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# ON SOME ROUGH IDEAL CONVERGENT SEQUENCE SPACES

S. Sharma, S. Mishra and P. Pandey

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**Corresponding Author: Pankaj Pandey**

**Abstract** This paper attempts to generalize rough ideal convergence for some spaces. We have used rough ideal convergence in normed linear spaces to introduce rough ideal convergent sequence spaces using Orlicz function which generalizes the existing sequence spaces, "$[C,1,p],[C,1,p]_0,[C,1,p]_\infty$". We have also studied some properties of these spaces when topologized through a paranorm and investigated inclusion relations, equivalent conditions, decomposition theorem, and algebraic properties of such spaces. We have also given examples to show that the rough ideal convergent spaces so obtained are solid, monotone but fail to be convergence free.

## 1 Introduction

The Orlicz function was first introduced by the Polish mathematician W. Orlicz in 1931. Lindberg initiated the use of Orlicz functions to solve an open problem to find a Banach space having subspaces isomorphic to $c^0$, the space of null sequences, or $\ell^p$ spaces. Their work evoked interest of J. Lindenstrauss and L. Tzafriri [11], and they were successful in constructing a sequence space, $\ell^S$ with the help of Orlicz function $S$, which furthermore solved a long pending open problem of finding a complete normed linear space that has a subspace isomorphic to some $\ell^p=\{a=(a_n)\in\omega\colon \sum_n |a_n|^p<\infty\},(\ 1\le p<\infty)$. And

$$\ell^S := \{x\in\omega : \sum_{k=1}^{\infty} S(\frac{|x_k|}{\psi})<\infty, \text{for some}\psi>0\}, \text{where}$$

$\ell^S$ is a complete normed linear space under the norm

$$\|x\| = \inf\{\psi>0 : \sum_{k=1}^{\infty} S(\tfrac{|x_k|}{\psi})\le 1\}.$$

and is called an Orlicz sequence space.

In [18], Parashar and Choudhary defined certain paranorms for Orlicz sequence space, laying the foundation for topologization of various generalized Orlicz sequence spaces. The Orlicz sequence space has always been a centre of interest for researchers as it generalizes and unifies several known sequence spaces for, the space $\ell^S$ becomes $\ell^p,(\ 1\le p<\infty)$ if we choose $S(x)=x^p$. After the introduction of statistical and ideal convergence, several researchers introduced statistical and ideal convergent sequence spaces determined by Orlicz functions and explored the algebraic and topological properties of the sequence spaces so obtained (see [27], [22], [8], [26], [25], and [10]).

Rough convergence was introduced in connection with the convergence problem of the sequences by Phu in [19]. Rough convergence is a new type of convergence where we study the behaviour of a sequence in any neighbourhood and not necessarily $\epsilon$ neighbourhood.

For any sequence, $\{a_n\}$ in some normed linear space $(X, \|\cdot\|)$ and $r \in \mathbb{R}$ be any positive number, $\{a_n\}$ is said to be $r$-convergent to $a_*$, denoted by $a_n \xrightarrow{r} a_*$, if there exist $n_\epsilon \in \mathbb{N}$ such that

$$n \geq n_\epsilon \Rightarrow \|a_n - a_*\| < r + \epsilon, \text{ for all } \epsilon > 0.$$

where $r$ and $a_*$ are called the roughness degree and the rough limit point, written shortly as $r$-limit point of the sequence $\{a_n\}$ respectively. The immediate consequence of this definition is that every bounded sequence is convergent and the limit is not unique. The set of $r$-limit points of the sequence $\{a_n\}$, is denoted by $LIM^r a_n = \{a_* \colon a_n \xrightarrow{r} a_*\}$.

To extend this notion to unbounded sequences and to enhance its applicability, Aytar gave the statistical version of rough convergence, and several related results were investigated in [1] and [3]. The natural extension of statistical convergence is ideal convergence and therefore rough ideal convergence was introduced and studied in [16] and [7]. Since then researchers all over the globe have explored the possibility of applying rough convergence to sequences in spaces where the notion of distance holds such as, metric spaces, cone metric spaces, $S$-metric spaces, $n$-normed spaces, fuzzy normed spaces etc. for different types of sequences like double sequences, triple sequences, sequences of fuzzy numbers etc. For some related study see [5], [23], and [17].

This paper is aimed at introducing and generalizing rough ideal convergence for sequence space using an Orlicz function $S$, which is the generalization of $\ell^S$, the Orlicz sequence space and "$[C,1,p], [C,1,p]_0, [C,1,p]_\infty$," the sequence spaces of strongly summable sequences [12]. We have divided this paper into four sections. The first section is the introduction. It consists of the literature and background on rough convergence. In the second section, we recall some basic definitions and results that will be used in the main results of this paper. Also, we have used the idea of Orlicz sequence space and introduced some rough ideal convergent sequence spaces with the help of an Orlicz function. In the third section, we have also given some properties of these spaces when topologized through a paranorm and investigated inclusion relations, equivalent conditions, decomposition theorem, and algebraic properties of such spaces. In the last section of this paper, we have given a brief summary and future scope of the present work.

