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Numerical Simulation of Nonlinear SG Equation via DQM and Significant Applications in Josephson Junctions

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Abstract Partial differential equations (PDEs) serve as powerful tools for simulating and mathematically modeling a diverse range of phenomena, including elastic behavior, fluid flows, shallow water waves, epidemiology, biostatistics, gene mutation, and flow turbulence. It is not always feasible to solve these problems analytically when they are mathematically described. Therefore, it could be difficult for researchers to find analytical or exact solutions to these differential equations. Due to the complexity of these differential equations, there are several advanced numerical methods that may be used to solve them. Among these methods, the differential quadrature method (DQM) stands out as a pivotal technique, demonstrating exceptional effectiveness in obtaining numerical solutions for such differential equations. In this study, nonlinear Sine-Gordon (SG) equation is numerically solved using the exponential cubic B-spline differential quadrature method. The accuracy and efficiency of this method are shown by the findings, which are equivalent to those reported in the literature and close to exact solution. The results, presented as figures and tables, are deemed positive. Additionally, this paper provides an insightful discussion on the significant application of the SG equation in Josephson junctions.

1 Introduction

Partial differential equations (PDEs) plays an important role in understanding phenomena in science and engineering. Researchers seek solutions to modeled PDEs through various analytical and numerical approaches. However, it is not always feasible to solve the partial differential equations using available analytical techniques to provide an exact solution. This is where numerical approaches come into play, helping solve PDEs with the required accuracy. With the advancement of technology, various options are available for obtaining numerical solutions to differential equations using software such as MATLAB, Maple, and Mathematica. Researchers continually work on modifying numerical schemes to provide solutions efficiently and accurately. This paper presents one such attempt to modify a numerical technique. The work introduces a scheme developed to find more accurate solutions for a well-known differential equation using the exponential modified cubic B-spline basis function with the differential quadrature method (DQM). DQM introduced by Richard Bellman in the 1970s [1, 2], is a highly efficient numerical technique for solving differential equations, especially PDEs. Due to its efficiency and accuracy, DQM has found widespread use in solving problems related to beam and plate vibrations, fluid flow, heat transfer, and other engineering applications [3, 4]. Its ability to handle complex boundary conditions and provide accurate results with fewer computational resources makes it a preferred method in many applications. Quan, Chang, and Shu [5] have made significant contributions to the development and refinement of the DQM, enhancing its accuracy, applicability, and efficiency in solving various engineering and scientific problems. In the early stages of DQM, Quan and Chang [6] worked extensively on improving the weighting coefficients used in the method. Their work was particularly focused on enhancing the stability and precision of DQM when applied to complex boundary conditions. They also extended the applicability of DQM to a wider range of differential equations, including those with non-linear terms. Their improvements helped in making the method more robust and reliable, particularly for structural and mechanical engineering problems. Shu further extended the work by systematically developing the generalized DQM, which is an extension of the original DQM. Shu's work focused on the application of DQM to various fields, such as fluid mechanics, vibration analysis, and thermal problems. He contributed to making DQM more versatile by improving the technique's accuracy for higher-order derivatives and irregular grids. Shu's contributions helped establish DQM as a powerful tool in both academic research and industrial applications, enabling the method to be used in the analysis of more complex systems. Together, their contributions have significantly advanced the practical utility of DQM in solving complex partial differential equations with high efficiency and precision [7, 8, 9, 10]. DQM has also been applied with various basis functions, including exponential cubic B-splines to solve complex equations like the Burgers equation, convection-diffusion equations, and the nonlinear Schrödinger equation, among others [11, 12, 13, 14, 15, 16]. These advancements highlight DQM's broad applicability and efficiency in solving challenging PDEs across different scientific and engineering domains. This demonstrates the broad applicability of DQM with different spline-based basis functions in tackling complex PDEs across a wide range of fields.

