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ON $2t$-PEBBLING PROPERTY OF A BIPARTITE GRAPH WITHIN THE FRAMEWORK OF GRAHAM'S CONJECTURE

Sreedevi S, Preety Kalra and Parthiban A

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Corresponding Author: Preety Kalra

Abstract This paper examines the $2t$-pebbling property of a bipartite graph in the context of Graham's conjecture, while also discussing the role of bipartite graphs in analyzing the property of $2t$-pebbling, of a graph. When a graph H is connected to another graph S, there must be a minimum of $'n'$ pebbles at every vertex of H to facilitate the pebbling process. Pebbling operations involve eliminating 2 pebbles from one of the vertex and adding one to the nearby vertex. A bipartite graph is characterized by each vertex being connected to every other vertex, with the vertices partitioned into two sets U and V. The property of $2t$-pebbling is applicable to all bipartite graphs, including the bipartite graphs which are complete, Here each vertex in one set is linked to every single vertex in the other set. This property is upheld for any graph G connected to any graph H, as per Graham's conjecture. This approach offers insights into the transferability of pebbles within interconnected graphs and the structural properties of complete bipartite graphs.

1 Introduction

Pebble allocation or simply pebbling, a new advancement in graph theory credited to Lagarias and Saks, attracted significant research attention and yielded notable discoveries. While Chung[1] is often recognized as the pioneer in introducing pebbling concepts in literature, numerous other authors have also contributed to this area. Hulbert's pebbling survey[3] has been instrumental in disseminating key findings. In a connected network, pebbles can be strategically placed on vertices in different configurations. In graph pebbling, we have a graph composed of vertices (nodes) connected by edges. Each vertex can hold a certain number of pebbles, and the goal is to move these pebbles around the graph following predefined rules.

- The movement of a pebble involves removing two pebbles from a vertex and transferring one to a neighboring vertex.connected by an edge.
- The goal is often to reach a target vertex or a collection of vertices by placing pebbles strategically and efficiently.

Applications: It has applications in algorithmic analysis, network optimization, fault tolerance, and resource allocation.

✳ Algorithmic Analysis[12]: Graph pebbling is used to analyze the efficiency of algorithms, particularly in distributed computing and parallel processing.

✳ Network Optimization[8]: It models resource allocation and communication flow in networks, aiding in network optimization and routing strategies.

Research Areas[14, 15, 16]: Graph pebbling has led to the study of pebbling numbers, pebbling strategies, and their implications in various graph structures, such as bipartite graphs, product graphs, and more.

2 Preliminaries

In this section, we revisit some relevant definitions and results to ensure completeness.

Definition 2.1. [2] The pebbling number of a graph, symbolized as $f(G)$, refers to the minimal amount of pebbles needed to establish a pebbling configuration.A pebble transfer consists of removing two pebbles from a vertex and shifting one to an adjacent vertex through an edge. The goal is often to distribute pebbles in a way that satisfies certain conditions or reaches specific vertices.

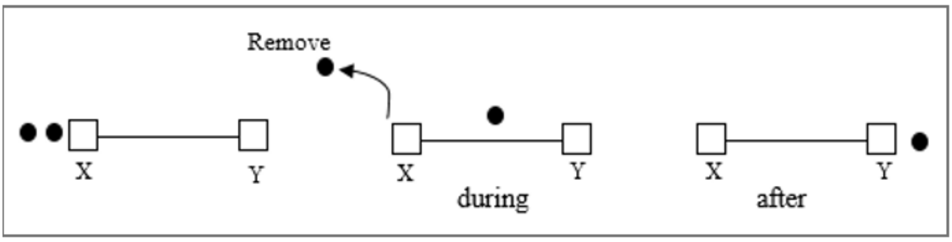

Figure 1. An example

Definition 2.2. [7] The number associated with $2t$-pebbling of a graph G is the minimal integer m such that for any arrangement D on the graph G and any vertex v, there exists a series of pebble shifts that results in minimum $2t$ pebbles on v, provided $|D| \geq m$. After a pebbling move, there are two pebbles on the rightmost vertex, satisfying the condition of 2-pebbling that each be reached from any other vertex using no more than $2t$ pebbles.

Definition 2.3. [4, 9] The Cartesian product of two graphs named S and T, represented as $S \times T$, has vertices (u, v) where u is a vertex in S and v is a vertex in T, and there is an edge between any two vertices (u, v) and (u', v') if, and only if there are edges among u and u' in S and among v and v' in T.

Definition 2.4. [13] For positive integers "t", a product graph $G \times H$ has the property of $2t$ pebbling, provided that there is a pebbling configuration on $G \times H$ in which, for any vertex (u, v) in the product graph, there is a series of pebbling moves that places minimum $2t$ pebbles on (u, v).