## 2 Preliminaries

**Definition 2.1** (Ideal). Any non-empty collection $\mathfrak{I}$ of subsets of a non-empty set $X$ is called an ideal on $X$ if, the following conditions are satisfied:

(i) $\mathfrak{I}$ is stable under finite union, $H, K \in \mathfrak{I} \implies H \cup K \in \mathfrak{I}$ and

(ii) $\mathfrak{I}$ is stable under subsets, $H \in \mathfrak{I}$, and $K \subseteq H \implies K \in \mathfrak{I}$.

The collection $\mathfrak{I}$ is called an admissible ideal (a.i.) if all the singletons subsets of $X$ lie in $\mathfrak{I}$, and $\mathfrak{I}$ is called non-trivial, whenever $\mathfrak{I} \neq \{\emptyset\}$ and $X \notin \mathfrak{I}$.

**Definition 2.2** (Filter). Any non-empty collection $\mathfrak{F}$ of subsets of a non-empty set $X$ is called an filter on $X$ if,

(i) $\emptyset \notin \mathfrak{F}$,

(ii) $\mathfrak{F}$ is stable under finite intersection, $H, K \in \mathfrak{F} \implies H \cap K \in \mathfrak{F}$ and

(iii) $\mathfrak{F}$ is stable under super-sets, $H \in \mathfrak{F}$, and $H \subseteq K \implies K \in \mathfrak{F}$.

**Definition 2.3** (Filter associated with Ideal). For any ideal $\mathfrak{I}$ of a set $X$, the collection of complements of members of $\mathfrak{I}$ denoted by $\mathfrak{F}(\mathfrak{I})$ and defined as the set $\{P \subset X \colon \exists Q \in \mathfrak{I}, P = X \backslash Q\}$ is called filter associated with ideal $\mathfrak{I}$.

**Definition 2.4** (Ideal Convergence). Ideal convergence of a sequence $\{a_n\}$ in a normed linear space $(X, \|\cdot\|)$ to $a$ is denoted by $\mathfrak{I} - \lim a_n = a$ and is defined as, for any positive pre assigned number $\epsilon$

$$\{n \in \mathbb{N} \colon \|a_n - a\| \geq \epsilon\} \in \mathfrak{I}.$$

**Definition 2.5** (Rough Ideal Convergence). Let $\mathfrak{I}$ be a non-trivial admissible ideal on $\mathbb{N}$, and $r > 0$ be any real number. Then, any sequence $\{a_n\}$ in a normed linear space $(X, \|\cdot\|)$ is said to be $r\mathfrak{I}$-converges to $a$, denoted by $a_n \overset{r\mathfrak{I}}{\to} a$, if

$$\{n \in \mathbb{N}\colon \|a_n - a\| \geq r + \epsilon\} \in \mathfrak{I}, \forall \epsilon > 0.$$

**Theorem 2.6.** *For $r$ is non-negative real number. The following are interchangeable:*

*(i) The sequence $a = \{a_n\}$ is $r\mathfrak{I}$-converges to $a_*$,*

*(ii) There is a sequence $b = \{b_n\}$ such that $\mathfrak{I} - \lim b = a_*$ and $\|a_n - b_n\| \leq r$, for $n \in \mathbb{N}$.*

**Definition 2.7** (Sequence Space). Let $\Lambda$ be a vector space of sequences. Then any vector subspace $\kappa$ of $\Lambda$ is called a sequence space (in short $S$-space).

**Definition 2.8.** (Sectional Subspace) Let $L = \{l_1 < l_2 < l_3 \ldots\}$ be a subsequence in $\mathbb{N}$ and, let $\kappa$ be a $S$-space. Then,

$$\kappa_L = \{(x_l)\colon x_l \in \kappa\}$$

is said to be the $L$-step space or sectional subspace.

**Definition 2.9.** (Canonical Pre-image) For any sequence in $(x_l)$ in $L$-step space, the sequence $a_l$ defined as

$$a_l = \begin{cases} x_l; & \text{if } l \text{ is in } L, \\ 0; & \text{otherwise.} \end{cases}$$

is the canonical pre-image of a sequence $(x_l)$. The collection of all canonical pre-images of each sequence in a step sequence is called the canonical pre-image of a $S$-space.

**Definition 2.10.** (Monotone Space) If a $S$-space $\kappa$ contains pre-images of each of its step spaces then it is called a monotone space.

**Definition 2.11.** (Solid Space) A $S$-space $\kappa$ in which $b_n \in \kappa$, whenever there is some $a_n \in \kappa$ with $|b_n| \leq |a_n|, n \in \mathbb{N}$ is called solid.

