Palestine Journal of Mathematics

Vol 14(Special Issue I)(2025) , 149–159

A STUDY ON DEG-CENTRIC GRAPHS OF SOME GRAPH FAMILIES

Timmy Tomy Thalavayalil and Sudev Naduvath

MSC 2010 Classifications: Primary 33C20; Secondary 33C65.

Keywords and phrases: Distance, eccentricity, graph operations, deg-centric graph, deg-centrication graphs.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Corresponding Author: Timmy Tomy Thalavayalil

Abstract The *deg-centric graph* of a simple, connected graph G, denoted by G_d, is a graph constructed from G such that, $V(G_d) = V(G)$ and $E(G_d) = \{v_iv_j : d_G(v_i, v_j) \leq deg_G(v_i)\}$. This paper presents the properties and structural characteristics of deg-centric graphs of some graph families; the deg-centrication of graph operations are also discussed.

1 Introduction

For a basic terminology of graph theory, we refer to [11]. For further topics on graph classes, [14, 13]. A graph will be assumed to be a simple, connected, and undirected graph G. The size of a graph is the number of edges and is denoted by $\varepsilon(G)$. Recall that the distance between two distinct vertices v_i and v_j of G, denoted by $d_G(v_i, v_j)$, is the length of the shortest path joining them. The eccentricity of a vertex $v_i \in V(G)$, denoted by $e(v_i)$ (or $e_G(v_i)$, is the furthest distance from v_i to some vertex of G. Vertices at a distance $e(v_i)$ from v_i are called the eccentric vertices of v_i. An *eccentric graph* of a graph G, denoted by G_e, is obtained from the same set of vertices as G with two vertices v_i and v_j being adjacent in G_e if and only if v_j is an eccentric vertex of v_i or v_i is an eccentric vertex of v_j (see[1, 2]). The *iterated eccentric graph* of G, denoted by G_{e^k}, is defined in (see[3]) as the derived graph obtained by taking the eccentric graph successively k-times; that is, G_{e^k}= $((Ge)e \ldots)e$, (k-times).

The *degree centric graph* or *deg-centric graph* of a graph G is the graph G_d with $V(G_d) = V(G)$ and $E(G_d) = \{v_iv_j : d_G(v_i, v_j) \leq deg_G(v_i)\}$ (see[4]). Let G be a graph and G_d be the deg-centric graph of G. Then, the successive iteration *deg-centric graph* of G, denoted by G_{d^k}, is defined as the derived graph obtained by taking the deg-centric graph successively k times; that is, G_{d^k}= $((G_d)_d \ldots)_d$, (k-times). This process is known as *deg-centrication process* (see[4]). Let $\varphi(G)$ denote the number of iterations required to transform a graph G to completion. The *exact degree centric graph* or *exact deg-centric graph* of a graph G and denoted by G_{ed}, is the graph with $V(G_{ed}) = V(G)$ and $E(G_{ed}) = \{v_iv_j : d_G(v_i, v_j) = deg_G(v_i)\}$. This graph transformation is called exact deg-centrication (see[5]). Let G be a graph and G_{ed} be the exact deg-centric graph of G. Then the iterated *exact deg-centric graph* of G, denoted by G_{ed^k}, is defined as the graph obtained by applying *exact deg-centrication* successively k-times; That is, G_{ed^k}= $((G_{ed})_{ed}...)_{ed}$, (k-times) (see[5]). The *coarse degree centric graph* or *coarse deg-centric graph* of a graph G, denoted by G_{cd}, is the graph with $V(G_{cd}) = V(G)$ and $E(G_{cd}) = \{v_iv_j : d_G(v_i, v_j) > deg_G(v_i)\}$. Then the *iterated coarse deg-centric graph* of G, denoted by G_{cd^k}, is defined as the graph obtained by applying *coarse deg-centrication* successively k-times; that is, G_{cd^k}= $((G_{cd})_{cd}...)_{cd}$, (k-times) (see[6]).

The *upper degree centric graph* or *upper deg-centric graph* of a graph G, denoted by G_{ud}, is the graph with $V(G_{ud}) = V(G)$ and $E(G_{ud}) = \{v_iv_j : d_G(v_i, v_j) \geq deg_G(v_i)\}$. This graph transformation is called upper deg-centrication (see[9]). Let G be a graph and G_{ud} be the upper deg-centric graph of G. Then the *iterated upper deg-centric graph* of G, denoted by G_{ud^k}, is defined as the graph obtained by applying *upper deg-centrication* successively k-times; that is,G_{ud^k}= $((G_{ud})_{ud}...)_{ud}$, (k-times) (see[9]). The *lower degree centric graph* or *lower deg-centric graph* of a graph G, denoted by G_{ld}, is the graph with $V(G_{ld}) = V(G)$ and $E(G_{ld}) = \{v_iv_j : d_G(v_i, v_j) < deg_G(v_i)\}$. Then, the *iterated lower deg-centric graph* of G, denoted by G_{ld^k}, is defined as the graph obtained by applying *lower deg-centrication* successively k-times; that is, G_{ld^k}= $((G_{ld})_{ld}...)_{ld}$, (k-times) (see[10]).

Motivated by the above-mentioned studies, we investigate the properties and structural characteristics of deg-centrication of graph operations and certain graph classes in this paper.

