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MATHEMATICAL ANALYSIS OF SOME (3+1)-D FRACTIONAL MODELS OF PDES VIA FORMABLE TRANSFORM HPM

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Abstract

The main objective of this paper is to address the mathematical models arising in applied sciences, specifically three- dimensional time- fractional Klein- Gordon equation (TFKGE), three-dimensional time- fractional sine-Gordon equation (TFSGE), and three- dimensional time- fractional Rosenau- Hyman equation (TFRHE) by using a hybrid technique: the formable transform homotopy perturbation method (STHPM). The homotopy perturbation method is an effective tool for solving both linear and nonlinear partial differential equations, and its combination with formable transform enhances its efficiency by reducing the computational complexity. The caputo derivative is used to compute the fractional derivative, while He's polynomial is employed to tackle nonlinearity. The outcomes of the numerical experiments reflect the efficacy of this technique and offering valuable insights for solving complex physical systems. This study advances the application of fractional calculus methods to challenging problems in theoretical and applied physics.

1 Introduction

Nonlinear partial differential equations (PDEs) play an important role in describing various processes in science and engineering, such as wave motion, solitons, and field dynamics. Equations like the Klein-Gordon and sine-Gordon are often used to model wave behavior and field interactions, especially in physics. The Rosenau-Hyman equation is also important for understanding wave phenomena, particularly in fluid dynamics, where both nonlinear and dispersive effects are present. Fractional derivatives extend the concept of ordinary derivatives and provide better tools to model complex systems that have hereditary properties. For this reason, the time-fractional versions of the Klein-Gordon, sine-Gordon equations and Rosenau-Hyman equation are becoming more popular in research. These versions of the equations are not easy to solve due to their complexity, especially in three dimensions.

In this paper, we solve the 3D time-fractional Klein-Gordon, sine-Gordon equations and Rosenau-Hyman equations using the formable transform homotopy perturbation method (FTHPM). This method combines the advantages of two powerful techniques: the formable transform and the homotopy perturbation method (HPM). The FTHPM helps to manage the nonlinearity of the equations and efficiently handles the time fractional derivatives.

In [1, 2, 3] an introduction of Homotopy Perturbation method is discussed and explores its applications to handle various type differential equations. The solution of fractional type PDEs through the homotopy perturbation transformation method yields a series solution along with convergence and error estimation of the solution in [4, 5]. The origin of fractional derivatives can be traced about 17th century when mathematician L' Hospital explored the interpretation of derivatives for non-integral order. The applications of fractional differential equations to various engineering problems, including its development and implementation, has been examined and discussed in [6, 7, 8]. The Rosenau-Hyman equation has been analyzed in terms of its mathematical formulation in [9]. Various effective methods have been devised and put into practice to determine solutions for the general and fractional Rosenau-Hyman equations in [10, 11]. In [12] author used Elzaki transform and HPM to tackle PDEs of fractional order type. Various analytical and semi-analytical techniques are devised to tackle time-fractional sine-Gordon equation is employed with reduce differential transform method [13], space-time spectral method [14], finite difference scheme [15]. Whereas time-fractional Klein-Gordon equations has been handled by a lie group approach [16], wavelet method [17], Elzaki transform homotopy perturbation method [18], reduce differential equation [19], quadruple Laplace transform [20]. In [21], Sumudu transform homotopy perturbation method is used to tackle Klein-Gordon equation and sine-Gordon equation. A discussion of He's polynomial to handle nonlinearities of differential equations is presented in [22]. In [23] the convergence analysis of the homotopy perturbation method (HPM) for PDEs is presented. The introduction and applications of formable transform to various types of PDEs are discussed in [24, 25]. In [26], the authors applied the formable transform decomposition method to solve time fractional PDEs. In [27], the authors have presented a hybrid homotopy perturbation method for solving 2D mathematical models arising in various applications of sciences. In [28], the authors have implemented an efficient technique for solving (2+1)D and (3+1)D fractional nonlinear Schrodinger equations.