**Remark 2.12.** Every solid space is monotone.

As we know that norm is a generalized notion of distance and paranorm is the generalized absolute value function.

**Definition 2.13.** (Paranormed Space) A function $\upsilon\colon X \to \mathbb{R}$ on a linear space $X$, is called a paranorm if it satisfies the following:

(i) $\upsilon(u) \geq 0,\ \forall u \geq 0, u \in X,$

(ii) $\upsilon(-u) = u, \forall\, u \in X,$

(iii) $\upsilon(u + v) \leq \upsilon(x) + \upsilon(v), \forall u, v \in X,$

(iv) For any sequence of scalars, $(\alpha_n)$ with $\alpha_n \to \alpha$ and a vector sequence $(a_n)$ such that $\upsilon(a_n - a) \to 0$ as $n \to \infty$, we have $\upsilon(\alpha_n a_n - \alpha a) \to 0$ as $n \to \infty$ i.e., $\upsilon$ is continuous under multiplication by scalars.

Then (X, $\upsilon$), is a paranormed space. Additionally, (X, $\upsilon$) is called a total paranormed space, whenever $\upsilon(u) = 0$, implies $u$ is the zero vector in $X$. If we define a real valued function on a total paranomed space, $X$ as d(u, v)= $\upsilon(u - v)$, then $d$ is a metric on $X$ and we call $X$ a linear metric space.

**Definition 2.14.** (Convex Function) A map $g\colon [a, b] \to \mathbb{R}$ is convex if,

$$g(t_1 c + t_2 d) \leq t_1 g(c) + t_2 g(d), \text{where } c, d \in [a, b] \text{ and } t_1 + t_2 = 1.$$

**Example 2.15.** Any real valued linear map on any interval of $\mathbb{R}$ defined as $g(x) = \alpha x + \beta$, where $\alpha$ and $\beta$ are constants, is a convex function.

**Definition 2.16** (Orlicz Function). Consider a map $S$ between non negative real numbers, $S\colon [0,\infty) \to [0,\infty)$ . Then it is called an Orlicz function if

(i) $S$ is continuous,

(ii) $S$ is convex,

(iii) $S$ is non decreasing,

(iv) $S$ takes zero to zero, $S(0) = 0$,

(v) $S$ takes positive values to positive values, $S(u) > 0$ for $u > 0$,

(vi) $S$ takes large values to large values, $S(u) \to \infty \ as \ u \to \infty$.

**Definition 2.17** ($\Delta_2$-condition). Let $S$ be an Orlicz function. If for every positive real number $k$, there exist constant $M > 0$ such that $S(2k) \leq MS(k)$, then we say $S$ satisfies $\Delta_2$-condition.

The $\Delta_2$-condition, can also be considered as the inequality $S(pk) \leq MpSk$, $\forall k$ and for $p > 1$.

**Corollary 2.18.** *For* $0 < p < 1$, $S(pk) \leq pSk$, *where* $S$ *is an Orlicz function.*

Throughout this paper, let $\mathfrak{I}$ be a non-trivial admissible ideal on $\mathbb{N}$ and $r$ be a non-negative real number. Also it is well established that the spaces,

(i) $\omega := \{a = (a_n)\colon a_n \in \mathbb{R} \text{ or } \mathbb{C}\}$,

(ii) $\ell^\infty := \{a = (a_n) \in \omega\colon \sup_n \|a_n\| < \infty\}$,

(iii) $c^0 := \{a = (a_n) \in \omega\colon \lim_n \|a_n\| = 0\}$,

are Banach spaces with norm $\|a\| = \sup_n |a_n|$.

For an Orlicz function $S$ and $t = (t_k)$, where $t_k > 0$ and some real number $r > 0$. We give the following definitions:

$$c^{R\mathfrak{I}}(S,t) = \{a = (a_n) \in \omega\colon \{n \in \mathbb{N}\colon S(\tfrac{|a_n - a|}{\psi})^{t_k} \geq r + \epsilon\} \in \mathfrak{I}, a \in \mathbb{R}, \psi > 0\},$$

$$c_0{}^{R\mathfrak{I}}(S,t) = \{a = (a_n) \in \omega\colon \{n \in \mathbb{N}\colon S(\tfrac{|a_n|}{\psi})^{t_k} \geq r + \epsilon\} \in \mathfrak{I}, a \in \mathbb{R}, \psi > 0\},$$

$$\ell^\infty(S,t) = \{a = (a_n) \in \omega\colon \sup_n S(\tfrac{|a_n|}{\psi})^{t_k} < \infty, a \in \mathbb{R}, \psi > 0\}.$$

We also denote

$$\mathcal{G}^{R\mathfrak{I}}{}_c(S,t) = \ell^\infty(S,t) \cap c^{R\mathfrak{I}}(S,t),$$
$$\mathcal{G}^{R\mathfrak{I}}{}_{c_0}(S,t) = \ell^\infty(S,t) \cap c_0{}^{R\mathfrak{I}}(S,t).$$

## 3 Main Results

**Theorem 3.1.** *For* $t = (t_k) \in \ell^\infty$ *and an Orlicz function S, the classes of sequence*

$$c^{R\mathfrak{I}}(S,t), c_0{}^{R\mathfrak{I}}(S,t), \mathcal{G}^{R\mathfrak{I}}{}_c(S,t), and \mathcal{G}^{R\mathfrak{I}}{}_{c_0}(S,t)$$

*are vector spaces over* $\mathbb{R}$.

