D^{\circ}{}_{\beta\overline{\eta}}\,D^{\circ}{}_{\beta\overline{\eta}}\,\mathbf{M})(\overline{X},\overline{Y},\overline{Z}) = -k\,F^2\mathbf{M}(\overline{X},\overline{Y},\overline{Z}). \qquad (3.10)
$$

Now, for a generalized P-reducible Finsler manifold (M,F) and applying Lemma 3.6, we get

$$
(D^{\circ}{}_{\beta\overline{\eta}}\,D^{\circ}{}_{\beta\overline{\eta}}\,\mathbf{M})(\overline{X},\overline{Y},\overline{Z}) = (D^{\circ}{}_{\beta\overline{\eta}}\,\lambda + \lambda^2)\mathbf{M}(\overline{X},\overline{Y},\overline{Z}). \qquad (3.11)
$$

Plugging (3.10) into (3.11), we have

$$
(D^{\circ}{}_{\beta\overline{\eta}}\,\lambda + \lambda^2 + k\,F^2)\mathbf{M}(\overline{X},\overline{Y},\overline{Z}) = 0.
$$

Hence, by assumption $D^{\circ}{}_{\beta\overline{\eta}}\,\lambda + \lambda^2 + k\,F^2 \neq 0$, then the Matsumoto torsion vanishes. Then by definition 3.1, (M,F) is a C-reducible manifold. □

As a direct consequence of Theorem 3.5, we retrieve the following result.

Corollary 3.8. *A P-reducible Finsler manifold of non-zero scalar curvature $k(x,y)$ is a C-reducible.*

4 Generalized P-reducible Spaces with zero Stretch Curvature

As a generalization of Landsberg curvature, L. Berwald created locally stretch curvature [2]. He proved that a vector's length stays constant under the parallel displacement along an infinitesimal parallelogram, and only then does the stretch curvature of a Finsler manifold vanish. This section examines intrinsically generalized P-reducible Finsler manifolds and gives special consideration to the situation where stretch curvature vanishes.

Let us begin with the following definition by using the intrinsic version of the stretch curvature $\sum$ given by (2.3).

Definition 4.1. Consider a Finsler manifold (M, F), with the corresponding stretch curvature being $\sum$. If the stretch curvature $\sum$ vanishes, the manifold (M, F) is referred to be a stretch manifold.

Theorem 4.2. *Assume that (M, F) is a stretch manifold. If (M, F) is a generalized P-reducible manifold, then it is C-reducible.*

Proof. Suppose that (M, F) is a generalized P-reducible Finsler manifold with vanishing stretch curvature $\sum$. Then, by the above Definition taking into account (2.3), we obtain

$$(D^{\circ}_{\beta\overline{W}}\,\mathbf{L})(\overline{X}, \overline{Y}, \overline{Z}) = (D^{\circ}_{\beta\overline{Z}}\,\mathbf{L})(\overline{X}, \overline{Y}, \overline{W}).$$

Putting $\overline{W} = \overline{\eta}$ and using the fact that $i_{\overline{\eta}}\,\mathbf{L} = 0$, we conclude that

$$(D^{\circ}_{\beta\overline{\eta}}\,\mathbf{L})(\overline{X}, \overline{Y}, \overline{Z}) = 0. \tag{4.1}$$

Also, as (M, F) is a generalized P-reducible Finsler manifold, then by Definition 3.3, we get

$$\mathbf{L}(\overline{X}, \overline{Y}, \overline{Z}) = \lambda(x, y)\,\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) + \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X}, \overline{Y})\,\mathbf{A}(\overline{Z})\}.$$

Taking horizontal covariant derivatives along geodesics, the above relation gives

$$\begin{aligned}(D^{\circ}_{\beta\overline{\eta}}\,\mathbf{L})(\overline{X}, \overline{Y}, \overline{Z}) &= (D^{\circ}_{\beta\overline{\eta}}\,\lambda(x, y))\,\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) + \lambda\,\mathbf{L}(\overline{X}, \overline{Y}, \overline{Z}) \\ &\quad + \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X}, \overline{Y})\,(D^{\circ}_{\beta\overline{\eta}}\,\mathbf{A})(\overline{Z})\} \\ &= (D^{\circ}_{\beta\overline{\eta}}\,\lambda + \lambda^2)\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) + \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X}, \overline{Y})\,\{\lambda\,\mathbf{A}(\overline{Z}) + (D^{\circ}_{\beta\overline{\eta}}\,\mathbf{A})(\overline{Z})\}\},\end{aligned}$$

From which and taking into account (4.1), we deduce that

$$\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) = \frac{-1}{(D^{\circ}_{\beta\overline{\eta}}\,\lambda + \lambda^2)}\,\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X}, \overline{Y})\,\{\lambda\,\mathbf{A}(\overline{Z}) + (D^{\circ}_{\beta\overline{\eta}}\,\mathbf{A})(\overline{Z})\}\}. \tag{4.2}$$

Contracting $\overline{Y}$ with $\overline{Z}$, we have

$$C(\overline{X}) = \frac{-(n+1)}{(D^{\circ}_{\beta\overline{\eta}}\,\lambda + \lambda^2)}\,\{\lambda\,\mathbf{A}(\overline{Z}) + (D^{\circ}_{\beta\overline{\eta}}\,\mathbf{A})(\overline{Z})\}.$$

From which together with (4.2), we conclude that (M, F) is C-reducible.
This completes the proof. □

In the case where the Chern connection and the Berwald connection coincide, a Finsler manifold is referred to as a Landsberg manifold. The class of generalized Landsberg metrics is a new class of Finsler metrics that Bejancu and Farran [3] introduced by using this concept of Landsberg manifolds. Here, we study intrinsically the generalized Landsberg manifolds.

