Palestine Journal of Mathematics

Vol 14(Special Issue I)(2025) , 160–167

OPTIMIZING ALLELOPATHIC INTERACTIONS: STABILITY ANALYSIS WITH TIME DELAY IN PLANT POPULATIONS

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Keywords: Plant population, Allelopathy, Delay, Hopf-Bifurcation, Stability.

Abstract A delay-differential mathematical model involving two competing plant populations is proposed to investigate the effect of allelochemicals on one another. The time delay is introduced in one plant population to describe the allelopathic effect between plant populations. The influence of time delay on the stability of the dynamics system has been investigated. The Equilibrium point is determined and the stability analysis is done at a point of non-zero equilibrium using RouthHurwitz's theorem. Delay disrupted the equilibrium of the system's internal balance and when the value of delay $\tau < 1.99$ system shows asymptotically stable or when the value of delay $\tau \geq 1.99$ the hopf bifurcation occurs. The influence of allelochemicals may be monitored and plant development improved using the intended mathematical model. MATLAB software is used to verify the numerical results.

1 Introduction

Allelopathy, a term introduced by Molisch in 1937, originates from the Greek words "allelon" (meaning "mutual") and "pathos" (meaning "suffering" or "feeling"). It describes the phenomenon where one plant affects another through the release of allelochemicals. Alveologenesis, synonymous with allelopathy, involves a combination of both biotic and abiotic stressors imposed by one species onto another. Allelopathy is an organic substance that interferes with negative resistance among plants or microbes through explicit or implicit effects [1]. The rise in the population of one species can affect the development of other specie by realizing harmful allelochemicals. Monaco et al. observed allelopathy is a dynamic occurrence involving a multitude of environmental and physiological systems that are interconnected [2]. The measurement of phytotoxicity for the first phase in the identification of allelochemicals is possible, but it is not enough for confirmation of an allelopathic interaction [3]. Amb and Ahluwalia observed that many rice kinds produce high antimicrobial allelochemicals that have an impact on major herbicides and fungal and bacterial diversity in the rice plant's environment or even soil types. These allelopathic interactions have been proven to be beneficial, and they may thus be exploited as a valuable component of an environmental & agricultural system [4].

Rice allelopathic ability is found in almost all plant components, including roots, varying forms and varieties of shells, and along. But a key drawback or restriction found in comparable research is that the quantity of allelopathic chemicals existing in the extracting or residual combination might be significantly higher than those in the new area in the production [5-6]. Naresh et al. investigated the impact of toxic effect on plant growth across time [7]. Whenever dealing with group or consecutive production of a range of plants, allelopathy may also be regarded as a significant lead number crops [8]. Dipesh and Kumar investigated non-linear delay differential equation system to investigate the allelopathic effect using delay parameter on plant growth dynamics [9-13].

The effects of environmental factors on plant growth are significant and multifaceted. Scientists are closely observing and working hard to grasp how plants respond to different environmental changes, while also formulating plans to mitigate their detrimental impacts, particularly those caused by abiotic stressors [14]. The mechanism of the exponentially characteristic equation's zeros was investigated in detail [15]. To further identify the basic biological processes, Wu et al. designed a framework for sensitivity analysis of functional structural plant models in his research work [16]. Ruan and Rihan integrated a time delay of some type in biological models by numerous researchers working with the dynamics of single-species populations [17-18].

Climate change's variable climatic stress circumstances contribute significantly to public health, crop production, and agricultural sustainability. Rising increasing population and ecosystem destruction demand the creation of crops that can respond to extremes in the environment [19]. Yuan and Cheng explore the presence and asymptotically stability of non-local evolutionary equations of trivial facets with a single hypothesis [20]. Felpeto et al. observed that allelopathy has the potential to reduce exclusion due to competitiveness and increase phytoplankton variety in aqueous conditions characterized by intensive rivalry among diverse species for limited resources [21]. Hayyat et al. studied the allelopathic effect of herbicides blooming in deserts on crops using residues from the foliage and soils in their rhizospheres. Furthermore, they discovered that the presence of phenolic chemicals in the soil hindered plant root development more than mycorrhizal conditions' reduction in shoot formation [22].

Mathematical models in dynamics may be used to characterize the soot-root interface and asset allocation that significant role in plant development [24]. Four tree species were examined for their allelochemical effects on herbaceous plants and soil microbial communities [25]. Mira et al. studied the germination and growth of inexpensive weeds under the effect of allelochemicals of Pteridium aquilinum, which gives inhibitory effects on other weeds [25].