Definition 1.1. [4] The *degree centric graph* or *deg-centric graph* of simple graph G, denoted by G_d, is the graph with $V(G_d) = V(G)$ and $E(G_d) = \{v_iv_j : d_G(v_i, v_j) \leq deg_G(v_i)\}$. This graph transformation is called *deg-centrication* of the graph.

Definition 1.2. [4] The *iterated deg-centric graph* of a graph G, denoted by G_{d^k}, is defined as the graph obtained by applying *deg-centrication* successively k-times. That is, G_{d^k}= $((G_d)_{d...})_d$, (k-times).

Theorem 1.3. [4] *The deg-centric graph of a non-star graph G with $\delta(G) \geq diam(G)$ is complete.*

Theorem 1.4. [4] *The iterated deg-centric graph of a graph G is complete if and only if G is not a star graph.*

Corollary 1.5. [4] *The deg-centric graph G_d of a non-star graph G with $deg_G(v_i) \geq e(v_i)$ is complete.*

Theorem 1.6. [4] *For a non-star graph G with $m = diam(G)$, $\varphi(G) \leq \varphi(P_m)$.*

2 Bounds for the Size of Deg-centric Graphs

In this section, we explore an upper bound for the size of deg-centric graphs.

Remember that $G+e$ means adding an edge e to G. If two or more edges say, $e_1, e_2, e_3, \ldots, e_k$ are added to G it is denoted by $G + (e_1, e_2, e_3, \ldots, e_k)$.

Proposition 2.1. *For any tree T and $G = T + e$ it follows that, $\varepsilon(G_d) \geq \varepsilon(T_d)$.*

Proof. The above statement immediately follows from the fact that if e is added between vertices v_i, v_j, then $deg_T(v_i) < deg_G(v_i)$ and $deg_T(v_j) < deg_G(v_j)$. □

Observation 2.1. For any graph G and $H = G + e$ it follows that, $\varepsilon(H_d) \geq \varepsilon(G_d)$.

Proposition 2.2. *Let distinct graphs G and H both be of order n.*

(a) If G is a spanning subgraph of H then, $\varepsilon(H_d) \geq \varepsilon(G_d)$.

(b) If G is not a spanning subgraph of H and, G and H share a common spanning tree as well as $\varepsilon(G) < \varepsilon(H)$ then, $\varepsilon(H_d) \geq \varepsilon(G_d)$.

Proof. (a) Clearly, by Observation 2.1, adding edges in an appropriate manner iteratively to G one at a time to obtain H. For each iteration, Observation 2.1, remains valid. This settles the result.

(b) Clearly, as per Proposition 2.1, adding edges in an appropriate manner iteratively to two copies of a common spanning T of G and H, one at a time to first obtain G and after that obtain G. For each iteration, Proposition 2.1, remains valid. This settles the result.

□

Proposition 2.3. *Let S be the set of distinct spanning trees of a graph G. Then, $\varepsilon(G_d) \geq \varepsilon(T_d^-)$ where, $\varepsilon(T_d^-) = \{\varepsilon(T_d) : T \in S\}$.*

Proof. In view of Proposition 2.1, it follows for any spanning tree T of G that, for $H = T + (e_1, e_2, e_3, \ldots, e_k)$, $\varepsilon(H_d) \geq \varepsilon(T_d)$. Amongst the finite number of distinct spanning trees of G there exists some T^- such that $\varepsilon(T_d^-) = \{\varepsilon(T_d) : T \in S\}$. This settles the result. □

3 Deg-centric Graphs of Some Graph Families

In this section, we discuss the deg-centric graph of certain interesting graph classes. Let $\varphi(G)$ denote the number of iterations required to transform a graph G to complete. A non-trivial *bistar graph*, denoted by $S_{a,b}$, is a graph obtained by joining the centers of two non-trivial star graphs $K_{1,a}$, $a \geq 1$ and $K_{1,b}$, $b \geq 1$ with the edge v_0u_0.

Proposition 3.1. *For $a, b \geq 1$,*

$$\varepsilon((S_{a,b})_d) = 2(a + b) + 1.$$

Proof. For a bistar graph $S_{a,b}$; $a, b \geq 1$, let the pendant vertices of $k_{1,a}$ be the set $X = \{v_1, v_2, \ldots, v_a\}$ and let the pendant vertices of $k_{1,b}$ be the set $Y = \{u_1, u_2, \ldots, u_b\}$. Finally, let $W = \{v_0, u_0\}$ be center vertices. By Definition 1.1, it follows that both v_0, u_0 are adjacent with all other $a + b + 1$ vertices. Elements of sets X and Y are pendant vertices. Hence, by Definition 1.1, one edge forms from these pendant vertices, which implies $a + b$ pendant vertices with degree two in $(S_{a,b})_d$. Therefore,

$$\begin{aligned} \varepsilon((S_{a,b})_d) &= \frac{2(a + b + 1) + 2(a + b)}{2} \\ &= 2(a + b) + 1. \end{aligned}$$

□

Proposition 3.2. *For $m, n \geq 2$, $\varphi(K_{m,n}) = 1$.*

Proof. For $m, n \geq 2$, $K_{n,m}$ is a graph whose vertex set can be partitioned into two independent sets X, $|X| = n$ and Y, $|Y| = m$. Let $X = \{v_1, v_2, \ldots, v_n\}$ and $Y = \{u_{1,2}, \ldots, v_m\}$. In view of Definition 1.1, all the vertices of $k_{n,m}$ are adjacent in $(K_{n,m})_d$; that is, the deg-centric graph is complete. Hence, $\varphi(K_{m,n}) = 1$. □

A *djembe graph*, denoted by $D_{1,n}$, is obtained by joining the vertices u_i's; $1 \leq i \leq n$ of a closed helm graph $CH_{1,n}$ to its central vertex v_0.