Definition 2.5. [5, 13] Let "t" be any positive integer. A bipartite graph G has the property of $2t$ pebbling if there is a pebbling configuration on G such that, for any vertex v in G, there is a series of pebbling moves that places minimum $2t$ pebbles on v.

Definition 2.6. [3] A path, denoted as P_n, is a simple graph consisting of n vertices arranged in a straight line, with edges connecting adjacent vertices.

Definition 2.7. [13] The square of P_n, denoted as P_n^2, represents the Cartesian product of the path with itself. The vertices ofP_{2k+12} can be labeled as pairs (i, j). Here $1 \leq i, j \leq 2k+1$. The edges of P_{2k+12} is determined by the Cartesian product operation. Specifically, two vertices (i, j), (i', j') are connected by an edge if, and only if $i = i'$ or $j = j'$ (i.e., they share a coordinate).

The following results are used in the main research.

Lemma 2.8. *[13] If $P^2_{(2k+1)}$ is a graph with 2^k pebbles and the vertex x_1 with even pebble count, then a pebble may be shifted to $x_{(2k+1)}$.*

Lemma 2.9. *[13]* $f(P_{2k}^2) = 2^k, f(P_{2k+1}^2) = 2^k + 1$.

Conjecture 2.10. [4] For any pair of graphs G_1 and G_2, we have $f(G_1 \times G_2) = f(G_1)f(G_2)$.

Lemma 2.11. *[13] If H is a graph with the property of $2t$-pebbling, then $f(P_{2k}^2 \times H) \leq 2^k f(H)$.*

In the context of graph pebbling, Graham's conjecture is significant because it provides a maximum limit on the number of pebbles required to establish a pebbling configuration that allows reaching any vertex from any other vertex in the graph. Despite progress, Graham's conjecture remains open, especially in the context of bipartite graphs. Exploring unresolved aspects, such as understanding the behavior of pebbling numbers in specific subclasses of bipartite graphs is carried out in this research.

3 Main Results

Theorem 3.1. *A graph G will satisfy $2t$ pebbling property, if the configuration of the graph is $f(P_{2k}^2 \times G) \geq 2^k f(G)$.*

Proof. Assuming that $k = 1$, and having $P_{2k}^2 = P_2$, we are going to create the inequality $f(P_{2k}^2 \times G) \leq 2^k f(G)$ which is proved using Lemma 2.11. Now $p_i = p(v_i(G))$, and considering that $1 \leq i \leq 2k + 2$. The occupied vertices will be q_i in the configuration of $v_i(G)(2k + 2 \geq i \geq 3)$. Setting v as the target vertex in the configuration of $P_{2k}^2 \times G$, we will introduce the vertex $v = (v_i, x)$ belongs to $v_i(G)(2k + 2 \geq i \geq 4)$ while having x belongs to $V(G)$. To simplify the configuration, we will introduce a vertex A for the sub-graph of G, where $A = \langle v_3, v_4 \ldots\ldots\ldots\ldots v_{(2k+1)}, v_{(2k+2)}\rangle$. We, therefore, can use induction, to get the following configuration: $f(A^2 \times G) \leq 2^k f(G)$. The relation between A, and the number of pebbles can be defined in the form: $A^2 \cong P_{(2k-1)}^2$. We will therefore introduce j_1 and j_2 which are also sub-graph vertex into the equation. Having j_1 as odd sub-graph vertex and j_2 even sub-graph vertex then a pebble movement of $\frac{(j_1-1)}{2}$ pebbles to the vertex of v_3 from the vertex of v_1 will occur using Lemma 2.9. Therefore, our new configuration will have $\tilde{p}(A_2) = \frac{(j_1-1)}{2}$ pebbles shifted to v_3 from v_1. The configuration will be $\tilde{p}(A_2) = \frac{(j_1-1)}{2} + j_2 = \frac{(j_1+j_2+1)+(j_2-2)}{2} \geq 2^{(k-1)}$. Since $\frac{(j_1-1)}{2}$ pebbles can be shifted to v_3 from v_1, we will have the configuration as shown: $\tilde{p}(v_2) = j_2 - 1$ and $\tilde{p}(A^2) = \frac{(j_1-2)}{2} + j_2 + 1 = \frac{(j_1+j_2+1)+(j_2-1)}{2}) \geq 2^{(k-1)}$. Hence $f(P_{2k}^2 \times G) \geq 2^k f(G)$. □

Corollary 3.1.1. By Lemma 2.11 and from Theorem 3.1, we have; $f(P_{2k}^2 \times G) = 2^k f(G)$.