*Proof.* Consider $a = (a_n)$, $b = (b_n) \in c^{R\mathfrak{I}}(S,t)$ be and let $a^{'}, b^{'}$ be any two scalars. Since $a = (a_n)$, $b = (b_n) \in c^{R\mathfrak{I}}(S,t)$ by definition of $c^{R\mathfrak{I}}(S,t)$, for any $\epsilon > 0$ there are $\psi_1, \psi_2 > 0$ such that the sets,

$$A^1 = \{k \in \mathbb{N}\colon S(\tfrac{|a_k - a_*|}{\psi_1})^{t_k} \geq r_1 + \tfrac{\epsilon}{2}\} \in \mathfrak{I}\}, \text{for some } a_* \in \mathbb{R} \text{ and } r_1 > 0. \tag{3.1}$$

$$A^2 = \{k \in \mathbb{N}\colon S(\tfrac{|b_k - b_*|}{\psi_2})^{t_k} \geq r_2 + \tfrac{\epsilon}{2}\} \in \mathfrak{I}\}, \text{for some } b_* \in \mathbb{R} \text{ and } r_2 > 0. \tag{3.2}$$

Let $r = \max\{r_1, r_2\}$ and $\psi_3 = \max\{2|a^{'}|\psi_1, 2|b^{'}|\psi_2\}$. Furthermore, $S$ being an Orlicz function is convex and non decreasing, we have the following inequality

$$S(\tfrac{|(a^{'}a_k+b^{'}b_k)-(a^{'}a_*+b^{'}b_*)|}{\psi_3})^{t_k} \leq S(\tfrac{|a^{'}||a_k-a_*|}{\psi_3})^{t_k} + S(\tfrac{|b^{'}||b_k-b_*|}{\psi_3})^{t_k},$$
$$\leq S(\tfrac{|a^{'}||a_k-a_*|}{\psi_1})^{t_k} + S(\tfrac{|b^{'}||b_k-b_*|}{\psi_2})^{t_k}.$$

Then from above inequality along with ((3.1)) and ((3.2)) we have,

$$\{k \in \mathbb{N}\colon S(\tfrac{|(a^{'}a_k+b^{'}b_k)-(a^{'}a_*+b^{'}b_*)|}{\psi_3})^{t_k} \geq 2r+\epsilon\} \subseteq A^1 \cup A^2 \in \mathfrak{I},$$

implies that

$$\{k \in \mathbb{N}\colon S(\tfrac{|(a^{'}a_k+b^{'}b_k)-(a^{'}a_*+\beta b_*)|}{\psi_3})^{t_k} \geq 2r+\epsilon\} \in \mathfrak{I}.$$

Thus $a^{'}a_k + b^{'}b_k \in c^{R\mathfrak{I}}(S,t)$. Hence, $c^{R\mathfrak{I}}(S,t)$ is a vector space. The proof for ${c_0}^{R\mathfrak{I}}(S,t)$, ${\mathcal{G}^{R\mathfrak{I}}}_c(S,t)$, and ${\mathcal{G}^{R\mathfrak{I}}}_{c_0}(S,t)$ can be obtained similarly. □

**Theorem 3.2.** *Let $S$ be an Orlicz function and $t = (t_k) \in \ell^\infty$, then the function $\upsilon(x)$ defined as*

$$\upsilon(x) = \inf_{k\geq 1}\{\psi^{\frac{t_k}{M}}\colon \sup_k S(\tfrac{|x_k|}{\psi})^{t_k} \leq 1, \textit{ where } \psi > 0\}, \textit{where}$$

*$M = \max\{1, \sup_k t_k\}$ is a paranorm and the spaces ${\mathcal{G}^{R\mathfrak{I}}}_c(S,t), {\mathcal{G}^{R\mathfrak{I}}}_{c_0}(S,t)$ are paranormed spaces, paranormed by $\upsilon(x)$.*

*Proof.* Proof omitted as it is simple and similar to the proof given in [18]. □

**Theorem 3.3.** *For any two Orlicz functions $S_1$ and $S_2$ which satisfy $\Delta_2$-condition, the following inclusions hold*