Proposition 3.3. *For $n \geq 3$, $\varphi(D_{1,n}) = 1$.*

Proof. The djembe graph $D_{1,n}$, $n \geq 3$, is of the order $2n+1$. Let $V(D_{1,n}) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. Since $\delta(D_{1,n}) = 4 > diam(D_{1,n}) = 2$, by Theorem1.3, $\varphi(D_{1,n}) = 1$. This settles the result. □

A *gear graph*, denoted by G_n,$n \geq 3$, is a graph obtained by inserting an extra vertex between each pair of adjacent vertices on the outer cycle of a wheel graph W_n.

Proposition 3.4. *For $n \geq 3$,*

$$\varphi(G_n) = \begin{cases} 1, & \textit{for } n = 3 \\ 2, & \textit{for } n \geq 4. \end{cases}$$

Proof. The gear graph G_n, $n \geq 3$, is of the order $2n + 1$. Let $V(G_n) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$.

(a) If $n = 3$, by Definition1.1, $(G_3)_d \cong k_7$.

(b) If $n \geq 4$, we have $\delta(G_n) = 2$ and $diam(G_n) = 4$. Thus $\varphi(G_n) > 1$. The graph is not a star, by Theorem1.6, $\varphi(G_n) = 2$. Therefore, the result follows.

□

Proposition 3.5. *For $n \geq 4$, $\varepsilon((G_n)_d) = \frac{1}{2}(3n^2 + 5n)$*

Proof. The gear graph, $n \geq 4$, is of the order $2n+1$. Let $V(G_n) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. Since $deg_G(v_0) = n > e(v_0) = 2$, by Definition1.1, $2n$ edge forms from v_0 in $(G_n)_d$. Vertices v_i are adjacent to the center vertex v_0 However, since $\deg_{G_n}(v_i) = 3$, by Definition1.1, $2n$ edges form from a vertex v_i in $(G_n)_d$. Since $\deg_{G_n}(u_i) = 2$, then a distance less than or equal to two vertices forms edges from u_i, that is $\deg_{(G_n)_d(u_i)} = n + 3$. Finally,

$$\varepsilon((G_n)_d) = \frac{1}{2} \sum_{w_i \in V((G_n)_d)} deg_(w_i) = \frac{1}{2}(3n^2 + 5n)$$

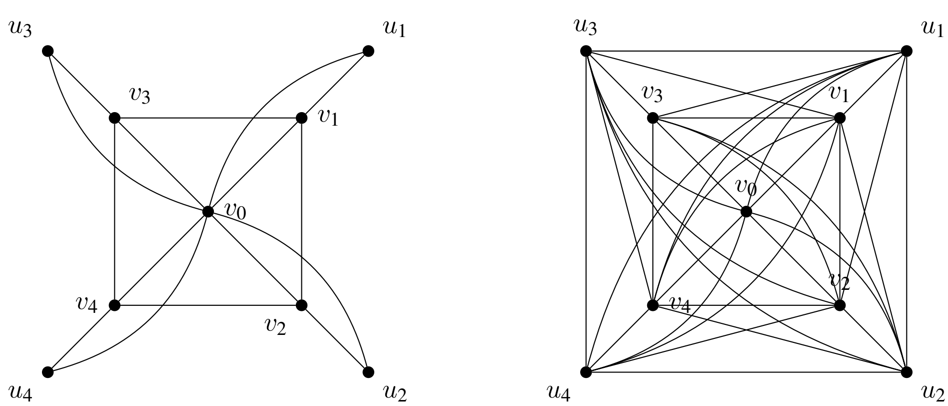
. □

A *flower graph*, $F_{1,n}$, $n \geq 3$ is a graph obtained from a helm graph $H_{1,n,}$ by joining each of its pendant vertices u_i's to its central vertex v_0.

Proposition 3.6. *For $n \geq 3$, $\varphi(F_{1,n}) = 1$.*

Proof. The flower graph $F_{1,n,}$, $n \geq 3$ is of the order $2n + 1$. Let $V(F_{1,n,}) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. Since $\delta(F_{1,n,}) = 2 = diam(F_{1,n,}) = 2$, by Theorem1.3, $\varphi(F_{1,n}) = 1$. □

An illustration of a proposition 3.6 is given in Figure 1.



(a) $F_{1,4}$ (b) $(F_{1,4})_d$

Figure 1: $\varphi((F_{1,4})_d) = 1$.

The *sunflower graph*, denoted by $SF_{1,n}$, $n \geq 3$ is obtained from the wheel $W_{1,n}$ by attaching n vertices u_i, $1 \leq i \leq n$ such that each u_i is adjacent to v_i and v_{i+1} and count the suffix is taken modulo n.

Proposition 3.7. *For $n \geq 3$,*

$$\varphi(SF_{1,n}) = \begin{cases} 1, & \text{for } n = 3. \\ 2, & \text{for } n \geq 4. \end{cases}$$

Proof. The sunflower graph $SF_{1,n}$, $n \geq 3$, is of the order $2n + 1$. Let $V(SF_{1,n,}) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$.

(a) If $n = 3$, since $\delta(SF_{1,n}) = 2 = diam(SF_{1,n}) = 2$, by Theorem1.3, $(SF_{1,n})_d \cong k_7$.

