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Investigate the Impact of Protected Zones on Prey-Predator Interactions using Delay Differential Equations

Shiv Raj, Ahmed Ali Mohsen and Pankaj Kumar

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Corresponding Author: Shiv Raj

Abstract In this work, the stability of a mathematical model for the dynamics of prey-predator model with protected zones, dividing the species into three compartments. Through qualitative analysis, the presence and local stability of all feasible equilibria with and without time delay are studied. Sufficient criteria are established for the orientation of Hopf bifurcation and the regional stability of the internal equilibrium point under the influence of delay as a variable parameter. Ultimately, MATLAB numerical simulations were used to derive and validate theoretical results.

1 Introduction

Mathematical models are a major part of modern research in many domains, such as ecological studies, biology, health care, finance, neuroscience, computer science, and so forth. The mathematical model is becoming more and more popular, especially in the fields of wildlife and microbiology. The progression of any unidentified variable can be described by developing an appropriate model that encompasses all conceivable aspects of the variable. Mathematics-based models can be used to simulate how systems in reality behave. Moreover, a deeper understanding of the practical problems system is frequently obtained through the examination of mathematical models. The comprehension of intricate, nonlinear biological processes is aided by recent developments in mathematical techniques such as chaos theory. The calculations and simulations that were previously unavailable are made possible by higher computational capacity. In certain cases, information is accurate now, but it lags ahead because the phenomena have not been publicly available. In this situation, evaluating the recent actions of the data allows one to ascertain the past behaviors of a variable. Many times, we would like to know how an actual structure will operate in the future, but we are unable to actually test with the system because it is too costly, complex, or both. However, the subsequent actions of the framework can be predicted via mathematical modelling.

Several researchers (Yaseen et al. [1], Sreerag et al. [2], Dai and Tang [3]) have explored the effects of constant-rate prey extraction on the dynamics of predator-prey systems, yielding a wide array of dynamic phenomena. Ji and Wu [4] investigated a predator-prey model featuring steady prey harvesting and a Holling type II functional response. The dynamic response model was employed by Kar and Chakraborty [5] to examine the kinetic behavioral patterns of the system. In order to model the interactions among three biological species, Feng [6] explored a framework of differential equations incorporating both mobility and temporal delays. Nakaoka et al. [7] investigated a predator-prey scenario characterized by two delays within the Lotka-Volterra framework. They concluded that, with minimal delays, the structure's favourable equilibria is worldwide rapidly stable; nevertheless, when the period of delay increases, the whole thing be-

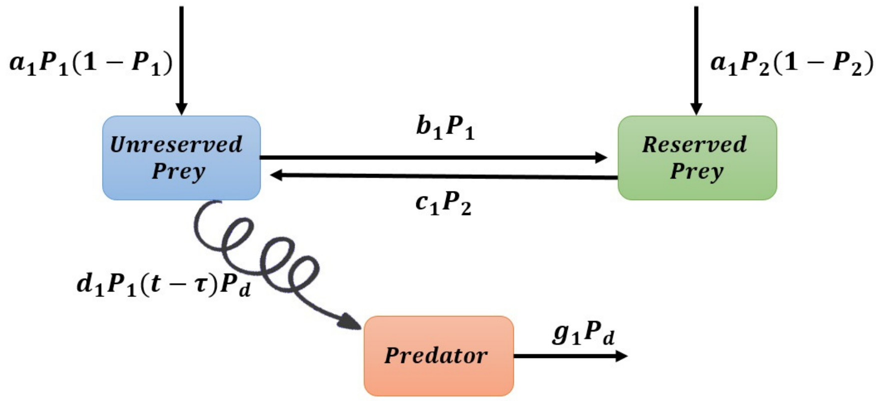

Figure 1. The block diagram of predator-prey model.

gins to behave chaotically. They determined the crucial time delay values needed for a Hopf bifurcation to occur in the framework. An SIR pandemic model with transcritical and Hopf bifurcations was put forth by Kar and Mondal [8]. Experts also talked about the global and local stability of the system. Cannibalism and pattern formation were investigated by Fasani and Rinaldi [9] in a geographically prolonged predator-prey scenario. In a predator-prey structure, Pei et al. [10] investigated population mortality and retention. Tax-related harvesting approaches were studied by AL-Husseiny [11], Conard [12], Lenzini and Rebaza [13]. Numerous delay theories regarding the predation of mature prey, commensal species, and the creation of prey teams to fend against predators were put up by Pankaj Kumar and Shiv Raj [14], [15], [16], [17], [18], [19]. Our research paper presents a predator-prey system with delay, where the prey habitat is under protection. The attack on prey is delayed due to the presence of the protected zone. We explore the existence and stability of all potential equilibria in the absence of time delay. When delay is introduced as a variable parameter, we adequately ascertain the orientation of Hopf bifurcation and the local stability of the internal equilibrium point. The subsequent sections of the article are as follows. A comprehensive explanation of the proposed system is presented in Section 2. The detection of the equilibrium points, existence, and its stability analysis are discussed in Sections 3 and 4. Numerical illustrations of the proposed work are provided in Section 5. Finally, the comprehensive conclusion of the present work is drawn in Section 6.