Theorem 3.2. *Let G be a graph satisfying the $2t$-pebbling attribute, then it will have to hold:*
1) $tf(G) \geq f(K_t \times G)$,
2) $tf(G) = f(K_t \times G)$.
Therefore, $K_t \times G$ also satisfies the property of $2t$ pebbling.

Proof. Initially introducing the t pebbling number we will consider the configuration $f_t(T, v)$, and use it to denote m which is the smallest integer. Assigning m pebbles on the T vertices will allow pebble movement of t pebbles to move to the vertex v. From the t pebbling tree, we get $f_t(T, v) = (t2^{a_1}) + (2^{a_2}) + \cdots + (2^{a_t}) - (t) + 1$. Here, $a_1, a_2 \ldots a_t$ is the sequence as provided by the size of the path existing, with the path partition that is maximum having the configuration T_v. From the above tree, we are going to define the pebbling upper and lower bounds . Our target vertex will be r in this case. If you place 1 pebble at each of the other vertex, apart from the target vertex r, and a pebble movement can't take place. Also considering u as a distance taken as d from the vertex r, and place 2^{d-1} pebbles at u; then, it will restrict the movement of pebbles to the vertex of r. Therefore, the lower bound for $f(G)$ can be given by $max(n(G), 2^{diam(G)}) \leq f(G)$[10]. To determine the upper bound for graph G's pebbling number, let us consider a graph G having 'n'' vertices with a diameter d. Therefore, using the configuration $(n - 1).(2^d - 1) + 1 \geq f(G)$, the above can be proved using the Pigeonhole principle if $f(G) = (n - 1).2^{d-1} + 1$. This configuration can allow at least 1 vertex in G to have at least 2^d pebbles. Also, the mentioned configuration can allow at least 1 pebble to occupy any target vertex in the graph G. The upper bound on our Bipartite graph can be described as sharp if G is K_n. Also, the upper bound is way off if the graph $G = P_n$ forms the path on vertex n.

Therefore, for the bipartite graph of diameter d the configuration $f(G) \leq (n-d)(2^{d-1})+1$ satisfy the $2t$-pebbling property. Hence we have $K_t \times G$ satisfy the property of $2t$- pebbling. □

Theorem 3.3. *If the bipartite graph G fulfills the property of $2t$ pebbling,then the following configuration will be realized which is $f(K_{m,n})f(G) \leq f(K_{m,n} \times G)$.*

Proof. Let $v_1 \ldots, v_m$ and $w_1 \ldots\ldots, w_n$ be vertices of $K_{m,n}$. The vertices must be labelled in a manner that all the vertices v_i are next to every vertex of w_j. Here, $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Without limiting the general case, our destination vertex is set as (v_1, γ) for all γ. Therefore, we will let $mf(G) - nf(G)$ pebbles occupy the bipartite graph $K_{m,n} \times G$. We, therefore, partition the $K_{m,n} \times G$ graph into Q_1 and Q_2. Here, $Q_1 = C \times G$ and $Q_2 = D \times G$. Through induction, C can be induced in the graph on $v_1, w_1, \ldots, w_{n-1}$ which is the vertex subset of $K_{m,n}$. On the other hand, D is the induced sub-graph on $w_n, v_2, \ldots, v_m$ which is vertex subset of $K_{m,n}$. The above will create the following configuration $A = K_{1,n-1} = K_2$ and $B = K_{1,m-1} = K_2$. Assuming that Q_i has p_i pebbles; therefore, the r_i vertices will have an odd pebble number where $i = 1, 2$. For $p_1 \geq nf(G)$, we have a single pebble that can go through the pebble movement to the vertex (v_1, γ). Assuming that the integer t which is positive and $p_1 = nf(G) - t$. Also $p_2 = mf(G) + t$, this allows us to analyze 2 possible cases.
Case 1: Let us consider $mf(G) - r_2 \geq t$. We are going to apply the steps of pebbling in all the vertices of K_2. This allows for at least $\frac{(p_2 - r_2)}{2}$ pebbles to be placed on the vertices of Q_1. Thus, in total, we have $p_1 + \frac{p_2 - r_2}{2} \geq nf(G) - t + \frac{m.f(G)+t-m.f(G)+t}{2} = n.f(G)$ number of pebbles on Q_1. Therefore, from the configuration we can place a single pebble on (v_1, γ)[6].
Case 2: Let us consider $t > mf(G) - r_2$. By use of induction, we get $p_2 + r_2 = m.f(G) + t + r_2 > 2m.f(G)$. From the above configuration, a 2 Pebble, pebbling movement to (w_n, γ) occurred. Also (v_1, γ) and (w_n, γ) are adjacent, making it possible for 1 pebble to move to (v_1, γ) from (w_n, γ). The Graham's conjecture allows the above configuration to satisfy the $2t$-pebbling property can be configured in the form: $f(K_n \times G) \leq n.f(G)$[4]. This is because G can satisfy the property of $2t$ pebbling and thus verified. □

Corollary 3.3.1. A graph G satisfies the $2t$-pebbling property then the configuration of G will be $f(K_{1,n} \times G) \leq 2f(G)$.