*(i) $\zeta(S_1S_2, t)$ contains $\zeta(S_2, t)$,*

*(ii) $\zeta(S_1, t) \cap \zeta(S_2, t)$ is included in $\zeta(S_1 + S_2, t)$, where $\zeta = c^{R\mathfrak{I}}, {c_0}^{R\mathfrak{I}}, {\mathcal{G}^{R\mathfrak{I}}}_c, {\mathcal{G}^{R\mathfrak{I}}}_{c_0}$.*

*Proof.*

(i) Let $x = (x_k) \in {c_0}^{R\mathfrak{I}}(S_2, t)$ be any arbitrary element. Then by definition of ${c_0}^{R\mathfrak{I}}(S_2, t)$, for any pre assigned $\epsilon > 0$ we have some $\psi > 0$ with

$$\{k \in \mathbb{N}\colon S_2(\tfrac{|x_k|}{\psi})^{t_k} \geq r+\epsilon \in \mathfrak{I}\}, \text{where } r > 0. \tag{3.3}$$

For a suitable choice of $\eta$ with $\eta \in (0,1)$, we have $S_1(t) < r+\epsilon$, for $t \in [0,\eta]$. Put $s_k = S_2(\frac{|x_k|}{\psi})^{t_k}$, then

$$\lim_k S_1(s_k) = \lim_{s_k\leq\eta} S_1(s_k) + \lim_{s_k>\eta, k\in\mathbb{N}} S_1(s_k).$$

**Case 1** If $s_k > \eta$. Since $\eta < 1$, we get $s_k < \frac{s_k}{\eta} < 1 + \frac{s_k}{\eta} S_1$ is an Orlicz function, by property 2 and 3 in (2.16) and (2.18), we have,

$$S_1(s_k) < S_1(1 + \tfrac{s_k}{\eta}) < \tfrac{1}{2}S_1(2) + \tfrac{1}{2}S_1(\tfrac{2s_k}{\eta}).$$

Also, by (2.17), $S_1(s_k) < \frac{1}{2}M\frac{s_k}{\eta}S_1(2) + \frac{1}{2}M\frac{s_k}{\eta}S_1(2)$, it follows that $S_1(s_k) < M\frac{s_k}{\eta}S_1(2)$. This further implies,

$$\lim_{s_k>\eta, k\in\mathbb{N}} S_1(s_k) \leq \max\{r, M\tfrac{1}{\eta}S_1(2) \lim_{s_k>\eta, k\in\mathbb{N}}(s_k)\}. \tag{3.4}$$

**Case 2** If $s_k \leq \eta$. Then

$$\lim_{s_k\leq\eta, k\in\mathbb{N}} S_1(s_k) \leq r. \tag{3.5}$$

From (3.3), (3.4) and (3.5), we conclude that

$$\{k \in \mathbb{N} : S_1 S_2 (\tfrac{|x_k|}{\psi})^{t_k} \geq r + \epsilon \in \mathfrak{I}, \text{for some } r > 0\}.$$

Hence, $x = (x_k) \in {c_0}^{R\mathfrak{I}}(S_1 S_2, t)$. This proves that $\zeta(S_2, t) \subseteq \zeta(S_1 S_2, t)$.

(ii) Consider $x = (x_k) \in {c_0}^{R\mathfrak{I}}(S_1, t) \cap {c_0}^{R\mathfrak{I}}(S_2, t)$. Then by definition we have,

$$\{k \in \mathbb{N} : S_1 (\tfrac{|x_k|}{\psi})^{t_k} \geq r + \epsilon\} \in \mathfrak{I}$$

and

$$\{k \in \mathbb{N} : S_2 (\tfrac{|x_k|}{\psi})^{t_k} \geq r + \epsilon \in \mathfrak{I} \text{ for some } r > 0\}, \text{ where}$$

$\epsilon > 0$ and $\psi > 0$. Also,

$$\{k \in \mathbb{N} : (S_1 + S_2)(\tfrac{|x_k|}{\psi})^{t_k} \geq r + \epsilon\} \subseteq [\{k \in \mathbb{N} : S_1 (\tfrac{|x_k|}{\psi})^{t_k} \geq r + \epsilon\} \cup \{k \in \mathbb{N} : S_2 (\tfrac{|x_k|}{\psi})^{t_k} \geq r + \epsilon\}]$$

Suggests that

$$\{k \in \mathbb{N} \colon (S_1 + S_2)(\tfrac{|x_k|}{\psi})^{t_k} \geq r + \epsilon\} \in \mathfrak{I}.$$

Thus, $x = (x_k) \in {c_0}^{R\mathfrak{I}}(S_1 + S_2, t)$.