(a) 3D time fractional Klein-Gordon equation:

$$D_\tau^\alpha \sigma(\check{x},\check{y},z,\tau) - c^2\left[\frac{\partial^2\sigma}{\partial\check{x}^2}(\check{x},\check{y},z,\tau)+\frac{\partial^2\sigma}{\partial\check{y}^2}(\check{x},\check{y},z,\tau)+\frac{\partial^2\sigma}{\partial z^2}(\check{x},\check{y},z,\tau)\right] + a\sigma(\check{x},\check{y},z,\tau) + N(\sigma(\check{x},\check{y},z,\tau)) = \psi(\sigma(\check{x},\check{y},z,\tau)), \quad (1.1)$$

with initial conditions are:

$$\sigma(\check{x},\check{y},z,0) = f_1(\check{x},\check{y},z), \frac{\partial\sigma}{\partial\tau}(\check{x},\check{y},z,0) = f_2(\check{x},\check{y},z).$$

Here $D_\tau^\alpha = \frac{\partial^\alpha}{\partial\tau^\alpha}$, $1<\alpha\le 2$, $\tau>0$. The parameters c and a are real numbers, $N(\sigma(x,y,z,\tau))$ is non-linear term, and $\psi(\sigma(x,y,z,\tau))$ is a source term. It is worth mentioning that for $\alpha=2$, the equation (1) is reduced to the 3D Klein-Gordon equation.

(b) 3D time fractional sine-Gordon equation:

$$D_\tau^\alpha \sigma(x,y,z,\tau) + bD_\tau^{\alpha-1}\sigma(x,y,z,\tau) - c\left(\sigma_{xx}(x,y,z,\tau)+\sigma_{yy}(x,y,z,\tau)+\sigma_{zz}(x,y,z,\tau)\right) + \phi(x,y,z)\sin(x,y,z,\tau) = \psi(\sigma(x,y,z,\tau)) \quad (1.2)$$

with initial conditions are:

$$\sigma(x,y,z,0) = f_3(x,y,z), \quad \sigma_\tau(x,y,z,0) = f_4(x,y,z).$$

Here $D_\tau^\alpha = \frac{\partial^\alpha}{\partial\tau^\alpha}$, $1<\alpha\le 2$, $\tau>0$ here $\phi(\hat{x},\hat{y},z)$ is Josephson current density, b is dissipative term, and $b\ge 0$. When $b=0$, equation (2) reduces to un-damped sine Gordon equation, and damped for $b>0$ c is a non negative real number. It is worth mentioning that for $\alpha=2$ the equation (2) is reduced to 3D sine-Gordon equation.

(c) 3D time fractional Rosenau-Hyman equation:

$$D_\tau^\alpha \sigma(x,y,z,\tau) + a\left(\frac{\partial}{\partial x}(\sigma^n)+\frac{\partial}{\partial y}(\sigma^n)+\frac{\partial}{\partial z}(\sigma^n)\right)+\left(\frac{\partial^3}{\partial x^3}(\sigma^n)+\frac{\partial^3}{\partial y^3}(\sigma^n)+\frac{\partial^3}{\partial z^3}(\sigma^n)\right) = 0, \quad (1.3)$$

$0 < \alpha \leq 1$
in some continuous domain with initial conditions $\sigma(x, y, z, 0) = f_5(x, y, z)$ here, α is any real number.

The outline of paper is: In section 2, some definitions related to fractional derivatives are presented. In subsection 2.1, introduction of formable transform is discussed. In subsection 2.2, homotopy perturbation method has been discussed. Subsection 2.3, the convergence analysis of HPM is elaborated. In section 3, methodology is discussed. In subsection 3.1, the FTHPM is applied to the 3D TFKGE, in subsection 3.2, FTHPM is applied to the 3D TFSGE, whereas in subsection 3.3, FTHPM is applied to the TFRHE. Various examples are presented in section 4. The concluding remarks are presented in section5.

2 Some basic definitions of fractional calculus

Fractional order calculus expands classical calculus to encompass non integral order. It introduces fractional order derivatives and integrals, redefining traditional notion of differentiation and integration. In this paper, we make use of foundational concepts and properties derived from the theory of fractional calculus.

Definition: A real function $g(\tau) \in C_\mu$ for $\tau > 0$ and $\mu \in \mathbb{R}$ if there exists a real number $q \in \mathbb{R}$ and $q > \mu$, such that $g(\tau) = \tau^q m(\tau)$, where $m(\tau) \in C[0, \infty)$ and $g(\tau) \in C_\mu^n$ if $g^{(n)} \in C_{\mu, n\in\mathbb{N}}$.