2 Mathematical Model

A predator-prey scenario takes into account the impact of the habitat and time delay in the preceding paragraph. Given the two types of prey (reserved and unreserved), the model under consideration is a predator-prey system depicted in the subsequent figure (see Figure 1), along with ODEs.

$$\begin{aligned} \frac{dP_1}{dt} &= a_1P_1\left(1-P_1\right) - b_1P_1 + c_1P_2 - d_1P_1\left(t-\tau\right)P_d, \\ \frac{dP_2}{dt} &= e_1P_2\left(1-P_2\right) + b_1P_1 - c_1P_2, \\ \frac{dP_d}{dt} &= f_1P_1P_d - g_1P_d. \end{aligned} \tag{2.1}$$

With initial condition $P_1 > 0, P_2 > 0$ and $P_d > 0$.

Symbols	Meanings
a_1	Natural Growth rate of P_1
b_1	Prey movement from reserve zone to unreserved zone
c_1	Prey migration from reserve zone to unreserved zone
d_1	Predator attack rate on prey in free zone
e_1	Natural birth rate of prey in reserve zone
f_1	Prey conversion rate in the free zone to a predator
g_1	Natural death rate of predator

3 System's Equilibrium Points

We determine the system's equilibrium point (2.1) Equating 3rd equation of model (2.1) to zero

$$f_1 P_1 P_d - g_1 P_d = 0,$$
$$P_d\left(f_1 P_1 - g_1\right) = 0.$$

As $P_d \neq 0$, so value of the variable P_1

$$P_1 = \frac{g_1}{f_1}. \tag{3.1}$$

Equating 1st equation of model (2.1) to zero
$a_1 P_1\left(1 - P_1\right) - b_1 P_1 + c_1 P_2 - d_1 P_1 P_d = 0$, substitute the value of P_1

$$a_1 \tfrac{g_1}{f_1}\left(1 - \tfrac{g_1}{f_1}\right) - b_1 \tfrac{g_1}{f_1} + c_1 P_2 - d_1 \tfrac{g_1}{f_1} P_d = 0$$
$$- d_1 \tfrac{g_1}{f_1} P_d = -a_1 \tfrac{g_1}{f_1}\left(1 - \tfrac{g_1}{f_1}\right) + b_1 \tfrac{g_1}{f_1} - c_1 P_2,$$
$$P_d = \left(\frac{a_1}{d_1}\left(1 - \frac{g_1}{f_1}\right) - \frac{b_1}{d_1}\right) + \frac{f_1 c_1}{d_1 g_1} P_2. \tag{3.2}$$

Equating 2nd equation of model (2.1) to zero
$e_1 P_2\left(1 - P_2\right) + b_1 P_1 - c_1 P_2 = 0$, substitute the value of P_1

$$e_1 P_2\left(1 - P_2\right) + b_1 \frac{g_1}{f_1} - c_1 P_2 = 0,$$
$$P_2^2 + \left(-1 + \frac{c_1}{e_1}\right) P_2 - \frac{b_1 g_1}{f_1 e_1} = 0,$$
$$P_2 = \frac{-\left(-1 + \frac{c_1}{e_1}\right) \pm \sqrt{\left(-1 + \frac{c_1}{e_1}\right)^2 - 4 \times -\frac{b_1 g_1}{f_1 e_1}}}{2},$$
$$P_2 = \frac{\left(1 - \frac{c_1}{e_1}\right) \pm \sqrt{\left(-1 + \frac{c_1}{e_1}\right)^2 + 4 \times \frac{b_1 g_1}{f_1 e_1}}}{2}.$$

If $\left(-1 + \frac{c_1}{e_1}\right)^2 + 4 \times \frac{b_1 g_1}{f_1 e_1} > 0$ has two distinct roots. If $\left(-1 + \frac{c_1}{e_1}\right)^2 + 4 \times \frac{b_1 g_1}{f_1 e_1} = 0$ the equation has one repeated root. To find the value of P_d substitute the value P_2 in (3.2)

$$P_d = \left(\frac{a_1}{d_1}\left(1 - \frac{g_1}{f_1}\right) - \frac{b_1}{d_1}\right) + \frac{c_1 f}{d_1 g_1}\left\{\frac{\left(1 - \frac{c_1}{e_1}\right) \pm \sqrt{\left(-1 + \frac{c_1}{e_1}\right)^2 + 4 \times \frac{b_1 g_1}{f_1 e_1}}}{2}\right\}.$$

4 Local Stability

We will solely examine the interior equilibrium's stability and local Hopf-Bifurcation in the following part. The distinctive positive equilibrium of the structure composed of equations $1st, 2nd$ and $3rd$ of model (2.1), is evident.