(b) For $n \geq 4$, we have $\delta(SF_{1,n}) = 2$ and $diam(SF_{1,n}) = 4$. Thus, $\varphi(SF_{1,n}) > 1$. The graph G is not a star, then, $m = diam(G)$ then $\varphi(G) \leq \varphi(P_m)$, $\varphi(SF_{1,n}) \leq \varphi(P_4) = 2$, by Theorem1.6, $\varphi(SF_{1,n}) = 2$.

□

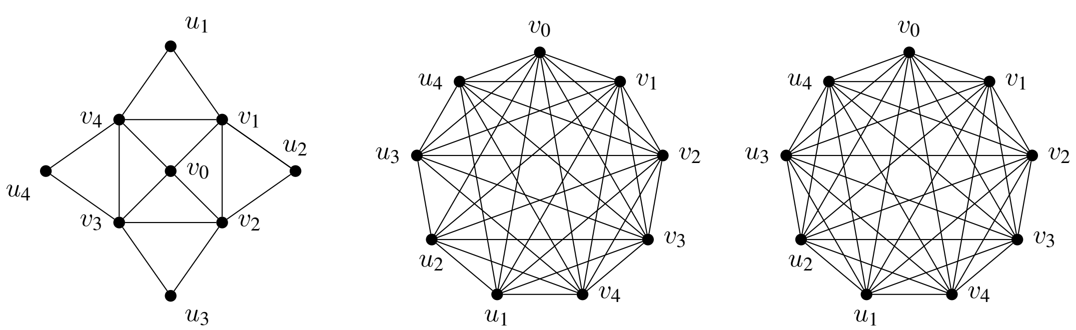
Proposition 3.8. *For $n \geq 4$, $\varepsilon((SF_{1,n})_d) = \frac{1}{2}(3n^2 + 5n)$.*

Proof. The sun flower graph $SF_{1,n,}$, $n \geq 3$, is of the order $2n + 1$. Let $V(SF_{1,n,}) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. Since $\deg_{SF_{1,n}}(v_0) = n > e(v_0) = 2$ and $\deg_{SF_{1,n}}(v_i) = n+1 > e(v_i) = 2$ and $\deg_{SF_{1,n}}(v_i) = n + 1 > e(v_i) = 2$, by the Definition1.1, $2n$ edges form from v_0 and v_i in deg-centric graph. Since $deg_{SF_{1,n}}(u_i) = 2$, by the Definition1.1, distance one or two edges forms from u_i, that is $\deg_{(SF_{1,n})_d(u_i)} = n + 3$. Finally,

$$\begin{aligned}\varepsilon((SF_{1,n})_d) &= \frac{1}{2}\sum_{w_i \in V((SF_{1,n})_d)} deg_(w_i) \\ &= \frac{1}{2}((n+1)(2n) + n(n+3)) \\ &= \frac{1}{2}(3n^2 + 5n).\end{aligned}$$

□

An illustration of a proposition 3.7 and proposition 3.8 is given in Figure 2.



(a) $G = SF_{1,4}$ (b) $(SF_{1,4})_d$ (c) $(SF_{1,4})_{d^2} = k_9$

Figure 2: Deg-centric graph of $SF_{1,4}$.

A *closed sunflower graph* $CSF_{1,n}$ is obtained by adding the edge u_iu_{i+1} of the sunflower graph, suffixes taken modulo n.

Proposition 3.9. *For $n \geq 3$, $\varphi(CSF_{1,n}) = 1$.*

Proof. The closed sunflower graph $CSF_{1,n,}$, is of the order $2n+1$. Let $V(CSF_{1,n,}) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. For $n \geq 3$, $\delta(CSF_{1,n,}) \geq diam(CSF_{1,n,})$, by Theorem 1.3, $(CSF_{1,n})_d$ is complete, that is $\varphi(CSF_{1,n}) = 1$. □

A *blossom graph*, denoted by $Bl_{1,n}$, is obtained by making each u_i adjacent to the central vertex of the closed sunflower graph.

Proposition 3.10. *For $n \geq 3$, $\varphi(Bl_{1,n}) = 1$.*

Proof. The blossom graph $Bl_{1,n,}$, $n \geq 3$ is of the order $2n+1$. Let $V(Bl_{1,n,}) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. Since $\delta(Bl_{1,n,}) = 5 > diam(Bl_{1,n,}) = 2$, by the Theorem1.3, $\varphi(Bl_{1,n}) = 1$. □

A *sunlet graph*, denoted by Sl_n, $n \geq 3$, is a graph obtained by attaching a pendant vertex to every vertex of a cycle graph c_n, $n \geq 3$. In other words, a sunlet graph on $2n$ vertices is obtained by taking the corona product $C_n \circ K_1$.

Proposition 3.11. *For $n \geq 3$,*

$$\varepsilon((Sl_n)_d) = \begin{cases} \frac{3n^2-n}{2}, & \text{if } 3 \leq n \leq 5 \\ \frac{3n^2-3n}{2}, & \text{if } n = 6 \\ 8n, & \text{if } n \geq 7. \end{cases}$$

Proof. Let $V(Sl_n) = \{v_1, v_2, \ldots, v_n, \underbrace{u_1, u_2, \ldots, u_n}_{\text{pendant vertices}}\}$. Then,

(a) If $3 \leq n \leq 5$, in view of Definition 1.1, $\deg_{Sl_n}(v_i) = 3 > e(v_i) = 2$ then all v_i vertices are adjacent with other $2n-1$ vertices. However, since all u_i are pendants vertices, by Definition 1.1, no edge forms from a u_i in the deg-centric graph. Then the number of edges equals $\frac{n(2n-1)+(n)(n)}{2}$. Finally, $\varepsilon((Sl_n)_d) = \frac{3n^2-n}{2}$.