Proof. Let us consider the bipartite graph $f(K_{1,n} \times G) \leq 2f(G)$ where $n > 1$. The graph is supposed to satisfy the property of $2t$ pebbling. Let our target vertex of the graph $f(K_{1,n} \times G) \leq 2f(G)$ be v_0 and n be the degree. Pebble the target vertex with the configuration of $v_0 \times G$; it will suffice where $[n-1] \times f(G)$ pebbles occupying $K_{1,n} \times G$. Let the vertices $v_0, v_1 \ldots, v_n$ be the vertices of $K_{1,n}$ where v_0 will be the vertex having the degree n. Therefore we will set our new target vertex in $f(K_{1,n} \times G) \leq 2f(G)$ to be (v_0, γ).

Let $f(K_{1,n} \times G) \leq 2f(G)$ graph has p_i pebbles that will occupy q_i vertices of $v_i \times G$ on each where $i = 0, 1, \ldots m$. If $p_0 + \sum_{i=1}^{n} \frac{p_i - q_i}{2} \geq f(G)$ then $f(G)$ pebbles will occupy $v_0 \times G$. The mentioned sub graph is considered to be isomorphic to the graph of G; and this will allow a pebble to be placed on (v_0, γ). Since our graph G contains the $2t$ pebbling property, the configuration will be $\frac{p_i - q_i}{2} < f(G)$ considering in $1, \ldots, m$. From the configuration we allow 2 pebbles to occupy the vertices of (v_i, γ). This will allow pebble movement of 1 pebble to the vertices of (v_0, γ).

The configuration will then create the inequalities as follows: $p_0 + \sum_{i=1}^{n} \frac{p_i - q_i}{2} < f(G)$; therefore, $\frac{p_i - q_i}{2} \leq f(G)$, $i = 1, \ldots, n$. Adding the two inequalities together it will give: $p_0 + p_1 + \ldots p_n < [n-1]f(G)$. Therefore, for any pebble distribution, you can't move a pebble on some vertex in $v_0 \times G$. We can start with fewer pebbles that is $[n-1].f(G)$ pebbles. From the results, let's derive the relation to Graham's Conjecture. Since our graph G satisfies the property of $2t$ pebbling, pebble the target vertex existing in the middle edge of G. It will allow $3f(G)$ pebbles to $f(K_{1,n} \times G) \leq 2f(G)$ pebbles and hence by theorem 3.3 the result is reached. □

Theorem 3.4. *In a bipartite graph which is complete, $m + n = f(K_{m,n})$ satisfies the property of $2t$ pebbling only when $m > 1$ and $n > 1$. Here m,n are integers which are positive and Km, n is our bipartite graph which is complete. Also we have $m - n = f(Km, n)$ only if $1 \leq m$, while $n \geq 1$.*

Proof. We start by letting the $K_{m,n}$ vertices to be $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$, where for every vertex v_i, it will always be adjacent to the vertex w_j of sub graph where i is from $1, \ldots, n$ and j is from $1, \ldots, m$. Therefore, without limiting the general case, our targeted vertex will be set as v_1. Thus, we have $p(v_1) = 0$. Using three cases, we are going to breakdown a configuration that is possible using the $(m+n)$[4] pebbles on the graph $K_{m,n}$ which is a complete bipartite one.
Case 1: If $2 \leq p(w_j)$ for some vertex in subgraph j, a single pebble will be set to v_1 from w_j.
Case 2: If $p(w_j) = 0$ for all the vertex in sub-graph j; therefore, $(n+m)$ pebbles can have a pebble movement to $v_2, \ldots, v_n$. This will create a new configuration on v_i of $p(v_i) \geq 2$. This means that the vertex v_i has at least 2 pebbles, and we can, therefore, put 1 pebble on w_1. The above pebble configuration will leave $m - n + 2 \geq m$ pebbles on the remaining $v_2, \ldots, v_n$ vertices. Also the other vertex v_{i_0} having $p(v_{i_0}) \geq 2$ configuration will have at least 2 pebbles. Therefore, v_{i_0}, w_1, v_1 will form the transmitting sub-graph of the complete bipartite graph.
Case 3: Considering $p(w_{j_0})$ is equal to 1 for j_0 and $p(w_j)$ is less than or equal to 1 for all the sub-graph vertex of j, then $p(v_i) \geq 2$. This means that the subgraph will have at least 2 pebbles, and i, v_i, w_{j_0}, v_1 will form a transmitting sub-graph. Hence the theorem. □