1.1 Sine-Gordon (SG) equation

The SG equation, a second-order partial differential equation, is commonly used across various scientific and technical domains, including mechanical transmission systems, Josephson junctions, and magnetic crystals. A significant characteristic of the SG equation is that its numerical solutions reveal soliton behavior—localized waveforms that maintain their shape during transmission [17]. It is also regarded as an extension of traditional Maxwell systems in optics, providing improved insight into light behavior [18]. Additionally, the SG equation is often employed in geometrical analyses of solitons within canonical field theory, linking soliton velocity with black hole temperature in the literature [19]. Another critical application of the SG equation is in modeling fault dynamics in phenomena like strain waves and earthquakes [20]. This highlights its relevance in understanding seismic distortions in the Earth's crust and provides insight into the origin of natural defects. The kink-like form of the SG soliton solution is particularly useful for exploring the underlying mechanics of these processes. One of the SG equation's most fascinating properties is that it supports soliton solutions, which act like particles due to their localized, wave-like structure [21]. These solitons play a crucial role in various scientific and engineering domains, particularly in Josephson junctions, a key application discussed in this paper.

Applications of Solitons in Josephson Junctions

Solitons are waves that preserve their physical characteristics and velocity as they propagate through a medium, and they have discovered applications across multiple areas of physics, such as in the study of Josephson junctions. Josephson junctions are components made up of two superconductors with a thin insulating barrier in between, allowing the flow of a supercurrent. These solitons are essential for various applications, including advanced electronics, quantum computing, and voltage standards. Below are some key applications of solitons in Josephson junctions [21, 22, 23]:
Voltage Standards: Josephson junctions are crucial in the development of precise voltage standards, which are essential for calibrating voltage-measuring devices across various industries like telecommunications, power grid management, and medical equipment. When exposed to a microwave frequency, a Josephson junction generates a voltage proportional to the frequency, known as the Josephson voltage. This highly stable and predictable voltage is used to create a Josephson Voltage Standard (JVS), which provides a reliable reference voltage for calibration and other purposes.
Voltage-Controlled Soliton Oscillators: Solitons in Josephson junctions can be used to generate stable microwave signals. By applying an external voltage, these solitons can oscillate at specific frequencies, enabling the development of highly coherent oscillators.

Digital Signal Processing: Josephson junctions allow for extremely fast digital signal processing, which is particularly useful in applications like high-speed data transmission, radar, and image processing.
High-Speed Communication: Since solitons can travel long distances without distortion, they are ideal for transmitting data efficiently through Josephson junctions, offering significant advantages in high-speed communication technologies.
Magnetic Flux Quantization: Solitons also influence magnetic flux quantization in Josephson junctions. As a soliton moves through the junction, it induces a phase shift in the superconducting order parameter, resulting in discrete quantized magnetic flux.
Quantum Computing: In this realm, solitons within Josephson junctions are being explored as potential components for qubits. By controlling solitons with external voltages, they could serve as building blocks for quantum information processing systems.
Superconductivity Research: Devices based on Josephson junctions are essential for studying high-temperature superconducting materials and phase transitions. These junctions help researchers understand the properties of layered superconductors, contributing to advancements in superconducting technology.

Role of Josephson Junctions in Emerging Technologies

Josephson junctions play an increasingly significant role in advancing technology that impacts everyday life. Below are some areas where Josephson junctions are making notable contributions [24, 25]:
Enhanced Efficiency in Electronics: Josephson junctions enable the creation of low-power, high-speed electronics, improving energy efficiency and extending the battery life of modern devices.
Improved Communication: By producing stable and accurate microwave signals, Josephson junctions contribute to faster, more reliable communication systems, resulting in better data transfer rates and connectivity.
Medical Imaging and Diagnostics: SQUIDs (Superconducting Quantum Interference Devices) are employed in medical imaging technologies such as MRI, providing detailed internal images and improving diagnostic accuracy.
Quantum Computing Innovations: Josephson junctions are fundamental components of superconducting qubits, which hold promise for solving complex problems in fields like material science, cryptography, and drug discovery through quantum computing.
High-Speed Data Processing: Josephson junctions are used in devices that facilitate ultrafast data processing, benefiting applications such as video and image processing, ultimately leading to more responsive technologies and enhanced productivity.
Josephson junctions continue to drive advancements in various technologies, from improving electronic efficiency to fostering progress in quantum computing and medical diagnostics.