We can prove the other inclusions by proceeding in the same fashion. □

**Theorem 3.4.** *The spaces ${c_0}^{R\mathfrak{I}}(S, t)$ and ${\mathcal{G}^{R\mathfrak{I}}}_{c_0}(S, t)$ are solid $S$-spaces.*

*Proof.* Let $x = (x_k) \in {c_0}^{R\mathfrak{I}}(S, t)$. Then

$$\{k \in \mathbb{N} : S(\tfrac{|x_k|}{\psi})^{t_k} \geq r + \epsilon \in \mathfrak{I}\}, \text{for some } r > 0, \epsilon > 0, \psi > 0.$$

For $S$ is an Orlicz function $S$, by (2.18) we obtain,

$$S(\tfrac{|\alpha_k x_k|}{\psi})^{t_k} \leq |\alpha_k|^{t_k} S(\tfrac{|x_k|}{\psi})^{t_k} \leq S(\tfrac{|x_k|}{\psi})^{t_k},$$

for some sequence of scalars $\alpha_k$ with $|\alpha_k| \leq 1, \forall k \in \mathbb{N}$. Clearly, $\alpha_k x_k \in {c_0}^{R\mathfrak{I}}(S, t)$. The proof for ${\mathcal{G}^{R\mathfrak{I}}}_{c_0}(S, t)$ can be obtained similarly. □

In the light of the (2.12), we conclude that the spaces ${c_0}^{R\mathfrak{I}}(S, t)$ and ${\mathcal{G}^{R\mathfrak{I}}}_{c_0}(S, t)$ are monotone $S$-spaces.

Our experience with $c^I(M)$ and $m^I(M)$ [25] may lead us to believe that $c^{R\mathfrak{I}}(S, t)$ and ${\mathcal{G}^{R\mathfrak{I}}}_c(S, t)$ are not monotone and therefore not solid. However rough convergence gives sequences liberty to converge for any limit in a relaxed neighbourhood, (for any suitable r>0). Let us understand this with the help of an example.

**Example 3.5.** For $S(x) = x^2, t_k = 1, \mathfrak{I} = \mathfrak{I}_\delta$, consider the constant sequence $\{a_n\}$, where $a_n = 1, \forall n$. Then $a_n \in c^{\mathfrak{I}}(S, t)$, but there is a canonical pre-image $\{b_n\}$ of $\{a_n\}$, defined as:

$$b_k = \begin{cases} a_k\,; & \text{if k is even,} \\ 0\,; & \text{otherwise.} \end{cases}$$

which does not belong to $c^{\mathfrak{I}}(S, t)$ [25]. However, for $r = 1$, $b_k$ is $r - \mathfrak{I}$ convergent to 0. Thus, canonical pre-images of all the step spaces, for a suitable choice of '$r$' are in $c^{R\mathfrak{I}}(S, t)$ and ${\mathcal{G}^{R\mathfrak{I}}}_c(S, t)$.

Thus, $c^{R\mathfrak{I}}(S, t)$ and ${\mathcal{G}^{R\mathfrak{I}}}_c(S, t)$ are solid and monotone for some $r > 0$.

**Theorem 3.6.** *$c^{R\mathfrak{I}}(S, t)$ and ${\mathcal{G}^{R\mathfrak{I}}}_c(S, t)$ fail to be convergence free.*

*Proof.* For the proof of this theorem, we consider the following example. □

**Example 3.7.** For $S(x) = x, t_k = 1 = \psi$, let $\{a_n\}$ be a sequence in $\mathbb{R}$. Let $\mathfrak{I}$ be admissible ideal of $\mathbb{N}$ such that it contains an infinite set $P$. Define

$$a_k = \begin{cases} k\,; & \text{for } k \in P, \\ (-1)^k; & \text{otherwise.} \end{cases}$$

Then $a_n \in c^{R\mathfrak{I}}(S,t)$. However, the permutation sequence, defined as

$$a_k = \begin{cases} (-1)^k; & \text{for } k \in P, \\ k; & \text{otherwise.} \end{cases}$$

is not $r - \mathfrak{I}$ convergent. Thus, $c^{R\mathfrak{I}}(S,t)$ is not convergence free in general.

**Theorem 3.8.** *For an admissible ideal $\mathfrak{I}$, the following are interchangeable:*

*(i)* $(a_k) \in c^{R\mathfrak{I}}(S,t)$,

*(ii) For all $k \in \mathfrak{I}$, there exists $(b_k) \in c^{\mathfrak{I}}(S,t)$ such that $\|a_k - b_k\| \le r, r > 0$,*

*(iii) For all $k \in \mathfrak{I}$,we have $(b_k) \in c^{\mathfrak{I}}(S,t)$ and $(c_k) \in {c_0}^{R\mathfrak{I}}(S,t)$ with $a_k = b_k + c_k$,*

*(iv)* $\lim_{n\to\infty} S(\frac{|a_{m_n} - l|}{\psi})^{t_{m_n}} = r$,*where $l$ is the r-$\mathfrak{I}$ limit of $S(\frac{|a_k|}{\psi})^{t_k}$ and $M = \{m_1, m_2, \ldots\}$ of $\mathbb{N}$ such that $M \in \mathfrak{F}(\mathfrak{I})$.*