Differentiation w.r.t P_1

$$t_1 = a_1 - 2a_1P_1 - b_1 - d_1e^{-\tau\lambda}P_d,$$
$$t_2 = b_1,$$
$$t_3 = f_1P_d.$$

Differentiation w.r.t P_2

$$t_4 = c_1,$$
$$t_5 = e_1 - 2P_2 - c_1,$$
$$t_6 = 0.$$

Differentiation w.r.t P_d

$$t_7 = 0,$$
$$t_8 = 0,$$
$$t_9 = f_1P_1 - g_1,$$

$$\begin{vmatrix} \xi - t_1 & t_2 & t_3 \\ t_4 & \xi - t_5 & t_6 \\ t_7 & t_8 & \xi - t_9 \end{vmatrix} = 0.$$

$$\begin{vmatrix} \xi - a_1 + 2a_1P_1 + b_1 + d_1e^{-\lambda\tau}P_d & b_1 & f_1P_d \\ c_1 & \xi - e_1 + 2P_2 + c_1 & 0 \\ 0 & 0 & \xi - f_1P_1 + g_1 \end{vmatrix} = 0.$$

$$(\xi - f_1P_1 + g_1)\begin{vmatrix} \xi - a_1 + 2a_1P_1 + b_1 + d_1e^{-\lambda\tau}P_d & b_1 \\ c_1 & \xi - e_1 + 2P_2 + c_1 \end{vmatrix} = 0.$$

$$\xi - a_1 + 2a_1P_1 + b_1 + d_1e^{-\lambda\tau}P_d\left\{\left(\xi - a_1 + 2a_1P_1 + b_1 + d_1e^{-\lambda\tau}P_d\right)(\xi - e_1 + 2P_2 + c_1) - c_1b_1\right\} = 0.$$

$$\begin{aligned}
&(\xi - f_1P_1 + g_1)\left\{\xi^2 + \xi\left(-a_1 + 2a_1P_1 + b_1 + d_1e^{-\lambda\tau}P_d - e_1 + 2P_2 + c_1\right)\right. \\
&\quad + \left(-a_1 + 2a_1P_1 + b_1 + d_1e^{-\lambda\tau}P_d\right)(-e_1 + 2P_2 + c_1) - b_1c_1 = 0 \\
&\xi^3 - \xi^2\left(f_1P_1g_1 - a_1 + 2a_1P_1 + b_1 + d_1e^{-\lambda\tau}P_d - e_1 + 2P_2 + c_1\right) \\
&\quad + \xi\left\{(-f_1P_1g_1)\left(-a_1 + 2a_1P_1 + b_1 + d_1e^{-\lambda\tau}P_d - e_1 + 2P_2 + c_1\right)\right. \\
&\quad + \left(-a_1 + 2a_1P_1 + b_1 + d_1e^{-\lambda\tau}P_d\right)(-e_1 + 2P_2 + c_1) - c_1b_1 \\
&\quad + (-f_1P_1g_1)\left\{\left(-a_1 + 2aa_1 + b_1 + d_1e^{-\lambda\tau}P_d\right)(-e_1 + 2P_2 + c_1) - c_1b_1\right\} = 0.
\end{aligned}$$

$$\xi^3 + X\xi^2 + Y\xi + Z + e^{-\lambda\tau}(W\xi + N) = 0. \tag{4.1}$$

Where,

$$\begin{aligned}
X &= \left(f_1P_1g_1 - a_1 + 2a_1P_1 + b_1 + d_1e^{-\lambda\tau}P_d - e_1 + 2P_2 + c_1\right), \\
Y &= \left\{(-f_1P_1g_1)\left(-a_1 + 2a_1P_1 + b_1 + d_1e^{-\lambda\tau}P_d - e_1 + 2P_2 + c_1\right) + \right. \\
&\quad \left(-a_1 + 2a_1P_1 + b_1 + d_1e^{-\lambda\tau}P_d\right)(-e_1 + 2P_2 + c_1) - c_1b_1, \\
Z &= (-f_1P_1g_1)\left\{\left(-a_1 + 2aa_1 + b_1 + d_1e^{-\lambda\tau}P_d\right)(-e_1 + 2P_2 + c_1) - c_1b_1\right\}, \\
N &= (-f_1P_1g_1)(d_1P_d)(-e_1 + 2P_2 + c_1) + (-f_1P_1g_1)(-c_1b_1)
\end{aligned}$$

Each root of the singular equation (4.1) must possess a negative real component to maintain the stability of the equilibrium point. Identifying the conditions that lead to all solutions of equation (4.1) having negative real components is challenging. Equation (4.2) is derived for $\tau = 0$. Equation (4.1) transforms into the characteristic equation of the previous system of equations of model (2.1), at the interior equilibrium $E^* = (N_1^*, N_2^*,\ N_3^*)$ When $\tau = 0$.

$$\xi^3 + X\xi^2 + (Y+W)\xi + Z + N = 0. \tag{4.2}$$

In line with the Routh-Hurwitz standard

$$\begin{aligned} &If X > 0,\ (Z+N) > 0, \\ &(Y+W) > (Z+N)\,. \end{aligned} \tag{4.3}$$

Therefore, the real components of each of the equation (4.2)'s solutions will be negative.

$$Z + N = 0$$

assuming that $\xi = 0$ is the answer to equation (4.1). Consequently, this condition runs counter to the second condition (4.3) that was previously stated. Consequently, $\xi = 0$ cannot be a correct solution for (4.1). Assuming there is a (4.1) solution for any ≥ 0 with μ > 0,

$$-i\mu - X\mu + iY\mu + Z + (cos\mu\tau - isin\mu\tau)\,(iW\mu + N) = 0. \tag{4.4}$$