Advantages and Disadvantages of Solitons in Josephson Junctions

Advantages: Stability: Solitons in Josephson junctions are highly stable and can persist without dissipating, making them valuable for applications in quantum computing and secure communication.
Nonlinear Properties: The nonlinear behavior of solitons allows them to interact in unique ways, offering possibilities for signal modulation and information processing.
High-Speed Transmission: Solitons can propagate through Josephson junctions at high speeds without losing shape, which is advantageous for data transmission in electronics.
Disadvantages: Challenging Generation: Creating solitons in Josephson junctions requires precise control over parameters such as voltage and temperature, which can be difficult and time-consuming.
Sensitivity to Noise: Solitons are susceptible to noise and environmental disturbances, which can degrade their signal quality or cause them to dissipate.
Limited Commercial Applications: Although solitons in Josephson junctions show great promise, their use remains confined to certain research areas, with limited commercial deployment so far,

especially in fields like quantum computing and high-speed data transmission.
Overall, solitons in Josephson junctions offer exciting potential across multiple areas, but challenges related to their generation and stability must be addressed for broader practical applications.
The SG equation is given by:

$$u_{tt} + \alpha u_t = \beta u_{xx} + \eta(x)\sin(x) \tag{1.1}$$

initial conditions:

$$u(x,0) = \phi_1(x) and u_t(x,0) = \phi_2(x)$$

and values defined at the boundaries. where $u = u(x,t)$ is a function of space x and time t. Due to its rich mathematical structure, the SG equation is also widely studied for its soliton solutions, integrability, and connections to other nonlinear models.
Applications:

- Josephson junctions in superconductivity, where it models the dynamics of magnetic flux quanta.
- Nonlinear optics, where it describes the propagation of light in certain materials.
- Field theory in physics, particularly in models involving scalar fields and kink solutions in particle theory.
- Mechanical systems such as the pendulum chain, where it governs the motion of coupled pendulums under certain conditions.

Different analytical and numerical approaches have been used to solve the Sine-Gordon (SG) problem and obtain its soliton solutions. These methods include the homotopy analysis method, modified cubic B-spline collocation, Legendre spectral element method, and cubic B-spline DQM, among others [26]- [38]. In this study, the focus is on applying the exponential modified cubic B-spline differential quadrature method (Expo-MCB-DQM) to numerically solve the SG problem, a nonlinear PDE. The structure of this paper is as follows: The organisation of this paper is as follows: Section 2 presents the numerical approach for constructing exponential cubic B-splines using the differential quadrature method. In Section 3, applies this method to solve numerical SG equation problems to assess its accuracy and effectiveness. Lastly, Section 4 discusses the results and key conclusions.

1.2 Numerical Scheme

Expo-MCB-DQM

DQM is a numerical technique developed to approximate the derivatives of a function by expressing them as weighted sums of function values at discrete points within a given domain. The origins of DQM date back to the late 1970s, when it was first introduced by Richard Bellman and his colleagues. It was inspired by the concept of quadrature, which involves approximating integrals using weighted sums, and extended this idea to the calculation of derivatives. Bellman, a mathematician known for his work in dynamic programming, saw the potential in using quadrature-like techniques to solve differential equations efficiently. DQM is based on the premise that the derivative at any given point in a domain can be represented as a linear combination of the function's values at all grid points within that domain. The accuracy of the method depends on determining appropriate weighting coefficients, which are influenced by the spatial distribution of the grid points [39]. These coefficients allow DQM to achieve high accuracy, even with a relatively small number of points, making it computationally efficient compared to traditional methods. Over the years, DQM has been refined and extended to handle various types of boundary conditions and more complex problems [40]. Its ability to produce accurate results with fewer grid points than methods like finite difference or finite element methods has made it a popular choice for problems requiring both precision and computational efficiency [41]. Here's a step-by-step explanation of how the DQM is applied using exponential cubic B-splines to approximate derivatives and solve differential equations [42]

Step 1: Domain Discretization

The solution domain $[a, b]$ is divided into a series of discrete grid points, denoted as $a = x_1 < x_2 < \cdots < x_N = b$. These grid points serve as the locations where the function and its derivatives will be approximated.