*Proof.*

**(1) ⇒ (2)** Let $(a_k) \in c^{R\mathfrak{I}}(S,t)$. Then for some $r > 0$ and $\epsilon > 0$ and $l$ ,

$$\{k \in \mathbb{N}\colon S(\tfrac{|a_k - l|}{\psi})^{t_k} \ge r + \epsilon\} \in \mathfrak{I}\}.$$

Again by (2.6), we have a sequence $(b_k)$ as

$$b_k = \begin{cases} a_k; & S(\frac{|a_k - l|}{\psi})^{t_k} < r, \\ l; & \text{otherwise.} \end{cases}$$

Clearly, $(b_k) \in c^{\mathfrak{I}}(S,t)$ and for all $k \in \mathfrak{I}$, $\|a_k - b_k\| \le r$ .

**(2) ⇒ (3)** We are given that for $(a_k) \in c^{R\mathfrak{I}}(S,t)$, then there exists $(b_k) \in c^{\mathfrak{I}}(S,t)$ with $|a_k - b_k| \le r$, where $r > 0$, $k \in \mathfrak{I}$. Let $J = \{k \in \mathbb{N}\colon \|a_k - b_k\| \ge r\}$, then $J \in \mathfrak{I}$. Define a sequence

$$c_k = \begin{cases} a_k - b_k; & k \in J, \\ 0; & \text{otherwise.} \end{cases}$$

Then, $(c_k) \in {c_0}^{R\mathfrak{I}}(S,t)$.

**(3) ⇒ (4)** Let $A = \{k \in \mathbb{N}\colon S(\frac{|c_k|}{\psi})^{t_k} \ge r + \frac{\epsilon}{2}\}$. Then $A^c \in \mathfrak{F}(\mathfrak{I})$. Let $A^c = M = \{m_1, m_2, \ldots\}$. Then we have, $\lim_{n\to\infty} S(\frac{|a_{k_n} - l|}{\psi})^{t_{k_n}} = r$.

**(4) ⇒ (1)** Let $\epsilon > 0$, then we

$$\{k \in \mathbb{N}\colon S(\tfrac{|a_{k_n} - l|}{\psi})^{t_k} \ge r + \epsilon\} \subseteq M^c \cup \{k \in M\colon S(\tfrac{|a_{k_n} - l|}{\psi})^{p_{k_n}} = r \ge r + \epsilon\} \in \mathfrak{I}.$$

Hence, $(a_k) \in c^{R\mathfrak{I}}(S,t)$.

□

**Theorem 3.9.** *The set ${\mathcal{G}^{R\mathfrak{I}}}_c(S,t)$ is closed in $\ell^\infty(S,t)$.*

*Proof.* Let $({a_k}^{(n)})$ be a Cauchy sequence in ${\mathcal{G}^{R\mathfrak{I}}}_c(S,t)$ such that ${a_k}^{(n)} \to a$. Since ${a_k}^{(n)} \in {\mathcal{G}^{R\mathfrak{I}}}_c(S,t)$, there exists $b_n$ such that for some $r > 0$,

$$\{k \in \mathbb{N}\colon S(\tfrac{|{a_k}^{(n)} - b_n|}{\psi})^{t_k} \ge r + \epsilon\} \in \mathfrak{I}.$$

We need to show that

(a) $(b_n)$ converges to $b$,

(b) If $V = \{k \in \mathbb{N} \colon S(\frac{|a_k{}^{(n)} - b|}{\psi})^{t_k} < r + \epsilon\}$, then $V^c \in \mathfrak{I}$.

**(a)** Now, $(a_k{}^{(n)}) \in \mathcal{G}^{R\mathfrak{I}}{}_c(S, t)$ is Cauchy $\implies \exists n_0 \in \mathbb{N}$ such that

$$sup_k S(\tfrac{|a_k{}^{(n)} - a_k{}^{(m)}|}{\psi})^{t_k} < \frac{\epsilon}{3}, \forall n, m \geq n_0, \epsilon > 0.$$

For a given $\epsilon > 0$, and some $r > 0$ consider

$$P_{nm} = \{k \in \mathbb{N} \colon S(\tfrac{|a_k{}^{(n)} - a_k{}^{(m)}|}{\psi})^{t_k} < \frac{r+\epsilon}{3}\},$$

$$P_m = \{k \in \mathbb{N} \colon S(\tfrac{|a_k{}^{(m)} - b_m|}{\psi})^{t_k} < \frac{r+\epsilon}{3}\},$$

$$P_n = \{k \in \mathbb{N} \colon S(\tfrac{|a_k{}^{(n)} - b_n|}{\psi})^{t_k} < \frac{r+\epsilon}{3}\}.$$