Separating the unreal from the real portions,

$$Z - X\mu^2 + Ncos\mu\tau + Wsin\mu\tau = 0, \tag{4.5}$$

$$Y\mu - \mu^3 + W\mu cos\mu\tau - Nsin\mu\tau = 0. \tag{4.6}$$

Which results in

$$\mu^6 + p\mu^4 + q\mu^2 + r = 0. \tag{4.7}$$

$$p = X^2 - 2Y,\ q = Y^2 - W^2 - 2XZ,\ r = Z^2 - N^2.$$

If we take $x = \mu^2$, eauation (4.7) becomes,

$$x^3 + px^2 + qx + r = 0. \tag{4.8}$$

And $j\,(x) = x^3 + px^2 + qx + r$
Lemma 1. Regarding the polynomial equation (4.8), the following outcomes are obtained:

If r is less than zero, equation (4.8) exhibits at least one positive solution

If r is greater than or equal to zero and $p^2 - 3q$, is less than or equal to zero, then equation (4.8) will have all negative solutions.

If r is greater than or equal to zero and $p^2 - 3q$, is strictly greater than zero, then equation (4.8) will have all negative solutions.

$$g = \frac{-p \pm (p^2 - 3q)}{3} > 0,\ and\ h(g) \leq 0.$$

Proof. Let's assume that equation (4.8) possesses at least one positive real solution

$$\mu_0 = \sqrt{x_0}$$

From the equation (4.5) and (4.6), we obtain,

$$cos\mu_0\tau = \frac{-\left(W\mu_0^2\left(Y - \mu_0^2\right) + \left(Z - X\mu_0^2\right)(N)\right)}{(N)^2 + (W\mu_0)^2}, \tag{4.9}$$

$$\tau_k = \frac{1}{\mu} arc\ cos\left(\frac{-\left(W\mu_0^2\left(Y-\mu_0^2\right)+\left(Z-X\mu^2\right)(N)\right)}{(N)^2+(W\mu)^2} + 2j\pi \right). \tag{4.10}$$

When k= 0,1,2,3...

Lemma 2. Let us consider $j\left(x_0\right)=\left(3x^2+2px_0+q_0\right)$ and the condition (4.3) are fulfilled. For $(k=0,\ 1,\ 2,\ 3\ldots)$ stand for $\mu\xi\left(\tau\right)=x\left(\tau\right)+i\mu(\tau)$ be the solution of the system (4.1) whenever the condition $x\left(\tau_k\right)=0,\ \mu\left(\tau_k\right)=\mu_0$ is satisfied, where

$$\tau_k = \frac{1}{\mu} arc\ cos\left(\frac{-\left(W\mu_0^2\left(Y-\mu_0^2\right)+\left(Z-X\mu_0^2\right)(N)\right)}{(N)^2+(W\mu_0)^2} + 2j\pi \right).$$

Let us consider $\pm i\mu_0$ are simple solution. If the transversally situation

$$x^k\left(\tau_k\right) = \left.\frac{R_e\xi\left(\tau\right)}{d\tau}\right|_{\xi=i\ \mu_0}.$$

Consider each level of equilibrium when the set of equations of model (2.1), has a Hopf bifurcation and $\tau=\tau_k$

Proof Upon substituting $\xi=\xi\left(\tau\right)$ into equation (4.1) and differentiating both sides of equation (4.1) with respect to , the following holds:

$$\left(3\xi^2+2X\xi+Y\right)+\left(\left(\xi^2\right)\right).$$

5 Conclusion and numerical simulation of system

The modelling framework (2.1) reflects the interacting behaviors of the role-reversal process among the three types of species: the predator, the protected prey, and the uncontrolled prey. Several critical requirements for the system stability surrounding all equilibrium points were identified by the theoretical study of the equilibrium positions (Sections 3 and 4). We used our proposed requirements to determine the dynamical behavior for all equilibrium locations. Keep in mind that MATLAB was used for the entire computational simulation process, and the dde23 code was used. Several tools in MATLAB can be used to solve the initial value differential equation systems numerically. Ordinary differential equations (ODEs) with non-stiff initial values were solved using this solver. In order to carry out more numerical experiments, we selected the subsequent set of values for parameters.

$$a_1=0.05\ ,\ b_1=0.001\ ,\ c_1=0.1\ ,\ d_1=0.1\ ,\ e_1=0.2\ ,\ f_1=0.01\ ,\ g_1=0.009,\ \tau=39.8 \tag{5.1}$$

Based on the information provided in equation (4.10) and certain initial conditions, the positive equilibrium point of our model is globally asymptotically stable, as depicted in Figure 2

It is clear that figure 2 according to the parameter values in equation (5.1) satisfies the globally asymptotically stable of the positive equilibrium point. Now, it is observed that for the parameters $b_1=0.00001$ and $e_1=0.02$ with keeping the all other values in equation (5.1), the solution of system (2.1)approaches globally asymptotically stable to reserved prey free equilibrium point see Figure 3.