Step 2: Function Representation

Assuming the function $u(x)$ is sufficiently smooth over the domain, the derivative of the function at each grid point x_i is approximated as a linear combination of function values at all grid points. This is done using the following general DQM formulation:

$$\left.\frac{d^{(r)}u}{dx^{(r)}}\right|_{x_i} = \sum_{j=1}^{N} p_{i,j}^{(r)} u(x_j), \quad i = 1, 2, \ldots, N, \quad r = 1, 2, \ldots, N-1 \tag{1.2}$$

Here, $p_{i,j}^{(r)}$ are the weighting coefficients that need to be determined, r is the order of the derivative, and N is the number of grid points.

Step 3: Choosing Basis Functions

In this method, third-degree exponential cubic B-splines are selected as the basis functions to determine the weighting coefficients. These spline functions offer smooth approximations and are especially effective for complex boundary conditions.

Step 4: Constructing Exponential B-Spline Functions

The exponential cubic B-spline functions $C_m(x)$ are constructed over sub-intervals of the domain. These splines are defined piecewise, with specific expressions for different segments of the interval, ensuring smoothness and continuity across the domain.

Step 5: Computing Spline Derivatives

The values and derivatives of the spline functions at each grid point x_i are computed. This helps in approximating the function's derivative at these points, which will be used in the DQM formulation.

Step 6: Modifying Spline Functions

To eliminate extra points introduced by the spline formulation, modified versions of the exponential cubic B-splines are used. These modified functions $G_k(x)$ are calculated at the grid points, ensuring that the total number of points remains manageable.

Step 7: Approximating First-Order Derivatives

With the modified splines in place, the first-order derivative at each point is approximated using the DQM equation:

$$G'_k(x_i) = \sum_{j=1}^{N} p_{i,j}^{(1)} G_k(x_j), \quad \text{for } i = 1, 2, \ldots, N, \quad k = 1, 2, \ldots, N. \tag{1.3}$$

Step 8: Solving System of Equations

The resulting system of equations, which approximates the spatial derivatives, is solved using MATLAB or another computational tool. The goal is to find the weighting coefficients $p_{i,j}^{(r)}$, which are essential for the numerical solution.

Step 9: Converting to Ordinary Differential Equations (ODEs)

Once the DQM-based approximations are in place, the spatial derivatives are replaced with the corresponding expressions using the spline basis functions. This transforms the original PDE into an ODE.

Step 10: Solving the ODE System

The final ODE system is solved numerically employing the reliable and stability-preserving Runge-Kutta (SSP-RK43) technique [43], which ensures stability and precision across time. This method is particularly suitable for handling stiff ODEs and guarantees reliable results.

2 Implementation of Scheme

The results are compared to exact solutions and previously proposed numerical schemes in order to show the accuracy of the numerical method that is used in this paper. This research use different error conditions to evaluate the effectiveness of the proposed method. Some of the key formulas used to calculate numerical errors include:

$$L_\infty = \max\left(|u_{\text{exact}}(x_i,t) - u_{\text{numerical}}(x_i,t)|\right), \tag{2.1}$$

$$L_2 = \sqrt{h\sum_{i=1}^{N} |u_{\text{exact}}(x_i,t) - u_{\text{numerical}}(x_i,t)|^2}, \tag{2.2}$$

$$\text{RMS} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} |u_{\text{exact}}(x_i,t) - u_{\text{numerical}}(x_i,t)|^2}. \tag{2.3}$$

The exact solution u_{exact} and the numerical solution $u_{\text{numerical}}$ are compared to assess the accuracy of the proposed method. The number of partitions of the domain is denoted by N.

In an iterative numerical method, the rate of convergence p indicates the rate at which the iteration approaches the exact solution as the number of iterations increases. The rate of convergence of the numerical method is calculated using the following formula [40]:

$$p \approx \frac{\log\left(\frac{E_N}{E_{2N}}\right)}{\log\left(\frac{2N}{N}\right)} \tag{2.4}$$

Where E_N and E_{2N} are the L_∞ errors with the number of partitions as N and $2N$, respectively.