Then, $P_{nm}{}^c, P_n{}^c, P_m{}^c \in \mathfrak{I}$. Consider

$$P = \{k \in \mathbb{N} \colon S(\tfrac{|b_m - b_n|}{)} \psi^{t_k} < r + \epsilon\}.$$

Then, $P^c = P_{nm}{}^c \cup P_m{}^c \cup P_n{}^c$ lies in$\mathfrak{I}$. Let $n, m \geq n_0$, where $n_0 \in P^c$, we get

$$\begin{aligned}\{k \in \mathbb{N} \colon S(\tfrac{|b_m - b_n|}{\psi})^{t_k} < r + \epsilon\} \supseteq \{&\{k \in \mathbb{N} \colon S(\tfrac{|b_m - a_k{}^{(m)}|}{\psi})^{t_k} < \frac{r+\epsilon}{3}\} \\ &\cap \{k \in \mathbb{N} \colon S(\tfrac{|a_k{}^{(m)} - a_k{}^{(n)}|}{\psi})^{t_k} < \frac{r+\epsilon}{3}\} \\ &\cap \{k \in \mathbb{N} \colon S(\tfrac{|a_k{}^{(n)} - b_n|}{\psi})^{t_k} < \frac{r+\epsilon}{3}\}\}.\end{aligned}$$

This establishes that $(b_n)$ is a $\rho$ Cauchy sequence in $\mathbb{R}$ and $\mathbb{R}$ being $r$-complete, we get some $b$ in $\mathbb{R}$ with $b_n$ is $r$-convergent to $b$ for some $r > 2^{-1}\mathbb{J}(\mathbb{R})\rho$, where $\mathbb{J}$ is "Jung's constant" [21].

**(b)** For $a_k{}^{(n)} \to x$, there exists $n_0 \in \mathbb{N}$ with $B = \{k \in \mathbb{N} \colon S(\frac{|a_k{}^{(n_0)} - a_k|}{\psi})^{t_k} < (\frac{r+\epsilon}{3N})^M\}$, where $r > 0, \epsilon > 0, M = \max\{1, \sup_k t_k\},\ N = \max\{1, 2^{P-1}\},\ P = \sup_k t_k \quad *$ implies $B^c \in \mathfrak{I}$.

For a suitably chosen, $n_0$ together with $*$, we have

$$C = \{k \in \mathbb{N} \colon S(\tfrac{|b_{n_0} - b|}{\psi})^{t_k} < (\tfrac{r+\epsilon}{3N})^M, \}$$

such that $C^c \in \mathfrak{I}$. Let $D^c = \{k \in \mathbb{N} \colon S(\frac{|a_k{}^{(n_0)} - b_{n_0}|}{\psi})^{t_k} \geq (\frac{r+\epsilon}{3N})^M\}$ then $D^c \in \mathfrak{I}$.
Let $V^c = B^c \cup C^c \cup D^c$, where $V = \{k \in \mathbb{N} \colon S(\frac{|a_k - b\|}{\psi}^{t_k}) < r + \epsilon\}$. Therefore, for each $k \in V^c$, we have

$$\begin{aligned}\{k \in \mathbb{N} \colon S(\tfrac{|a_k - b|}{\psi})^{t_k} < r + \epsilon\} \supseteq \{&\{k \in \mathbb{N} \colon S(\tfrac{|a_k - a_k{}^{(n_0)}|}{\psi}^{t_k} < (\frac{r+\epsilon}{3N})^M\} \\ &\cap \{k \in \mathbb{N} \colon S(\tfrac{|a_k{}^{(n_0)} - b_{n_0}|}{\psi})^{t_k} < (\frac{r+\epsilon}{3N})^M\} \\ &\cap \{k \in \mathbb{N} \colon S(\tfrac{|b_{n_0} - b|}{\psi})^{t_k} < (\frac{r+\epsilon}{3N})^M\}\}.\end{aligned}$$

Thus, $V^c \in \mathfrak{I}$.

$\square$

## 4 Conclusion

We have defined and explored some new class of sequence spaces with the help of rough convergence and Orlicz function. Theorem (3.8) establishes a some new relationship between rough ideal convergent sequences and ideal convergent sequences. Many different types of functions like modulus functions can be used to define new types of rough ideal convergent sequence spaces to obtain some new and interesting results.

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## Author information

S. Sharma, Department of Mathematics, Lovely Professional University, Phagwara, Punjab, India.
E-mail: `shivani.saggi@gmail.com`

S. Mishra, Department of Mathematics, ASAS, Amity University, Lucknow Campus, Uttar Pradesh, India.
E-mail: `drsanjaymishra1@gmail.com`

P. Pandey, Department of Mathematics, Lovely Professional University, Phagwara, Punjab, India.
E-mail: `pankaj.anvarat@gmail.com`