We assert that all techniques employed in the modeling process adhere to moderate and low levels, ensuring consistent responses to initial constraints across significant timeframes. The use of delayed differential equations in the study of allelopathy within competitive plant populations has been unprecedented until now. This analysis focuses on scenarios where one population releases allelochemicals, affecting another population that does not produce such chemicals, potentially leading to the demise of the affected population.

2 Mathematical Model

Let us assume an ecosystem in which competition dynamics interact at two different plant populations in the closed plane with an even boundary. We study the delay in competing plant population model. When one plant population produces an allelochemical affecting the other plant population and the concentration of allelochemicals is proportional to species density.

Where $P_1 \& P_2$ are the competiting plant populations.

$$\frac{dP_1}{dt} = a_1 P_1 \left(1 - \frac{P_1}{k}\right) - \alpha_1 P_1 P_2 \tag{1}$$

$$\frac{dP_2}{dt} = a_2 P_2 \left(1 - h\frac{P_2(t-\tau)}{P_1}\right) \tag{2}$$

Where $P_1(0) > 0, P_2(0) > 0 \forall t$ and $P_2(t-\tau) = \text{constant } t \epsilon [0, \tau]$.
$a_1 \& a_2$ are the growth rate of plant population 1st and 2nd respectively. k is the carrying capacity. α_1 is the mutual competition between plant population $P_1 \& P_2$ resp. $h\frac{P_2(t-\tau)}{P_1}$, this clearly indicates that the carrying capacity of P_2 plant population is directly proportional to P_1 plant population.

3 Equilibrium Points

At the steady state $P_2(t-\tau) = P_2$. The steady state of non-zero equilibrium points is E_1 (P_1^*, P_2^*):
$\frac{dP_1^*}{dt} = 0 \Rightarrow a_1 P_1^* \left(1 - \frac{P_1^*}{k}\right) - \alpha_1 P_1^* P_2^* = 0$
We get $P_1^* = \frac{a_1 k - \alpha_1 P_2^* k}{2a_1}$
& Similarly, $P_2^* = \frac{P_1^*}{2h}$ and the put the value of P_1^* in (3) we get, $P_2^* = \frac{a_1 k}{4a_1 h + \alpha_1 k}$

4 Biological Significance

$\mathbf{E}_1\left(\mathbf{P}_1^*, \mathbf{P}_2^*\right)$ and Hopf-bifurcation: At equilibrium, the stability of system (1)-(2) equations determining the plant population competition process

$$\frac{dP_1^*}{dt} = a_1 P_1^* \left(1 - \frac{P_1^*}{k}\right) - \alpha_1 P_1^* P_2^* \tag{4}$$

$$\frac{dP_2^*}{dt} = a_2 P_2^* \left(1 - h\frac{P_2^*(t-\tau)}{P_1^*}\right) \tag{5}$$

The characteristic equation for the mathematical model (4)-(5) is given by:
$\lambda^2 + a_1\lambda + b_1 + e^{-\lambda\tau}\left(a_2\lambda + b_2\right) = 0$
Where $a_1 = 2a_1\frac{P_1}{k} - a_2 - a_1 + \alpha_1 P_2, a_2 = \frac{h}{P_1}, b_1 = a_1a_2 - 2a_1a_2\frac{P_1}{k} - \alpha_1 a_2 P_2,$

$$b_2 = 2a_1\frac{h}{k} + \alpha_1\frac{hP_2}{P_1} + \alpha_1 a_2\frac{h}{P_1} - a_1\frac{h}{P_1}$$

Clearly a_1, a_2, b_1, b_2 all are positive.
When $\tau = 0$, the equation (6) becomes:
$\lambda^2 + \left(a_1 + a_2\right)\lambda + \left(b_1 + b_2\right) = 0$
Applying the Routh-Hurwitz criteria, Equation (7) have negative real component, which means stability will be formed in the system if:
$\left(X_1\right) : \left(a_1 + a_2\right) > 0$
$\left(X_2\right) : \left(b_1 + b_2\right) > 0$
Which implies $\left(X_1\right), \left(X_2\right)$ are correct.
Now, we'll explore how the roots' negative real aspect shifts towards the positive real aspect with varying values of τ.