Theorem 3.5. *A bipartite graph which is complete ($K_{m,n}$) satisfies the property of $2t$ pebbling if $p + q = 2m + 2n + 1$, where m, n are positive integers.*

Proof. For a complete bipartite graph, $K_{m,n}$ to satisfy the $2t$ pebbling property in relation to the Graham conjecture; first, we let our number of vertices to be q that has 1 pebble and $2.(m+n)+1 = p+q$ pebbles. By setting v_1 as our target vertex; when $p(v_1)$ is greater than or equal to 1, then the pebbles in the vertices except vertex v_1 will be $m - n \leq 2m + 2n - 1 + q - 1$[6]. This pebbles on the vertices will create the configuration: $f(K_{m,n}) = (m+n)$. This allows 1 more pebble to be moved to v_1 using the $2m+2n-1+q-1 \geq m-n$ number of pebbles. Considering $p(v_1) = 0$ configuration, analyze 3 cases as shown.
Case 1: The configuration of $p(w_j) \geq 2$ on the vertices of w_j allows 1 pebble to move from v_1 to w_j. Therefore, from $2m-2n-1+q-2$ number of pebbles, we will have a pebble movement of 1 pebble to the vertex of v_1.
Case 2: Having a configuration of $p(w_{j_0}) = 1$ for the vertex w_{j_0} then for all j we can say that $p(w_j) \leq 1$[11] . And for the vertex of v_i, it will have $q \leq m + n - 2$ pebbles. Therefore, v_i, w_{j_0}, v_1will form a transmitting sub-graph. From the sub-graph, we can use 3 pebbles on v_i and w_{j_0} to move 1 pebble to v_1. Alternatively, if $m+n-1 = q$ thus we can place $m-2$ number of pebbles on $v_2, \ldots, v_m$. This configuration will create transmitting sub-graphs v_i, w_1, v_1 and v_i, w_2, v_1 for some i.
Case 3: Suppose $p(w_j) = 0$ for all the subgraph of j. Thus, having $v_2, \ldots, v_m$ has $2m + 2n + 1 - q \leq m - 2n + 2$ number of pebbles, then from the 4 pebbles, it will allow for a 1 pebble move to the vertex v_1. And from the remaining $m + n \geq 2(m-n) - (1-q) - 4$ number of pebbles, we may utilise another 1 pebble to the vertex v_1. Therefore from the above cases we have $p + q = 2m + 2n + 1$; m,n are both positive integers. □

4 Conclusion

This study focuses on analyzing the $2t$-pebbling property and Graham's conjecture in bipartite graphs. According to Graham's conjecture, if a graph G is connected to a graph H, each vertex of G must have at least 'n' pebbles. The study examines graphs G and H regarding the property of $2t$-pebbling, where $V(G)$ together with $V(H)$ are the vertex sets, and $E(G)$,$E(H)$ are the edge sets. The $2t$-pebbling property is valid for all bipartite graphs, treating them as complete bipartite graphs. This property applies to any graph G connected to any graph H under Graham's conjecture. The study utilizes the $2t$-pebbling property to analyze the implications of Graham's conjecture leading to several significant findings.
Further research can be carried over on $2t$-pebbling property of several classes of graphs in the context of Graham's conjecture.

- A more curious question is the $2t$-pebbling and cross product. How do $2t$-pebbling numbers relate to pebbling of cross products of graphs? Will these connections enable us to further extend the reach of Graham's conjecture by addressing this question for even products $G \times H$?

- In graph theory, for which classes of graphs is it possible to find an upper and lower bound on the $2t$-pebbling number (where $t \in N$) with its corresponding Graham's conjecture-speaking bounds next to them?
- What structural parameters of the graph that are known isomorphic and testing for $2t$-pebble-ability (e.g. diameter or girth; degree) distribution which govern its $2t$-pebbling number?

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Author information

Sreedevi S, Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara, Punjab, India.
E-mail: sreedevi9931@gmail.com

Preety Kalra, Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara, Punjab, India.
E-mail: kalra.preety@gmail.com

Parthiban A, Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore, Tamil Nadu, India.
E-mail: parthiban.a@vit.ac.in