Definition: As defined in [26], The Mittag-Leffler function with two parameters a and b is:
$E_{(a,b)}(\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{\Gamma(an+b)}, \quad a, b > 0.$

Definition: The fractional order derivative (Caputo sense) of $h(\tau)$ (discussed [26]):

$$\frac{\partial^\alpha \sigma(\tau)}{\partial \tau^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^\tau (\tau - \Omega)^{n-\alpha-1} \sigma^n(\Omega)\, d\Omega, & n-1 < \alpha < n \\ \sigma^{(n)}(\tau), & \alpha = n \end{cases}$$

where $n \in \mathbb{N},\ \tau > 0,\ \frac{\partial^\alpha}{\partial \tau^\alpha}$ is fractional type derivative.

Definition: In [26], A function $h : [0, \infty) \to \mathbb{R}$ is said to be of exponential order $\alpha\,(\alpha > 0)$ if there exists a constant $M > 0$ such that for some $\tau_0 \geq 0, \quad |\sigma(\tau)| \leq M e^{\alpha\tau}, \quad$ for every $\tau \geq \tau_0$.

2.1 FORMABLE TRANSFORM

The formable transform for the function $\sigma(\tau)$ is formally defined as (see [26]):
$F\{\sigma(\tau)\} = B(s, u) = s \int_0^\infty e^{-s\tau} \sigma(u\tau) d\tau,$
$s > 0,\ u > 0,\ \tau \in [0, \infty).$
Formable transform of derivative is given below:

$$F\left[\sigma^{(n)}(\tau)\right] = \left(\frac{s}{u}\right)^n B(s, u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{n-k} \sigma^{(k)}(0), \quad n = 0, 1, 2, \ldots$$

Formable transform of the Caputo fractional derivative of order

$$n - 1 < \alpha \leq n$$

of function $\sigma(\tau)$ is given by

$$F\left[D_\tau^\alpha \sigma^{(n)}(\tau)\right] = \left(\frac{s}{u}\right)^\alpha B(s, u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{\alpha-k} \sigma^{(k)}(0)$$

Formable transform of some functions:

$$F(1) = 1, \quad F(\tau) = \frac{u}{s}, \quad F\left(\frac{\tau^n}{n!}\right) = \left(\frac{u}{s}\right)^n, \quad n \in \mathbb{N}$$

$$F(1)=1,\quad F(\tau)=\frac{u}{s},\quad F\left(\frac{\tau^n}{n!}\right)=\left(\frac{u}{s}\right)^n,\quad n\in\mathbb{N}$$

$$F\left(\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right)=\frac{u^\alpha}{s^\alpha},\quad n-1<\alpha<n$$

$$F(e^{a\tau})=\frac{s}{s-au},\quad F(\sin(a\tau))=\frac{asu}{s^2+a^2u^2}$$

$$F[\cos(a\tau)]=\frac{s^2}{s^2+a^2u^2}$$

2.2 INTRODUCTION OF HE's HPM

The homotopy perturbation method combines classical perturbation and homotopy techniques to overcome traditional limitations. To demonstrate its application in solving nonlinear differential equations, consider a differential equation (see [3])

$$\hat{A}(\eta)-f(r)=o,\quad r\in\Omega \tag{2.1}$$

Let the boundary condition is,

$$B(\sigma,\frac{\partial\sigma}{\partial n})=0,\quad r\in\Gamma$$

Here the differential operator is denoted as $\hat{A}$ boundary operator as B, $f(r)$ is known analytic function, Ω represents the domain with boundary Γ. Now $\hat{A}$ is divide into L, which is linear and N, which is non-linear. Now (1), is expressed as follow:

$L(\sigma)+N(\sigma)-f(r)=0$ Develop a homotopy $\hat{w}(r,p):\Omega\times[0,1]\to\mathbb{R}$, and satisfies

$$H(w,p)=(1-p)[L(w)-L(\sigma_0)]+\rho[\hat{A}(w)-f(r)]=0 \tag{2.2}$$
$$p\in[0,1],\quad r\in\Omega$$

Here, $p\in[0,1]$ is embedding parameter, and η_0 is initial approximation of equation which satisfies boundary conditions. The solution of equation (2.2), can be written as:

$$w=w_0+pw_1+p^2w_2+\cdots$$

Let p=1, the resulting approximation for equation (2.1), is:

$$\sigma=\lim_{p\to1}w=w_0+w_1+w_2+\cdots$$

2.3 CONVERGENCE ANALYSIS OF HOMOTOPY PERTURBATION METHOD

In this section, we have explored the theorems that illustrate the convergence of HPM (see [4], [5]).