Suppose $\lambda = i\theta$ be the root of Equation of (6), then Equation (6) become:

$$\begin{aligned}&(i\theta)^2 + a_1(i\theta) + b_1 + e^{-(i\theta)\tau}\left(a_2(i\theta) + b_2\right) = 0\\ &\quad\Rightarrow -\theta^2 + ia_1\theta + b_1 + i\theta a_2\cos(\theta\tau) + b_2\cos(\theta\tau) + \theta a_2\sin(\theta\tau) - i\theta b_2\sin(\theta\tau) = 0\end{aligned}$$

Separate real and imaginary parts we get:
$\theta^2 - b_1 = b_2\cos(\theta\tau) + \theta a_2\sin(\theta\tau)$
$a_1\theta = b_2\sin(\theta\tau) - \theta a_2\cos(\theta\tau)$
Squaring and adding (8)-(9) we get
$\theta^4 + \left(a_1^2 - a_2^2 - 2b_1\right)\theta^2 + \left(b_1^2 - b_2^2\right) = 0$
And the two roots of equation (10) are
$\theta_{1,2}^2 = \frac{\left(a_2^2+2b_1-a_1^2\right)\pm\sqrt{\left(a_1^2-a_2^2-2b_1\right)^2-4\left(b_1^2-b_2^2\right)}}{2}$

None of the two roots $\theta_{1,2}^2$ is positive if:

$$\left(X_3\right) : \left(a_2^2 + 2b_1 - a_1^2\right) < 0 \text{ and } \left(b_1^2 - b_2^2\right) > 0 \text{ or } \left(a_1^2 - a_2^2 - 2b_1\right)^{s2} < 4\left(b_1^2 - b_2^2\right)$$

That's mean Equation (11) have no positive root if constrain $\left(X_3\right)$ holds. From lemma [17].
Lemma 1: If $\left(X_1\right) - \left(X_2\right)$ hold, roots of Equation (6) are negative $\forall\tau \geq 0$.
On the contrary, if:
$\left(X_4\right) : \left(b_1^2 - b_2^2\right) < 0$ or $\left(a_2^2 + 2b_1 - a_1^2\right) > 0$ and $\left(a_1^2 - a_2^2 - 2b_1\right)^2 = 4\left(b_1^2 - b_2^2\right)$
Then positive root of Equation (8) is θ_1^2.
$\left(X_5\right) : \left(b_1^2 - b_2^2\right) > 0$ or $\left(a_2^2 + 2b_1 - a_1^2\right) > 0$ and $\left(a_1^2 - a_2^2 - 2b_1\right)^2 > 4\left(b_1^2 - b_2^2\right)$
Then, two positive roots of equation (8) are $\theta_{1,2}^2$.
In both $\left(X_4\right)$ and $\left(X_5\right)$, for different value of time lags, Equation (6) has imaginary root. The threshold value $\tau_j^{\pm}$of τ can be examined by using the Equation (8) - (9),

$$\tau_l^{\pm} = \frac{1}{\theta_{1,2}}\cos^{-1}\frac{\left(\theta_{1,2}^2 - b_1\right)}{a_2\lambda + b_2} + \frac{2l\pi}{\theta_{1,2}}, l = 0, 1, 2, \ldots \tag{12}$$

The preceding material can be summarized in the following lemma [17].
Lemma 2: (i) If $(X_1)-(X_2)$ and (X_4) is true and $\tau=\tau_l^+$, Equation (6) possess set of complex root $\pm i\theta_1$.
(ii) If $(X_1)-(X_2)$ and (X_5) is true and $\tau=\tau_l^-$ ($\tau=\tau_l^+$respectively), then Equation (6) will give a set of imaginary roots $\pm i\theta_2$ $(\pm i\theta_1)$ respectively.

Our proposition suggests that the roots of certain equations, initially characterized by a negative real component, will transition to a positive real component when $\tau>\tau_l^+\&\tau<\tau_l^+$. Let us have a look at this possibility:

$$\tau_l^{\pm}=\mu_l^{\pm}(\tau)+i\theta_l^{\pm}(\tau);l=0,1,2,3\ldots$$

From Equation (6) fulfil. $\mu_l^{\pm}\left(\tau_l^{\pm}\right)=0,\theta_l^{\pm}\left(\tau_l^{\pm}\right)=\theta_{1,2}$
The initial boundary condition can be verified.

$$\frac{d}{d\tau}\left(\operatorname{Re}\lambda_l^+\left(\tau_l^+\right)\right)>0 \text{ and } \frac{d}{d\tau}\left(\operatorname{Re}\lambda_l^-\left(\tau_l^-\right)\right)<0$$