Clearly, from Figure 3, if the values of prey movement and birth rate of prey reserved (b_1) and (e_1) are decreasing we have the positive point is unstable and the reserved prey free point became stable. Now, when the perdition rate of unreserved prey from predator (f_1) decreasing, we get the same above results but the solution of system (2.1) approaches globally stable to the predator free equilibrium point see Figure 4. But, if we change the parameters values of $b_1=0.00001,\ e_1=0.02$ and $f_1=0.001$ simultaneously, the solution of system (2.1) tends towards the global stability of the unreserved prey equilibrium point, as illustrated in Figure 5.

When decreasing the prey migration rate to $c_1=0.01$ and increasing the prey movement rate to $b_1=0.4$, while maintaining all other parameter values from equation (5.1), the solution of system (2.1) converges towards global stability at the reserved prey equilibrium point, as depicted in Figure 6.

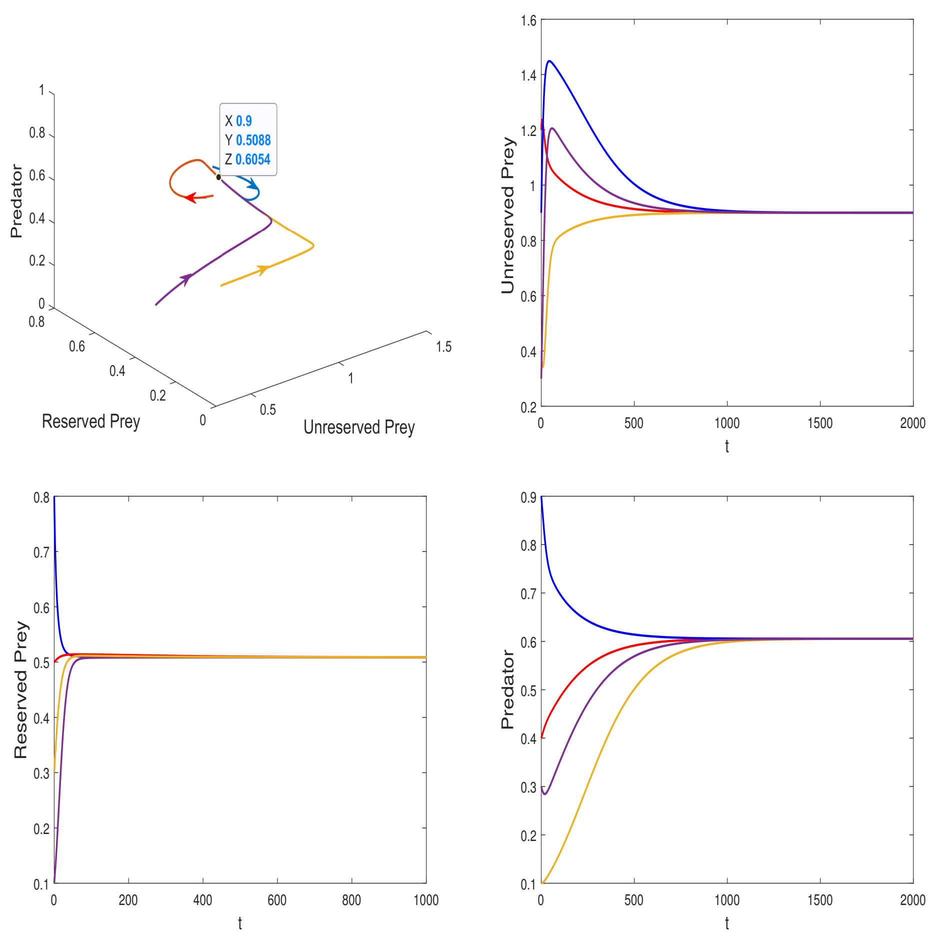

Figure 2. The trajectory of the positive equilibrium points of system (2.1) originates from various initial points utilizing the dataset outlined in equation (5.1).