Theorem: Let $\mathcal{H}$ and $\mathcal{K}$ be Banach spaces, consider a mapping $\Phi:\mathcal{H}\to\mathcal{K}$ that is contractive and non-linear, and for all v, $\tilde{v}\in\mathcal{H}$,

$$\|\Phi(v)-\Phi(\tilde{v})\|\le\gamma\|v-\tilde{v}\|,\quad 0<\gamma<1$$

As per the Banach fixed point theorem, the mapping ϕ possesses a unique fixed point u that is $\Phi(u)=u$ In the context of homotopy perturbation method

$V_n=\Phi(V_{n-1}),\quad V_{n-1}=\sum_{i=0}^{n-1}u_i,\quad n=1,2,3,\ldots$

Assume $V_0=v_0=u_0\in B_r(u)$ where $B_r(u)=\{u^*\in X:\|u^*-u\|<r\}$ then

(i): $\|V_n-u\|\le\gamma^n\|v_0-u\|$

(ii): $V_n\in B_r(u)$

(iii): $\lim_{n\to\infty}V_n=u$

Proof:

(i):By utilizing an inductive approach with the base case when $n=1$, we have:

$$\|V_1 - u\| = \|\Phi(V_0) - \Phi(u)\| \leq \gamma \|v_0 - u\|,$$

Assume that for $n = k$, the following holds:

$$\|V_{n-1} - u\| \leq \gamma^{n-1} \|v_0 - u\|,$$

Then for $n = k + 1$, we have:

$$\|V_n - u\| = \|\Phi(V_{n-1}) - \Phi(u)\| = \gamma \|V_{n-1} - u\| = \gamma^n \|v_0 - u\|.$$

Using the assumption (i):

$$\|V_n - u\| \leq \gamma^n \|v_0 - u\| \leq \gamma^n r < r.$$

$$\Rightarrow V_n \in B_r(u).$$

(ii) : Because of $\|V_n - u\| \leq \gamma^n \|v_0 - u\|$ and $\lim_{n\to\infty} \gamma^n = 0, \quad \lim_{n\to\infty} \|V_n - u\| = 0,$

that is $\lim_{n\to\infty} V_n = u.$

3 METHODOLOGY

Here we discuss the formable transform homotopy perturbation method to solve 3D time fractional Klein-Gordon equation and 3D time fractional sine-Gordon equation

3.1 Implementation on 3D time fractional Klein-Gordon equation

$$D_t^\alpha \{\sigma(x, y, z, \tau)\} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} + a\sigma(x, y, z, \tau) - b\sigma^2(x, y, z, \tau), \tag{3.1}$$

Initial conditions are:

$$\sigma(x, y, 0) = f_1(x, y, z), \quad \sigma_\tau(x, y, 0) = f_2(x, y, z),$$

where

$$D_\tau^\alpha = \frac{\partial^\alpha}{\partial \tau^\alpha}, \quad 1 < \alpha \leq 2, \quad \tau > 0.$$

By the solution of formable transform HPM, first of all apply formable transform on each side of eq. (3)

$$F[D_\tau^\alpha \sigma(x, y, z, \tau)] = F[\sigma_{xx} + \sigma_{yy} + \sigma_{zz} + a\sigma(x, y, z, \tau) + b\sigma^2(x, y, z, \tau)]$$

Use differential property of Caputo fractional derivative of Formable transform

$$\left(\frac{s}{u}\right)^\alpha B(s, u) - \left(\frac{s}{u}\right)^\alpha \sigma_0 - \left(\frac{s}{u}\right)^{\alpha-1} \sigma_\tau = F[\sigma_{xx} + \sigma_{yy} + \sigma_{zz} + a\sigma(x, y, z, \tau) + b\sigma^2(x, y, z, \tau)],$$

$$F[\sigma(x, y, z, \tau)] = f_1(x, y, z) + \frac{u}{s} f_2(x, y, z) + \left(\frac{u}{s}\right)^\alpha F[\sigma_{xx} + \sigma_{yy} + \sigma_{zz} + a\sigma(x, y, z, \tau) + b\sigma^2(x, y, z, \tau)]$$