Definition 4.3. A Finsler manifold (M, F) is called generalized Landsberg if the Berwald and Chern horizontal curvature tensors coincide (i.e. $R^{\circ} = \bar{R}$).

Focusing on the Landsberg curvature, we have the following proposition.

Proposition 4.4. *A Finsler manifold (M, F) is generalized Landsberg if and only if its Landsberg curvature $\mathbf{L}$ fulfills the condition*

$$(D^{\circ}_{\beta\overline{Y}}\,\mathbf{L})(\overline{X}, \overline{Z}, \overline{W}) - (D^{\circ}_{\beta\overline{X}}\,\mathbf{L})(\overline{Y}, \overline{Z}, \overline{W}) = 0.$$
$$\mathbf{L}(\overline{Y}, L(\overline{X}, \overline{Z}), \overline{W}) - \mathbf{L}(\overline{X}, L(\overline{Y}, \overline{Z}), \overline{W}) = 0,$$

where $\mathbf{L}(\overline{X}, \overline{Y}, \overline{Z}) =: g(L(\overline{X}, \overline{Y}), \overline{Z})$ *or* $L(\overline{X}, \overline{Y}) := \widehat{P}(\overline{X}, \overline{Y})$.

Proof. From Proposition 2.2, one can show that

$$\begin{aligned} R^\circ(\overline{X},\overline{Y})\overline{Z} &= \bar{R}(\overline{X},\overline{Y})\overline{Z} + (D^\circ_{\beta\overline{Y}}\widehat{P})(\overline{X},\overline{Z}) - (D^\circ_{\beta\overline{X}}\widehat{P})(\overline{Y},\overline{Z}) \\ &\quad +\widehat{P}(\overline{X},\widehat{P}(\overline{Y},\overline{Z})) - \widehat{P}(\overline{Y},\widehat{P}(\overline{X},\overline{Z})). \end{aligned} \tag{4.3}$$

$$(D^\circ_{\beta\overline{X}}\mathbf{L})(\overline{Y},\overline{Z},\overline{W}) = g((D^\circ_{\beta\overline{X}}\widehat{P})(\overline{Y},\overline{Z}),\overline{W}) - 2g(\widehat{P}(\overline{X},\widehat{P}(\overline{Y},\overline{Z})),\overline{W}). \tag{4.4}$$

For fixed $\overline{X}$ and $\overline{Y}$ and taking (4.3) and (4.4) into account, we figure out that (M,F) is generalized Landsberg if and only if the following equation holds:

$$\begin{aligned} \mathbb{E}(\overline{Z},\overline{W}) &:= (D^\circ_{\beta\overline{Y}}\mathbf{L})(\overline{X},\overline{Z},\overline{W}) - (D^\circ_{\beta\overline{X}}\mathbf{L})(\overline{Y},\overline{Z},\overline{W}) \\ &\quad +\mathbf{L}(\overline{Y},\widehat{P}(\overline{X},\overline{Z}),\overline{W}) - \mathbf{L}(\overline{X},\widehat{P}(\overline{Y},\overline{Z}),\overline{W}) = 0. \end{aligned} \tag{4.5}$$

In general $\mathbb{E}(\overline{Z},\overline{W})$ can be written as

$$\mathbb{E}(\overline{Z},\overline{W}) = \mathbb{E}^s(\overline{Z},\overline{W}) + \mathbb{E}^a(\overline{Z},\overline{W}), \tag{4.6}$$

where, $\mathbb{E}^s$ and $\mathbb{E}^a$ are the symmetric and alternating parts of $\mathbb{E}$, respectively. Hence, from the expression of $\mathbb{E}$, we have

$$\mathbb{E}^s(\overline{Z},\overline{W}) = (D^\circ_{\beta\overline{Y}}\mathbf{L})(\overline{X},\overline{Z},\overline{W}) - (D^\circ_{\beta\overline{X}}\mathbf{L})(\overline{Y},\overline{Z},\overline{W}), \tag{4.7}$$

$$\mathbb{E}^s(\overline{Z},\overline{W}) = \mathbf{L}(\overline{Y},\widehat{P}(\overline{X},\overline{Z}),\overline{W}) - \mathbf{L}(\overline{X},\widehat{P}(\overline{Y},\overline{Z}),\overline{W}). \tag{4.8}$$

On the other hand

$$\mathbb{E} = 0 \Leftrightarrow \mathbb{E}^s = 0 \quad \text{and} \quad \mathbb{E}^a = 0. \tag{4.9}$$

Therefore, from (4.5)-(4.9), the result follows. □

Owing to Definition 4.1, and the above Proposition, we conclude that

Theorem 4.5. *All generalized Landsberg manifolds are stretch manifolds.*

It is amazing that Matsumoto [10] proved that any positive definite C-reducible Finsler manifold (M,F) is a Randers metric. The following is the outcome of combining Theorems 4.2 and 4.5.

Corollary 4.6. *Assume that (M,F) is a generalized P-reducible Finsler manifold of dimension $n \geq 3$. If (M,F) is generalized Landsberg, then F is a Randers metric.*

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Received: 2024-10-17.
Accepted: 2025-01-13.