It implies that τ_l^+represents the bifurcation points. The distribution of zeros of equation (6) is determined by the following hypothesis [18].
Theorem: Suppose $\tau_l^+(l=0,1,2,3\ldots)$ calculated by Equation (12).
(1) If $(X_1),(X_2)$ are satisfied then equation (6) has negative real part $\forall\tau\geq 0$.
(2) If $(X_1),(X_2)$ and (X_4) are satisfied and when $\tau\epsilon\left[0,\tau_0^+\right)$, then equation (6) have negative real parts. When $\tau=\tau_0^+$, then equation (6) possess a pair of purely imaginary roots $\pm i\theta_1$. When $\tau>\tau_0^+$, (6) has at least one positive real part root.
(3) If $(X_1),(X_2)$ and (X_5) are satisfied, then positive integer n such that $0<\tau_0^+<\tau_0^-<\tau_1^+<\tau_1^+\ldots\ldots\ldots<\tau_{n-1}^-<\tau_n^+$and there are n number of fluctuation from steadiness to unsteadiness. this implies, when $\tau\epsilon\left[0,\tau_0^+\right),\left(\tau_0^-,\tau_1^+\right)\ldots\ldots\ldots\left(\tau_{n-1}^+,\tau_n^+\right)$then Equation (6) have all negative real root. When $\tau\in\left[0,\tau_0^+\right),\left(\tau_0^-,\tau_1^+\right)\ldots\ldots\ldots\left(\tau_{n-1}^+,\tau_{n-1}^-\right)$and $\tau>\tau_n^+$, Equation (6) contain at least one positive root.

Numerical Example: The complexities indicated by the mathematical model are visually represented using the following set of parametric values.

$$a_1=2,\alpha_1=0.2,k=60,a_2=50,h=2$$

For various values of the delay parameter, the nature of the dynamics system (1)-(2) changes from steady to complicated behavior around the equilibrium $E_1\left(P_1^*,P_2^*\right)$.

5 Results and Discussion

Figure1 represent the graph between plant population $P_1\&P_2$ time (t) respectively. It has been observed that at $a_1=2,\alpha_1=0.2,k=60,a_2=50,h=2$ both plant populations ($P_1\&P_2$) are stable in the absence of delay $(\tau=0)$. Bandyopadhyay [26] have used two competitive phytoplankton model to study the influence of toxic metal on the phytoplankton. By taking parametric values at $\alpha_1=2,\alpha_2=1,\beta_1=0.02,\beta_2=0.08,v_1=0.05,v_2=0.015,\gamma=0.0008$ he found stability in the system. Figure2 represent the graph between the plant population $P_1\&P_2$ time (t) respectively. It has been found that both plant population losses their stability and shifted toward asymptotically stability at $\tau<1.99$. Wang et al. [27] worked on phytoplankton-zooplankton model in order to check the effect of toxic metal using time lag. They have been examined that phytoplankton-zooplankton shows asymptotically stability at $\tau<4.5$. Gumus [28] worked on discrete time prey-predator model in order to check the allee effect on the prey population. He found that system show bifurcation with variation in parametric values. Murhy et al. [29] word on prey-predator model to verify the allelopathic effect on growth rates predation. They observed that system is shifting toward Hopf-bifurcation after losing the asymptotically stability with parametric value $a_1=0.1,a_2=0.9,\eta_1=0.008,\eta_2=0.005,c_1=0.1,c_2=0.2,H_1=30,H_2=40,K=500,L=400$ at $\tau\geq 0.36$.