(b) If $n = 6$, by Definition 1.1, then all v_i vertices are adjacent with other $2n-2$ vertices. However, since all u_i are pendant vertices, in view of Definition 1.1, no edge forms from a u_i in $(Sl_n)_d$. Finally, $\varepsilon((Sl_n)_d) = \frac{n(2n-2)+n(n-1)}{2} = \frac{3n^2-3n}{2}$.

(c) Consider $n \geq 7$, by Definition 1.1, then all v_i vertices are adjacent with eleven vertices. However, since all u_i are pendant vertices, no edge forms from a u_i in $(Sl_n)_d$. Then, all u_i have degree five. Then, $\varepsilon((Sl_n)_d) = \frac{11n+5n}{2}$. Finally, $\varepsilon((Sl_n)_d) = 8n$.

□

Corollary 3.12. *For $n \geq 3$, $\varphi(Sl_n) \geq 2$.*

Proof. In views of Definition 1.1, and Proposition 3.11, Sl_d is not complete; that is $\varphi(Sl_n) \geq 2$. □

An illustration of a proposition 3.11 is given in Figure 3.

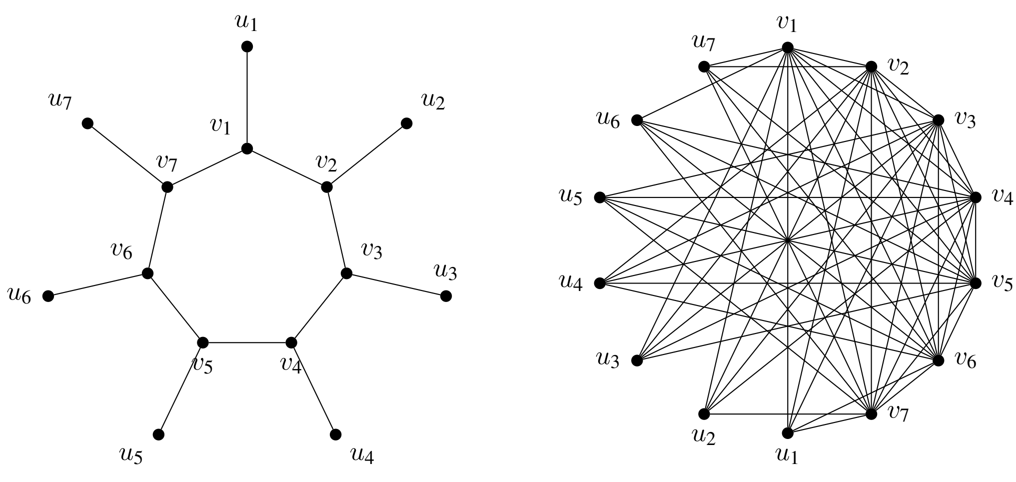


Figure 3: Deg-centric graph of Sl_7.

An *antiprism graph*, denoted by A_n, $n \geq 3$ is a graph obtained two cycles C_n and C'_n of order n with vertex sets $V = \{v_1, v_2, v_3, \ldots, v_n\}$ and $U = \{u_1, u_2, u_3, \ldots, u_n\}$ respectively. Join the vertices u_iv_i and u_iv_{i+1} to form the additional edges.

Proposition 3.13. *For $n \geq 3$,*

$$\varepsilon((A_n)_d) = \begin{cases} \varepsilon(K_{2n}) & \text{if } 3 \leq n \leq 8 \\ 16n & \text{if } n \geq 9. \end{cases}$$

Proof. For an antiprism graph A_n, $n \geq 3$, is of the order $2n$. Let $V(A_n) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$.
(i) If $3 \leq n \leq 8$, $\deg_{A_n}(v_i) = deg_{A_n}(u_i) = 4 > e(v_i) = e(u_i)$, by Corollary 1.5, $(A_n)_d \cong K_{2n}$.
(ii) If $n \geq 9$, $\deg_{A_n}(v_i) = \deg_{A_n}(u_i) = 4$, by Definition 1.1, $\deg_{(A_n)_d(v_i)} = \deg_{(A_n)_d(u_i)} = 16$. That implies $\varepsilon((A_n)_d) = 16n$.
□

Observation 3.1. For $n \geq 9$, $\varphi(A_n) \geq 2$.

Consider a complete graph K_n with the vertex set $V = \{v_1, v_2, v_3, \ldots, v_n\}$. Let $U = \{u_1, u_2, u_3, \ldots, u_n\}$ be a copy of $V(G)$ such that u_i corresponds to v_i. The *sun graph*, denoted by S_n, is a graph with vertex set $V \cup U$ and two vertices x and y are adjacent in S_n if $x \sim y$ in K_n and $x = u_i, y \in v_i, v_{i+1}$.

Proposition 3.14. *For $n \geq 4$, $\varphi(S_n) = 2$.*

Proof. For $n \geq 4$, we have $\delta(S_n) = 2$ and $diam(S_n) = 3$. Thus, $\varphi(S_n) > 1$. By Definition 1.1, $\varphi(S_n) = 2$. Therefore, the result follows. □

Observation 3.2. For $n = 3$, $\varphi(S_n) = 1$.