The SG equation has been numerically solved for three different cases to verify the accuracy and effectiveness of the proposed method by calculating the errors.

Example 1

Consider the SG equation (1) within the domain $x \in [-3, 3]$ with parameters $\alpha = 0$, $\beta = 1$, and $\eta(x) = -1$, and the following initial conditions:

$$\phi_1(x) = 4\tan^{-1}\left(\exp(\gamma x)\right)$$

and

$$\phi_2(x) = \frac{-4\gamma\exp(\gamma x)}{1+\exp(2\gamma x)}$$

The exact solution is provided, and the boundary conditions are derived from it:

$$u(x,t) = 4\tan^{-1}\left(\exp(\gamma(x-0.5t))\right)$$

Here, γ is a parameter that depends on the speed of the solitary wave, given by:

$$\gamma = \frac{1}{\sqrt{1-c^2}}$$

The computations are performed with parameters $c = 0.5$, $k = 0.0001$, a spatial step size of $h = 0.04$, and $N = 151$ node points. The results indicate that the proposed method is both accurate and comparable to those previously reported in the literature for $\epsilon = 0.5$. Error values are listed in Table 1, where they are benchmarked against the findings from studies [30], [35] to confirm the method's validity. Figure 1 provides a visual comparison of the exact and numerical solutions over time, highlighting the close match between them.

Example 2

Consider the SG equation (1) within the domain $x \in [-20, 20]$ with parameters $\alpha = 0$, $\beta = 1$, and $\eta(x) = -1$, and the following initial conditions:

Table 1. Comparison of error values in the solution for Example 1.

	-	0.25	0.50	0.75	1.0
Present Results	L_2	6.4669×10^{-6}	8.6894×10^{-6}	9.9925×10^{-6}	1.0807×10^{-5}
	L_∞	1.3759×10^{-5}	1.4149×10^{-5}	1.4242×10^{-5}	1.4056×10^{-5}
	RMS	2.1273×10^{-7}	2.8584×10^{-7}	3.2870×10^{-7}	3.5549×10^{-7}
Mittal and Bhatia [30]	L_2	3.66×10^{-5}	9.00×10^{-5}	1.60×10^{-4}	2.27×10^{-4}
	L_∞	4.90×10^{-5}	7.55×10^{-5}	1.43×10^{-4}	2.10×10^{-4}
Shukla and Tamsir [35]	L_2	5.67×10^{-6}	8.39×10^{-6}	1.05×10^{-5}	1.24×10^{-5}
	L_∞	9.61×10^{-6}	1.10×10^{-5}	1.26×10^{-5}	1.44×10^{-5}

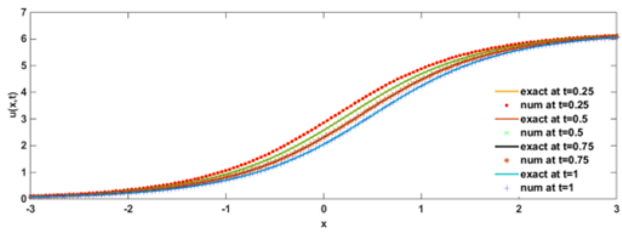

Figure 1. Graphical representation of Example 1 at time intervals t=0.25, 0.5, 0.75, and 1.

$$\phi_1(x) = 4\tan^{-1}\left(\exp(\gamma x)\right), \tag{2.5}$$

$$\phi_2(x) = \frac{-4\gamma c \exp(\gamma x)}{1+\exp(2\gamma x)}. \tag{2.6}$$

The exact solution is provided, and the boundary conditions are derived from it:

$$u(x,t) = 4\tan^{-1}\left(\exp\left(\gamma\left(x - ct\right)\right)\right) \tag{2.7}$$

Here, γ is a parameter that depends on the speed of the solitary wave, given by:

$$\gamma = \frac{1}{\sqrt{1-c^2}} \tag{2.8}$$

The computations are performed with parameters $c = 0.5$, $k = 0.01$, and $N = 501$ node points. The results indicate that the proposed method is both accurate and comparable to those previously reported in the literature for $\epsilon = 0.01$.