Apply inverse formable transform

$$\sigma(x, y, z, t) = f_1(x, y, z) + t f_2(x, y, z) + F^{-1} \left(\frac{u}{s}\right)^\alpha F[\sigma_{xx} + \sigma_{yy} + \sigma_{zz} + a\sigma(x, y, z, \tau) + b\sigma^2(x, y, z, \tau)]$$

Now apply homotopy perturbation method on each side of above equation

$$\sum_{n=0}^{\infty} p^n \sigma_n(x,y,z,\tau) = f_1(x,y,z) + \tau f_2(x,y,z)$$
$$+ F^{-1}\left(\frac{u}{s}\right)^{\alpha} F\left[\left(D_x^2 + D_y^2 + D_z^2\right)\sum_{n=0}^{\infty} p^n \sigma_n + a\sum_{n=0}^{\infty} p^n \sigma_n + b\sum_{n=0}^{\infty} p^n H_n\right], \quad (3.2)$$

Here we tackle the non-linear term with He's polynomial

$$N[u(x,y,z,\tau)] = \sum_{n=0}^{\infty} p^n H_n(\sigma),$$

Here $H_n(\eta)$ is He's polynomial [22] and is given as:

$$H_n(\eta_0,\eta_1,\eta_2,\dots,\eta_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{n} p^i \sigma_i\right)\right]\Bigg|_{p=0}, \quad n = 0,1,2,3,\dots$$

Equation (3.2) is combined form of formable transform and homotopy perturbation method. Compare the coefficients associated with corresponding indices of p,

$$p^0: \quad \sigma_0(\xi,\eta,z,\tau) = f_1(\xi,\eta,z) + \tau f_2(\xi,\eta,z),$$
$$p^1: \quad \sigma_1(\xi,\eta,z,\tau) = F^{-1}\left(\frac{u}{s}\right)^{\alpha} F\left[(\sigma_0)_{\xi\xi} + (\sigma_0)_{\eta\eta} + (\sigma_0)_{zz} + \sigma_0 + H_0\right],$$
$$p^2: \quad \sigma_2(\xi,\eta,z,\tau) = F^{-1}\left(\frac{u}{s}\right)^{\alpha} F\left[(\sigma_1)_{\xi\xi} + (\sigma_1)_{\eta\eta} + (\sigma_1)_{zz} + \sigma_1 + H_1\right],$$
$$p^3: \quad \sigma_3(\xi,\eta,z,\tau) = F^{-1}\left(\frac{u}{s}\right)^{\alpha} F\left[(\sigma_2)_{\xi\xi} + (\sigma_2)_{\eta\eta} + (\sigma_2)_{zz} + \sigma_2 + H_2\right].$$

Continuing the process, the solution is:

$$\sigma(\xi,\eta,z,\tau) = \lim_{p\to 1} \sigma_n(\xi,\eta,z,\tau)$$

This implies

$$\sigma(\xi,\eta,z,\tau) = \sigma_0(\xi,\eta,z,\tau) + \sigma_1(\xi,\eta,z,\tau) + \sigma_2(\xi,\eta,z,\tau) + \cdots$$

3.2 Implementation of FTHPM on 3D time fractional Sine-Gordon equation

$$D_\tau^\alpha \sigma(\xi,\eta,z,\tau) + D_t^{\alpha-1}\sigma(\xi,\eta,z,\tau) = \sigma_{\xi\xi} + \sigma_{\eta\eta} + \sigma_{zz} + \sin(\sigma) \quad (3.3)$$

Initial conditions are

$$\sigma(\xi,\eta,0) = f_3(\xi,\eta,z), \quad \sigma_\tau(\xi,\eta,0) = f_4(\xi,\eta,z)$$

where

$$D_\tau^\alpha = \frac{\partial^\alpha}{\partial \tau^\alpha}, \quad 1 < \alpha \le 2, \quad \tau > 0$$

By the solution of formable transform HPM, first of all apply formable transform on each side of eq. (3.3)

$$F\left[D_t^\alpha \sigma(\xi,\eta,z,\tau)\right] = F\left[\sigma_{\xi\xi} + \sigma_{\eta\eta} + \sigma_{zz} + \sin\sigma\right]$$