Proof. For a sun graph $S_n, n \geq 3$, is of the order $2n$. Let $V(S_n) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. For $n = 3$, $\delta(S_3) \geq diam(S_3)$, by Theorem 1.3, $(S_3)_d$ is complete; that is, $\varphi(S_3) = 1$. □

Proposition 3.15. *For $n \geq 3$, $\varepsilon((S_n)_d) = \frac{1}{2}(n(3n + 1))$.*

Proof. The sun graph $S_n, n \geq 3$, is of the order $2n$. Let $V(S_n) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. Since $deg_{S_n}(v_i) = n + 1 > e(v_i) = 2$, by Definition 1.1, $(2n - 1)$ edges form from v_i in $(S_n)_d$. However, since $\deg_{S_n}(u_i) = 2$, by Definition 1.1, distance one or two edges forms from u_i in $(S_n)_d$. Finally,

$$\begin{aligned}\varepsilon((S_n)_d) &= \frac{1}{2} \sum_{w_i \in V((S_n)_d)} deg_(w_i) \\ &= \frac{1}{2}(n(2n - 1) + n(n + 2)) \\ &= \frac{1}{2}(n(3n + 1)).\end{aligned}$$

□

Recall that the sequence of the second pentagonal numbers is generated by:

$$p_n = \frac{n(3n + 1)}{2}, n = 0, 1, 2, \ldots$$

In expanded form it is:

$$0, 2, 7, 15, 26, 40, 57, 77, 100, 126, 155, 187, \ldots$$

The relation between the size of the deg-centricated sun graphs and the second pentagon numbers follows immediately as a corollary 3.16.

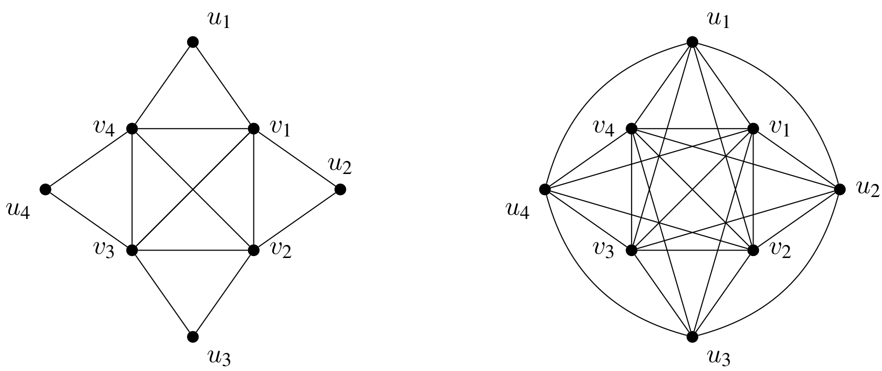
Corollary 3.16. *For $n \geq 3$, $\varepsilon((S_n)_d) = p_n$, $n \geq 3$.*

An illustration of a proposition 3.15 is given in Figure 4.

A *closed sun graph* CS_n, is the graph obtained from adding the edges u_iu_{i+1} in the sun graph. In views of Definition 1.1, and Theorem 1.3, for $n \geq 3$, $\varphi(CS_n) = 1$.

A *tree* denoted by T_n, $n \geq 1$ is a connected acyclic graph. It is known that a tree T_n has $n - 1$ edges.

Proposition 3.17. *For $n \geq 3$, $\varphi(T_n) \geq 2$.*



(a) S_4 (b) $(S_4)_d$

Figure 4: Deg-centric sun graph of S_4.

Proof. For a tree T_n, $n \geq 2$ is of the order n. Let $V(T_n) = \{v_1, v_2, \ldots, v_n, \}$. A tree with at least two vertices has at least two pendant vertices, in view of Definition 1.1, $(T_n)_d$ is not complete; that is, $\varphi(T_n) \geq 2$. □

The *ladder graph*, L_n, $n \geq 1$ is obtained by taking two copies of a path P_n with respective vertices say, $v_1, v_2, v_3, \ldots, v_n$ and $u_1, u_2, u_3, \ldots, u_n$ and adding the edges $v_i u_i$, $1 \leq i \leq n$. Note that $L_n \cong P_n \square K_2$ where $\square$ denotes the Cartesian product.

Proposition 3.18. *For a ladder $G = L_n$, $n \geq 1$ it follows that:*

$$\varepsilon(L_{1_d}) = 1,$$
$$\varepsilon(L_{2_d}) = 6,$$
$$\varepsilon(L_{3_d}) = 13,$$
$$\varepsilon(L_{4_d}) = 24,$$
$$\varepsilon(L_{5_d}) = 37,$$
$$\varepsilon(G_d) = \varepsilon(H_d) + 11 \text{ where } H = L_{n-1} \text{ and } n \geq 6.$$