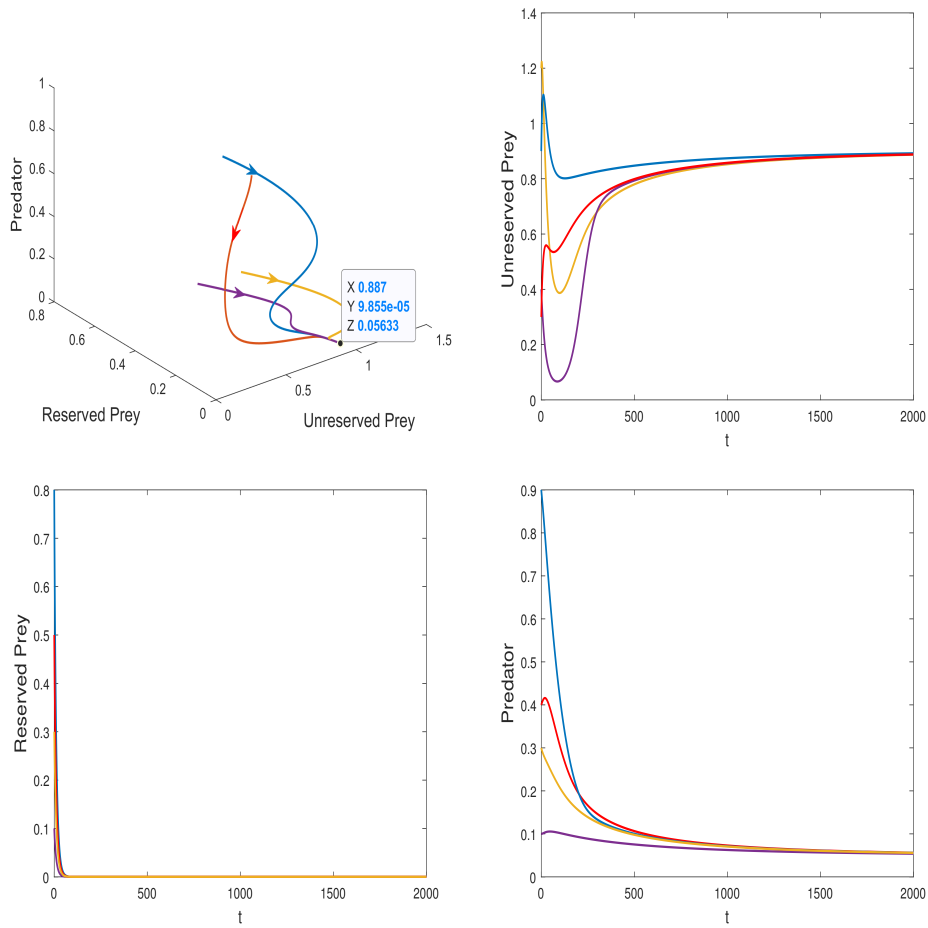

Figure 3. The global trajectory of the equilibrium point corresponding to reserved prey freedom in system (2.1) initiates from diverse initial points, employing the dataset provided in equation (5.1)

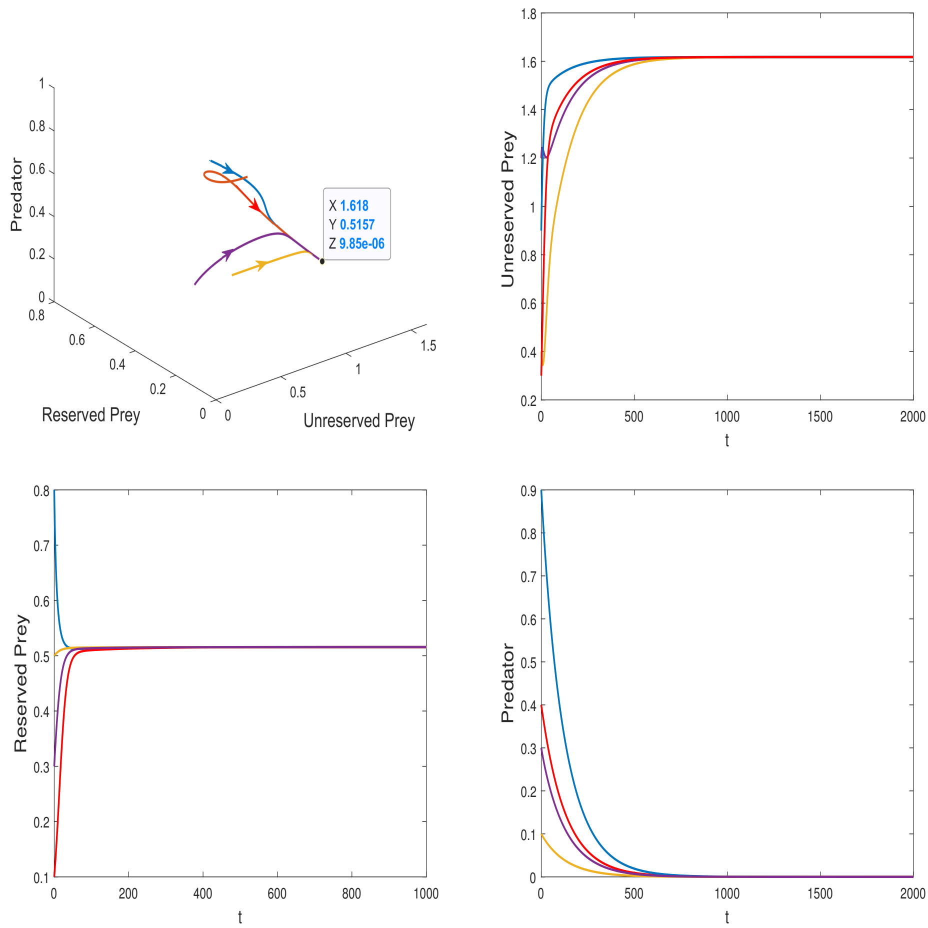

Figure 4. The global trajectory of the equilibrium point corresponding to a predator-free state in system (2.1) commences from various initial points, utilizing the dataset specified in equation (5.1)