$F[D_t^\alpha \sigma(\xi,\eta,z,\tau)] = F[\sigma_{\xi\xi} + \sigma_{\eta\eta} + \sigma_{zz} + \sin\sigma]$

Use differential property of Caputo fractional derivative of Formable transform

$$B(s,u) - \left(\frac{s}{u}\right)^{\alpha}\sigma_0 - \left(\frac{s}{u}\right)^{\alpha-1}\sigma_\tau + \left(\frac{s}{u}\right)^{\alpha-1} B(s,u) - \left(\frac{s}{u}\right)^{\alpha-1}\sigma_0 = F[\sigma_{\xi\xi} + \sigma_{\eta\eta} + \sigma_{zz} + \sin\sigma],$$

$$B(s,u) = \left(\frac{s}{s+u}\right)\left[f_3(\xi,\eta,z) + f_4(\xi,\eta,z)\right] + \left(\frac{u}{s}\right)^{\alpha}\left(\frac{s}{s+u}\right) F[\sigma_{\xi\xi} + \sigma_{\eta\eta} + \sigma_{zz} - \sin\sigma].$$

Apply inverse formable transform

$$\sigma(x,y,z,\tau) = e^{-t}\left[f_3(\xi,y,z) + f_4(\xi,y,z)\right] + F^{-1}\left(\frac{u}{s}\right)^{\alpha}\left(\frac{s}{s+u}\right) F[\sigma_{\xi\xi} + \sigma_{yy} + \sigma_{zz} - \sin\sigma],$$

Apply homotopy perturbation method

$$\sum_{n=0}^{\infty} p^n \sigma_n = e^{-\tau}\left[f_3(\xi,y,z) + f_4(\xi,y,z)\right]$$
$$+ pF^{-1}\left(\frac{u}{s}\right)^{\alpha}\left(\frac{s}{s+u}\right) F\left[\left(D_\xi^2 + D_y^2 + D_z^2\right)\sum_{n=0}^{\infty} p^n\sigma_n + \sum_{n=0}^{\infty} p^n H_n\right] \quad (3.4)$$

Here we tackle the non-linear term with He's polynomial

$$N[\sigma(x,y,z,t)] = \sum_{n=0}^{\infty} p^n H_n(\sigma)$$

Here $H_n(\sigma)$ is the He's polynomial [22] and is given as:

$$H_n(\sigma_0,\sigma_1,\sigma_2,\ldots,\sigma_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{n} p^i\sigma_i\right)\right]_{p=0}, \quad n = 0,1,2,3,\ldots$$

Equation (3.4) is combined form of formable transform and homotopy perturbation method. Compare the coefficients associated with corresponding indices of p,

$p^0 : \sigma_0(\xi,\upsilon,z,\tau) = e^{-t}\left[f_3(\xi,\upsilon,z) + f_4(\xi,\upsilon,z)\right],$
$p^1 : \sigma_1(x,y,z,t) = F^{-1}\left(\frac{u}{s}\right)^{\alpha}\left(\frac{s}{s+u}\right) F\left[(\sigma_0)_{xx} + (\sigma_0)_{yy} + (\sigma_0)_{zz} + H_0\right],$
$p^2 : \sigma_2(x,y,z,t) = F^{-1}\left(\frac{u}{s}\right)^{\alpha}\left(\frac{s}{s+u}\right) F\left[(\sigma_1)_{xx} + (\sigma_1)_{yy} + (\sigma_1)_{zz} + H_1\right],$
$p^3 : \sigma_3(x,y,z,t) = F^{-1}\left(\frac{u}{s}\right)^{\alpha}\left(\frac{s}{s+u}\right) F\left[(\sigma_2)_{xx} + (\sigma_2)_{yy} + (\sigma_2)_{zz} + H_2\right],$

Continuing the process, the solution is:

$$\sigma(x,y,z,t) = \lim_{p\to 1} p^n\sigma_n(x,y,z,t),$$

This implies

$$\sigma(x,y,z,t) = \sigma_0(x,y,z,t) + \sigma_1(x,y,z,t) + \sigma_2(x,y,z,t) + \cdots$$

3.3 Implementation on 3D time fractional Rosenau-Hyman equation

$$D_t^{\alpha}\sigma(x,y,z,t) + a\left(\frac{\partial}{\partial x}(\sigma^n) + \frac{\partial}{\partial y}(\sigma^n) + \frac{\partial}{\partial z}(\sigma^n)\right) + \left(\frac{\partial^3}{\partial x^3}(\sigma^n) + \frac{\partial^3}{\partial y^3}(\sigma^n) + \frac{\partial^3}{\partial z^3}(\sigma^n)\right) = 0, \quad (3.5)$$

with the initial condition

$$\sigma(x,y,0) = f_5(x,y,z)$$

.