Proof. By applying Definition 1.1, it easily follows that $\varepsilon(L_{1_d}) = 1$, $\varepsilon(L_{2_d}) = 6$, $\varepsilon(L_{3_d}) = 13$,$\varepsilon(L_{4_d}) = 24$ and $\varepsilon(L_{5_d}) = 37$. Now, besides the claimed result, it is valid that for any $n \geq 6$ and $H = L_{n-1}$ the size of H_d; that is, $\varepsilon(H_d)$ can be determined by applying Definition 1.1. Consider $H = L_{n-1}$ and assume that both H_d and $\varepsilon(H_d)$ has been determined. Now consider the extension from H to $G = L_n$. Some subgraph of H_d is a subgraph of G_d. Note that in G the degree of respectively v_{n-1}, u_{n-1} has increased to 3. Therefore, all edges found in H_d are in G_d, that is $\varepsilon(H) \subset \varepsilon(G)$. With regards to say, v_n the edges which forms are $v_n u_n, v_n u_{n-1}, v_n v_{n-1}, v_n v_{n-2}$. A similar thing can be applied to vertex u_n. The edges which forms are $u_n v_n, u_n v_{n-1}, u_n u_{n-1}, u_n u_{n-2}$. Also, the edges $v_{n-2} u_n, v_{n-3} v_n, u_{n-2} v_n, u_{n-3} u_n$ are formed. Hence,

$$\varepsilon(G_d) = \varepsilon(H_d) + (7 + 4)$$
$$= \varepsilon(H_d) + 11.$$

Finally, since an initial value, $\varepsilon(L_{5_d}) = 37$ is known, the result for $n \geq 6$. □

Observation 3.3. For $n \geq 3$, $\varphi(L_n) \geq 2$.

A *friendship graph*, denoted by $F_n, n \geq 1$, is obtained by joining n copies of the complete graph K_3 with a common vertex. In view of Definition 1.1, and Theorem 1.3,for $n \geq 1$, $\varphi(F_n) = 1$.

A *butterfly graph*, denoted by $BF_n, n \geq 1$, is constructed by joining two copies of the cycle graph C_n with a common vertex v and is therefore isomorphic to the friendship graph F_n. In view of Definition 1.1, and Theorem 1.3,for $n \geq 1$, $\varphi(BF_n) = 1$.

4 Deg-centrication process of Graph Operations

In this section, we discuss certain graph operations. For the elementary graph operation, $G+H$ and G or H is a non-trivial graph, the parameter $\varphi(G+H)=1$. Another elementary graph operation is the disjoint union of graphs denoted by $G\cup H$. Clearly, $\varphi(G\cup H)=max\{\varphi(G),\varphi(H)\}$ (see[4]).

Recall that the *Cartesian product* of two graphs G and H, denoted by $G\Box H$, has the following adjacency rule. The vertices (g,h) and (g',h') have edges between them if $g=g'$ and $hh'\in E(H)$ or $h=h'$ and $gg'\in E(G)$.

Proposition 4.1. *For a non trivial graph G and a graph H, it follows that*

(a) If $G\cong K_2$ and $H\cong K_2$, then $\varphi(G\Box H)=1$.

(b) If $G\cong P_n$, $n\leq 2$ and $H\cong K_2$, then $\varphi(G\Box H)=1$.

(c) If $G\cong P_n$, $n\geq 3$ and $H\cong K_2$, then $\varphi(G\Box H)\geq 2$.

Proof. (a) Since G is nontrivial, it has an order of at least 2. If $G\cong K_2$ and $H\cong K_2$, then $(G\Box H)_d=(C_4)_d$ is a complete graph. Hence, $\varphi(G\Box H)=1$.

(b) Let $G\cong P_n$, $n\leq 2$ and $H\cong K_2$. In view of Definition 1.1, $(G\Box H)_d$ is complete. Hence $\varphi(G\Box H)=1$.

(c) Let $G\cong P_n$, $n\geq 3$ and $H\cong K_2$. Since $G\Box H\cong P_n\Box K_n\cong L_n$ then, by Observation 3.3, $\varphi(L_n)\geq 2$.

□

Proposition 4.2. *For a grid graph $P_m\Box P_n$, m or $n\geq 3$, $\varphi(P_m\Box P_n)\geq 2$.*

Proof. Let G be a grid graph $P_m\Box P_n$, if m or $n\geq 3$. The grid graph is of the order mn. $P_m\Box P_n$, is a bipartite graph, as the cartesian product of two bipartite graphs is bipartite, then, in view of Definition 1.1, G_d is not complete; that is $\varphi(P_m\Box P_n)\geq 2$. □

Proposition 4.3. *For a rook's graph $K_m\Box K_n$, $\varphi(K_m\Box K_n)=1$.*

Proof. Let G be a rook's graph $K_m\Box K_n$, is of order mn, and the diameter is two. Then in view of Theorem 1.3, G_d is complete; that is $\varphi(K_m\Box K_n)=1$. □

The *corona product* of two graphs G and H, denoted by $G\circ H$, is obtained by taking n copies of H, and each vertex in G is adjacent to every vertex of the corresponding H. Every $i-th$ vertex of G is adjacent to each vertex of $i-th$ copy of H, where $1\leq i\leq n$.

Proposition 4.4. *For a nontrivial graph G and a graph H. Then,*

(a) If $G\cong K_n$ and $H\cong K_m$, then $\varphi(G\circ H)=1$.

(b) If $G\cong K_n$, and $H\cong C_m$, then $\varphi(G\circ H)=1$.

(c) If $G\cong K_n$, $n\geq 2$, and $H\cong P_m$, $m\geq 2$, then, $\varphi(G\circ H)\geq 2$

Proof. (a) If $G\cong K_n$ and $H\cong K_m$. Clearly, $\delta(K_n\circ K_m)\geq diam(K_n\circ K_m)$, then in view of Theorem 1.3, $(K_n\circ K_m)_d$ is complete. Hence, $\varphi(G\circ H)=1$.