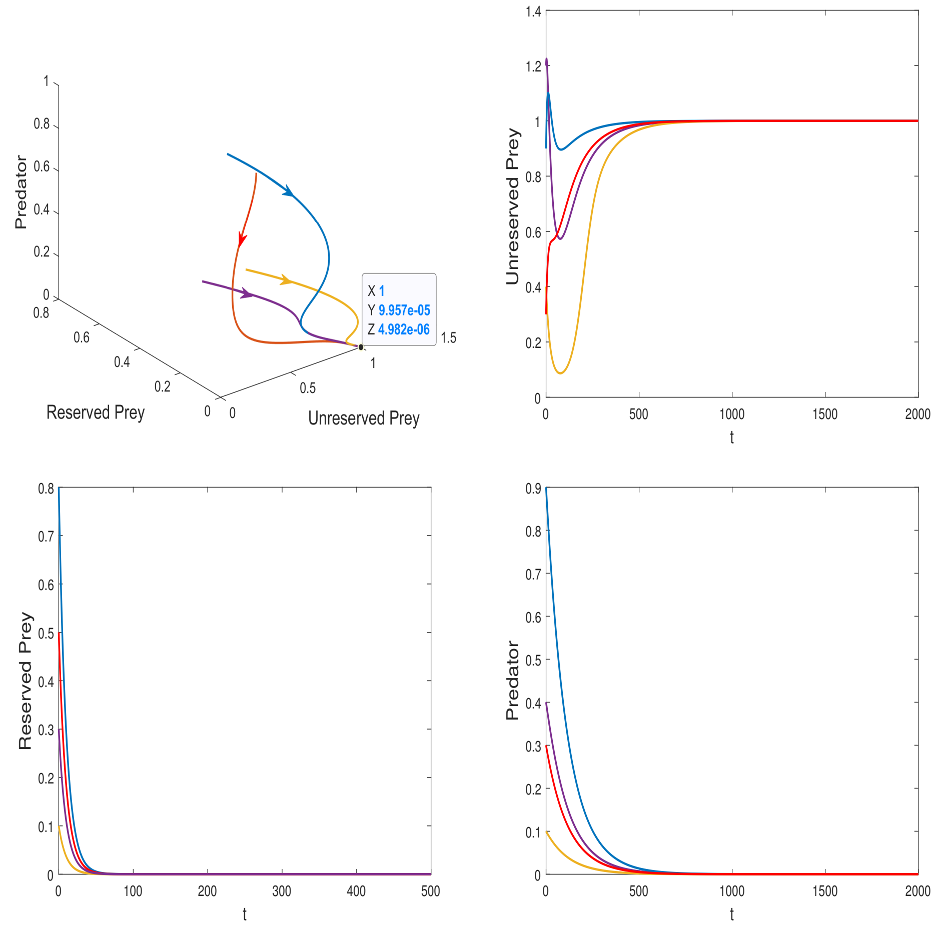

Figure 5. The trajectory of the unreserved prey equilibrium points in the system (2.1) originates from diverse initial points, utilizing the data set outlined in equation (5.1)

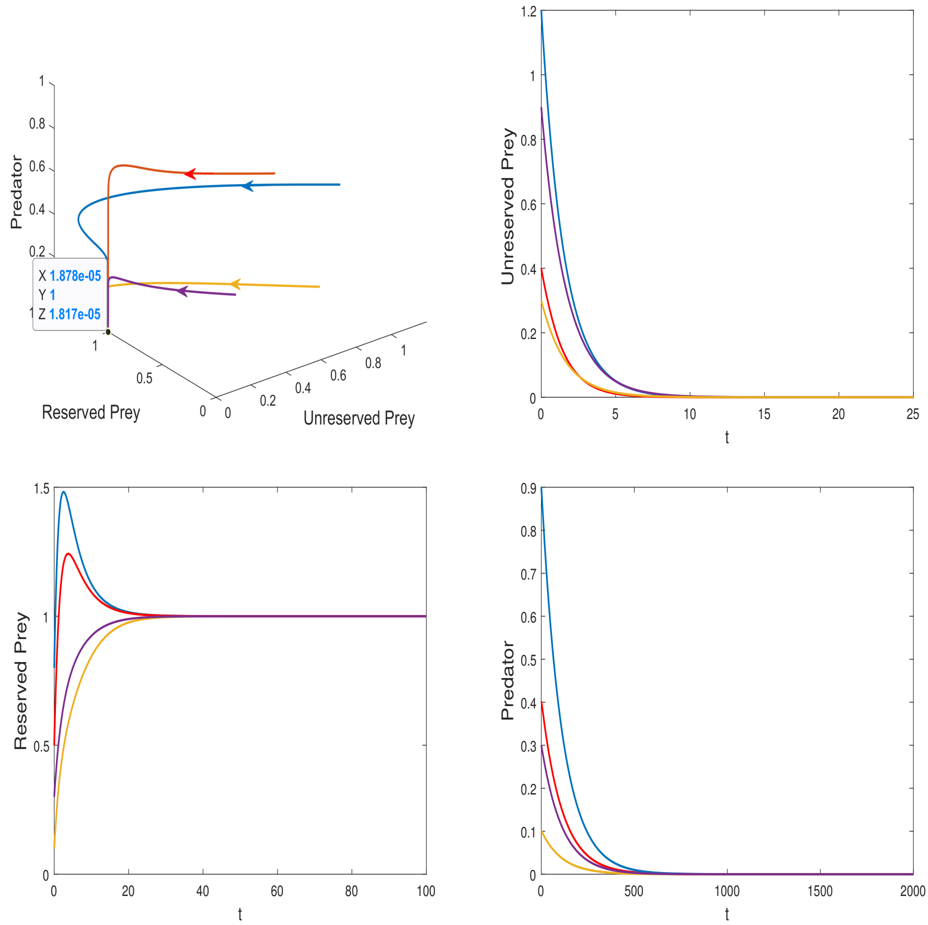

Figure 6. The trajectory of the reserved prey equilibrium points in the system (2.1) initiates from various initial points, utilizing the dataset specified in equation (5.1)