Error values are listed in Table 2, where they are benchmarked against the findings from study [38] to confirm the method's validity. Figure 2 provides a visual comparison of the exact and numerical solutions over time, highlighting the close match between them.

Table 2. Comparison of error values in the solution for Example 2.

		0.25	0.50	0.75	1.0	2.0	5.0	10.0	15.0	20.0
Present Results	L_2	1.1209×10^{-7}	3.5654×10^{-7}	5.6832×10^{-7}	7.0289×10^{-7}	9.9756×10^{-7}	1.6328×10^{-6}	3.2190×10^{-6}	5.2748×10^{-6}	7.8827×10^{-6}
	L_∞	1.2617×10^{-7}	3.9927×10^{-7}	6.3866×10^{-7}	8.2197×10^{-7}	1.1835×10^{-6}	1.4574×10^{-6}	2.7674×10^{-6}	4.2855×10^{-6}	6.1400×10^{-6}
	RMS	1.7688×10^{-8}	5.6261×10^{-8}	8.9680×10^{-8}	1.1092×10^{-7}	1.5741×10^{-7}	2.5765×10^{-7}	5.0795×10^{-7}	8.3235×10^{-7}	1.2439×10^{-6}
Shiralizadeh et al. [38]	L_2	2.4100×10^{-4}	3.4300×10^{-4}	4.1281×10^{-4}	4.6189×10^{-4}	5.1809×10^{-4}	4.3038×10^{-4}	5.1966×10^{-4}	6.5199×10^{-4}	8.4070×10^{-4}
	L_∞	1.1894×10^{-4}	1.2227×10^{-4}	1.2225×10^{-4}	1.2046×10^{-4}	1.1437×10^{-4}	1.3423×10^{-4}	1.7801×10^{-4}	2.3543×10^{-4}	3.1339×10^{-4}
	RMS	1.0778×10^{-5}	1.5339×10^{-5}	1.8461×10^{-5}	2.0657×10^{-5}	2.3170×10^{-5}	1.9247×10^{-5}	2.3240×10^{-5}	2.9158×10^{-5}	3.7579×10^{-5}

Example 3

Consider the SG equation (1) within the domain $x \in [-10, 10]$ with parameters $\alpha = 0$, $\beta = 1$, and $\eta(x) = -1$, and the following initial conditions:

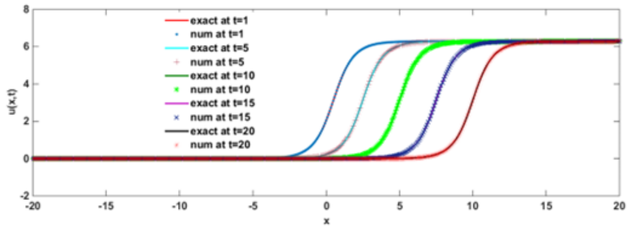

Figure 2. Graphical representation of Example 2 at time intervals t=1, 5, 10, 15, and 20.

$$\phi_1(x) = 0, \tag{2.9}$$
$$\phi_2(x) = 4\,\text{sech}(x). \tag{2.10}$$

The exact solution is provided, and the boundary conditions are derived from it:

$$u(x,t) = 4\tan^{-1}\left(\text{sech}(x)t\right) \tag{2.11}$$

The computations are performed with parameters $k = 0.01$ and $N = 401$ node points. The results indicate that the proposed approach is both accurate and comparable to those previously reported in the literature for $\epsilon = 1$.

Error values are listed in Table 3, where they are benchmarked against the findings from study [38] to confirm the method's validity. Table 4 shows the rate of convergence (ROC) of the proposed method is calculated using the L_∞ error norm for different time levels. Figure 3 provides a visual comparison of the exact and numerical solution over time, highlighting the close match between them.

Table 3. Comparison of error values in the solution for Example 3.