(b) If $G\cong K_n$ and $H\cong C_m$. Clearly, $\delta(K_n\circ C_m)\geq diam(K_n\circ C_m)$, then in view of Theorem 1.3, $(K_n\circ C_m)_d$ is complete. Hence, $\varphi(G\circ H)=1$.

(c) Let $G\cong K_n$, $n\geq 2$ and $H\cong P_m$, $m\geq 2$. Then, in view of Definition 1.1, $(K_n\circ P_m)_d$ is not complete; that is, $\varphi(K_n\circ P_m)\geq 2$.

□

Proposition 4.5. *For $n,m\geq 4$, $\varphi(C_n\circ C_m)\geq 2$.*

Proof. Let $G = C_n \circ C_m$, if $n, m \geq 4$. Then, in view of Definition 1.1, $(C_n \circ C_m)_d$ is not complete; that is, $\varphi(C_n \circ C_m) \geq 2$. □

Proposition 4.6. *For $n, m \geq 2$, $\varphi(P_n \circ P_m) \geq 2$.*

Proof. In view of Definition 1.1, $(P_n \circ P_m)_d$ is not complete; that is, $\varphi(P_n \circ P_m) \geq 2$. □

Proposition 4.7. *For $n \geq 3$, $\varphi(C_n \circ K_1) \geq 2$.*

Proof. In view of proposition 3.11 and corollary 3.12, $(C_n \circ K_1)_d$ is not complete; that is, $\varphi(C_n \circ K_1) \geq 2$. □

The *direct product* or tensor product of two graphs, two vertices (g, h) and (g', h') are adjacent if both $gg' \in E(G)$ and $hh' \in E(H)$. The direct product of G and H is denoted by $G \times H$.

Proposition 4.8. *If G and H are connected graphs with no odd cycles then $(G \times H)_d$ has exactly two components.*

Proof. Let two connected graphs G and H with no odd cycles. Clearly, the product $G \times H$ is connected if and only if either G or H contains an odd cycle(see[14]). Then it is clear that if G and H with no odd cycles then the direct product has exactly two components. If $G \times H$ has two components, it implies $(G \times H)_d$ has exactly two components. □

Proposition 4.9. *If $G \cong K_2$ and $H \cong K_n$, $n \geq 4$. then $\varphi(G \times H) = 1$.*

Proof. If $G \cong K_2$ and $H \cong K_n$, $n \geq 4$. In view of Theorem 1.3, $(G \times H)_d$ is complete; that is $\varphi(G \times H) = 1$. □

5 Conclusion

The graph transformation called deg-centrication has been discussed. Various exploratory results have been presented to establish some foundation for further research. As a scope of the study, the researchers can extend the study on graph theoretical parameters to deg-centric graphs of various class graphs and obtain fruitful results. Also, the researchers can extend the study to various types of domination, coloring, labelling, and decomposition to deg-centric graphs.

References

[1] J. Akiyama, K. Ando and D. Avis, *Eccentric graphs*, Discrete Math., **56** , 1-6, (1985).

[2] M.R Raja, T.A Mangam and S. Naduvath, *Eccentric completion of a graph*, Commun. Comb. Optim., **7**, 193–201,(2022).

[3] M.R Raja, J. Kok, T.A Mangam and S. Naduvath, *Cyclic property of iterative eccentrication of a grap*. Discrete Math. Algorithm. Appl., **15(07)**, 2250155, (2022).

[4] T.T Thalavayalil, J. Kok, and S. Naduvath, *A study on deg-centric graphs, Proyecciones.,***43(02)***, 911-926, (2024).*

[5] T.T Thalavayalil, J. Kok, and S. Naduvath, A study on exact deg-centric graphs*, J. Interconn. Networks., 2450004 ,(2024).*

[6] T.T Thalavayalil, and S. Naduvath, A study on coarse deg-centric graphs*, Gulf. J. Math.,* **16(2)***, 171-182,(2024).*

[7] Joseph, Anu and Dominic, Charles, Vertex induced 2-edge coloring and vertex incident 2-edge coloring of some graph products *, Palest. J . Math.,* **11(3)** *86-95,(2022).*

[8] Rajkumar, R and Devi, P, Permutability graph of cyclic subgraphs *, Palest. J. Math.,* **5(2)***,(2016).*

[9] T.T Thalavayalil, J. Kok, and S. Naduvath, A study on upper deg-centric graphs*, TWMS J. Appl. Eng. Math., to appear, (2024).*

[10] T.T Thalavayalil, and S. Naduvath, A study on lower deg-centric graphs*, Communicated, (2024).*

[11] D.B West, Introduction to graph theory*, Prentice Hall of India, New Delhi, (2001).*

[12] Sarkar, Ishita and Manjunath, N and Chaluvaraju, B, Bounds related to product variants of graphs *, Palest. J. Math.,* **12(2)***,(2023).*

[13] A. Brandstädt, V.B Le and J.P Spinrad, Graph classes: a survey*, SIAM. , (1999).*

[14] Weichsel, Paul M, The Kronecker product of graphs*, Proc. Amer. Math. Soci.,* **13(1)** *47-52 ,(1962).*

Author information

Timmy Tomy Thalavayalil, Department of Mathematics, Christ University, Bangalore-560029, India.
E-mail: `timmy.thalavayalil@res.christuniversity.in`

Sudev Naduvath, Department of Mathematics, Christ University, Bangalore-560029, India.
E-mail: `sudev.nk@christuniversity.in`