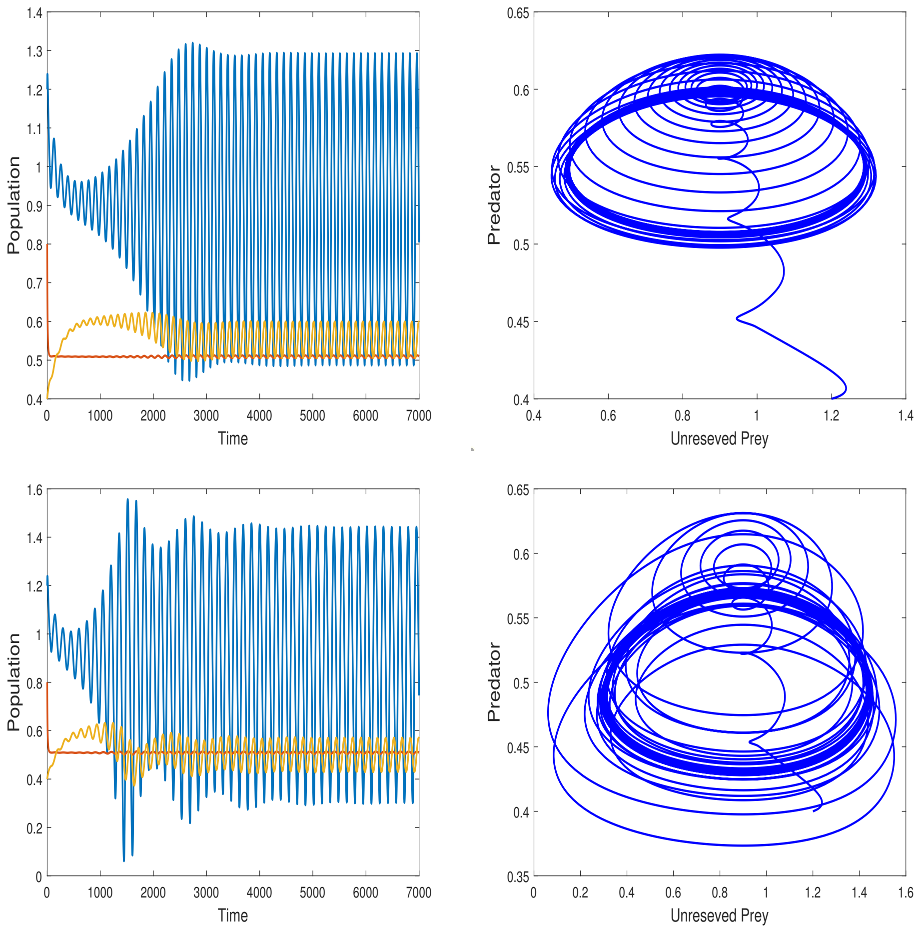

Figure 7. The periodic attractor associated with the positive equilibrium points of system (2.1), (a_1-b_1) when $\tau = 45.8$ and (c_1-d_1) when $\tau = 60.8$

From Figure 7, it is concluded that studying the effect of time delay (τ) on the dynamic behavior of the system (2.1) yields valuable insights, from range $\tau > \tau_0 \cong 41.8$ approached to periodic attractor and the positive point is losing the stability.

6 Discusion and results

The modelling framework (2.1) reflects the interacting behavior of the role-reversal process among the three types of species: the predator, the protected prey, and the uncontrolled prey. Several critical requirements for the system stability surrounding all equilibrium points were identified by the theoretical study of the equilibrium positions (Sections 3 and 4). We used our proposed requirements to determine the dynamical behavior for all equilibrium locations. Keep in mind that MATLAB was used for the entire computational simulation process, and the dde23 code was used. Several tools in MATLAB can be used to solve the initial value differential equation systems numerically. Therefore, the results of this work are variant, significant and so it is interesting and capable to develop its study in the future.

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Author information

Shiv Raj, Department of Mathematics, Vaish College Bhiwani, Haryana 127021, India.
E-mail: `shivrajswami86@gmail.com`

Ahmed Ali Mohsen, Department of Mathematics, Open Education College.
Department of Mathematics, College of Education for Pure Science (Ibn Al-Haitham), University of Baghdad, Iraq.
E-mail: `aamuhseen@gmail.com`

Pankaj Kumar, Mathematics Department, Lovely Professional University, Phagwara, Punjab (India)144411, India.
E-mail: `pankaj.kumar1@lpu.co.in`