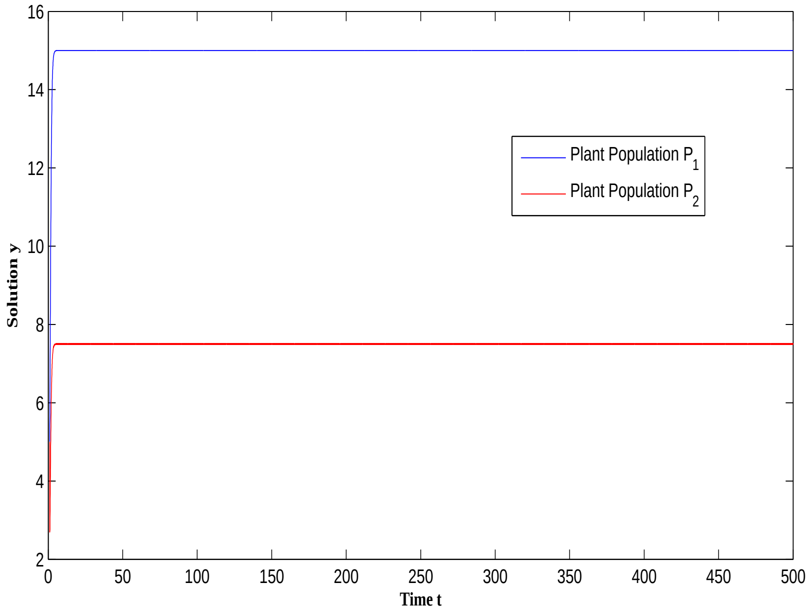

Figure 1. At $\tau = 0$, dynamic system is stable

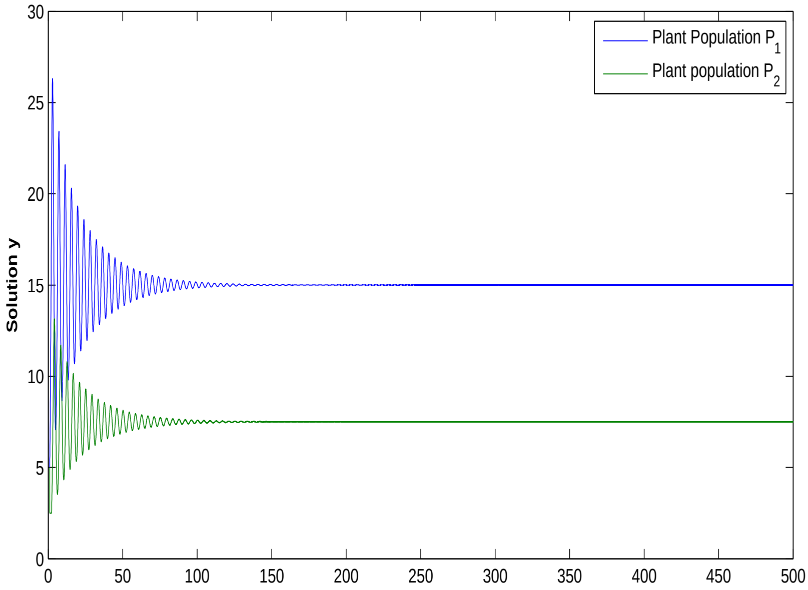

Figure 2. At $\tau < 1.99$, the plant population P_1 and P_2 is stable asymptotically.

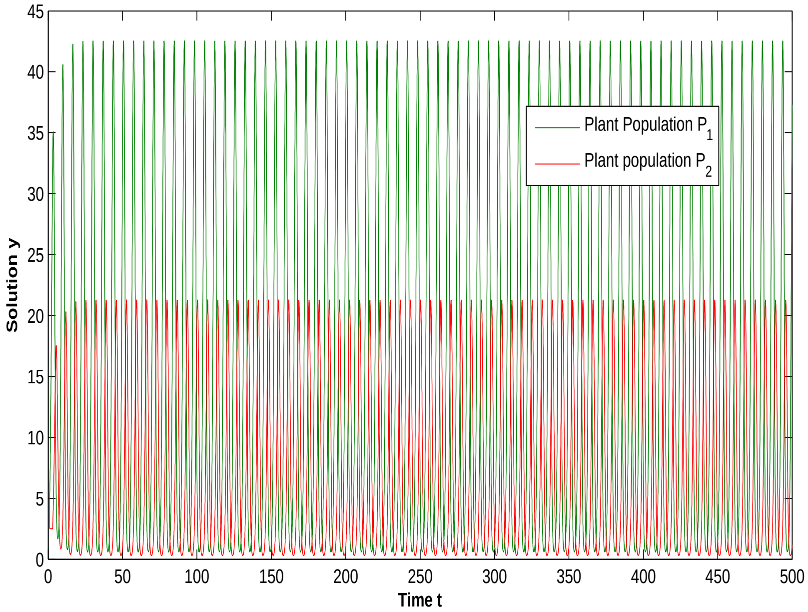

Figure 3. At $\tau \geq 1.99$, the plant population P_1 and P_2 loses stability and shows Hopf-bifurcation.

6 Conclusion

Study of the importance of time lags in plant population competing underneath the consequence of allelochemicals by using the proposed model has been done. By introducing delay in the model, imbalance in the system was found. Also, delay shows complex behavior along with stable periodic solution and limit cycle through hopf-bifurcation. Stability of equilibrium $\mathrm{E}^* (\mathrm{P}_1, \mathrm{P}_2)$ point is calculated. With the help of Routh-Hurwitz's theorem we calculate stability of the dynamic system. In the absence of delay none of plant populations affect adversely each other and grow at their normal rate. When the value of delay is less than the critical value system shows asymptotically stable. Actually, mean if there is a delay involved in the allelochemicals realize, still the system grows at their natural rate after few fluctuations in the beginning under the asymptotically stability. When the value of delay is greater than or equal to the critical point system loses its asymptotically stability and both the populations remain under the effect of allelochemicals forever. A sessional repetition of limit cycles will always occur after a particular time period showing Hopf-bifurcation. This all-graphical work done with matlab software using dde23 command.

Conflict of interest

The author declares no conflict of interest.

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