		0.25	0.50	0.75	1.0	2.0	5.0	10.0	15.0	20.0
Present Results	L_2	1.9450×10^{-6}	2.5475×10^{-6}	2.9748×10^{-6}	3.2777×10^{-6}	3.6014×10^{-6}	2.9577×10^{-6}	4.0544×10^{-6}	7.0675×10^{-6}	1.1713×10^{-5}
	L_∞	3.6320×10^{-6}	3.6320×10^{-6}	3.6320×10^{-6}	3.6320×10^{-6}	3.6320×10^{-6}	3.6320×10^{-6}	3.6320×10^{-6}	3.6320×10^{-6}	5.6180×10^{-6}
	RMS	4.3383×10^{-7}	5.6822×10^{-7}	6.6354×10^{-7}	7.3108×10^{-7}	8.0328×10^{-7}	6.5971×10^{-7}	9.0432×10^{-7}	1.5764×10^{-6}	2.6126×10^{-6}
Shiralizadeh et al. [38]	L_2	1.4400×10^{-4}	2.4339×10^{-4}	3.0422×10^{-4}	3.5484×10^{-4}	6.7163×10^{-4}	3.0000×10^{-3}	1.2600×10^{-2}	2.8900×10^{-2}	5.1700×10^{-2}
	L_∞	3.0169×10^{-5}	4.6806×10^{-5}	5.1706×10^{-5}	5.2994×10^{-5}	7.8976×10^{-5}	3.2159×10^{-4}	1.4000×10^{-3}	3.2000×10^{-3}	5.8000×10^{-3}
	RMS	7.1908×10^{-6}	1.2154×10^{-5}	1.5192×10^{-5}	1.7720×10^{-5}	3.3540×10^{-5}	1.4923×10^{-4}	6.2974×10^{-4}	1.4000×10^{-3}	2.6000×10^{-3}

Table 4. The ROC of numerical method with Example 3.

	$t = 1$		$t = 2$		$t = 5$	
N	L_∞	ROC	L_∞	ROC	L_∞	ROC
25	1.65×10^{-2}	—	1.85×10^{-2}	—	1.66×10^{-2}	—
50	3.32×10^{-4}	5.638675	6.27×10^{-4}	4.883128	4.27×10^{-4}	5.283404
100	2.53×10^{-5}	3.716241	4.15×10^{-5}	3.916740	2.96×10^{-5}	3.852788
200	3.63×10^{-6}	2.797904	3.63×10^{-6}	3.514449	3.63×10^{-6}	3.024371

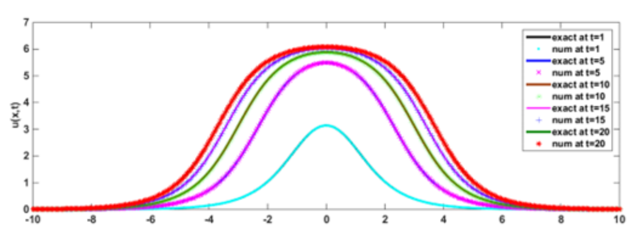

Figure 3. Graphical representation of Example 3 at time intervals t=1, 5, 10, 15, and 20.

3 Conclusion

Ensuring accurate voltage measurements is crucial for various applications like managing power grids, telecommunications, and medical devices. Josephson junctions provide highly precise voltage standards, ensuring reliability in measurements across different fields. Beyond this, Josephson junctions have broad applications in quantum computing, sensing, and digital electronics. Ongoing research is exploring new ways to maximize their unique properties. Solitons, significant in studying Josephson junctions, hold promise for future electronic advancements. The SG equation, widely applicable in physics and mathematics, remains an active area of research. In this study, we employed the DQM to numerically solve the nonlinear SG equation. We specifically utilized the exponential cubic B-spline basis function to compute the weighting coefficients in this approach. The accuracy and proficiency of the exponential cubic B-spline differential quadrature method have been verified using various error norms. This validation indicates that the numerical solutions obtained using this method closely align with the exact solutions, surpassing the accuracy of previously published numerical methods. These positive outcomes suggest the potential applicability of the technique to similar nonlinear partial differential equations, showcasing its adaptability in addressing complex problems. This research contributes to advancing our understanding of solitons in Josephson junctions and provides a robust numerical approach for studying nonlinear equations in various scientific fields.

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