Where

$$D_t^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}},\ 0 < \alpha \leq 1,\ t > 0.$$

By the solution of formable transform HPM, first of all apply formable transform on each side of eq. (3.5)

$F[D_t^{\alpha}\sigma(x,y,z,t)] = -F\left[a(D_x + D_y + D_z)\sigma^n + (D_{xxx} + D_{yyy} + D_{zzz})\sigma^n\right],$

Use differential property of Caputo fractional derivative of Formable transform

$$\left(\frac{s}{u}\right)^{\alpha} B(s,u) - \left(\frac{s}{u}\right)^{\alpha}\sigma_0 = -F\left[a(D_x + D_y + D_z)\sigma^n + (D_{xxx} + D_{yyy} + D_{zzz})\sigma^n\right],$$

$$B(s,u) = f_5(x,y,z) - \left(\frac{u}{s}\right)^{\alpha} F\left[a(D_x + D_y + D_z)\sigma^n + (D_{xxx} + D_{yyy} + D_{zzz})\sigma^n\right].$$

Apply inverse formable transform

$$\sigma(x,y,z,\nu) = f_5(x,y,z) - F^{-1}\left(\frac{u}{s}\right)^{\alpha} F\left[a(D_x + D_y + D_z)\sigma^n + (D_{xxx} + D_{yyy} + D_{zzz})\sigma^n\right],$$

Apply homotopy perturbation method

$$\sum_{n=0}^{\infty} p^n \sigma_n = f_5(x,y,z) - pF^{-1}\left(\frac{u}{s}\right)^{\alpha} F\left[\sum_{n=0}^{\infty} p^n H_n\right], \tag{3.6}$$

Here we tackle the non-linear term with He's polynomial

$$N[\sigma(x,y,z,t)] = \sum_{n=0}^{\infty} p^n H_n(\sigma),$$

Here $H_n(\sigma)$ is the He's polynomial [22] and is given as:

$$H_n(\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{n} p^i \sigma_i\right)\right]_{p=0}, \quad n = 0,1,2,3,\ldots$$

Equation (3.6) is combined form of formable transform and homotopy perturbation method. Compare the coefficients associated with corresponding indices of p,

$$p^0: \quad \sigma_0(x,y,z,t) = f_5(x,y,z),$$

$$p^1: \quad \sigma_1(x,y,z,t) = F^{-1}\left(\frac{u}{s}\right)^{\alpha} F\left[(H_0)_x + (H_0)_y + (H_0)_z + (H_0)_{xxx} + (H_0)_{yyy} + (H_0)_{zzz}\right],$$

$$p^2: \quad \sigma_2(x,y,z,t) = -F^{-1}\left(\frac{u}{s}\right)^{\alpha} F\left[(H_1)_x + (H_1)_y + (H_1)_z + (H_1)_{xxx} + (H_1)_{yyy} + (H_1)_{zzz}\right],$$

$$p^3: \quad \sigma_3(x,y,z,t) = -F^{-1}\left(\frac{u}{s}\right)^{\alpha} F\left[(H_2)_x + (H_2)_y + (H_2)_z + (H_2)_{xxx} + (H_2)_{yyy} + (H_2)_{zzz}\right].$$

Continuing the process, the solution is:

$$\sigma(x,y,z,\tau) = \lim_{p\to 1} p^n \sigma_n(x,y,z,\tau).$$

This implies

$$\sigma(x,y,z,\tau) = \sigma_0(x,y,z,\tau) + \sigma_1(x,y,z,\tau) + \sigma_2(x,y,z,\tau) + \ldots$$

4 Numerical Experiments

In this section we have presented some numerical experiments which will help us to show the efficacy of proposed scheme.

Example 4.1. The 3D time fractional Klein-Gordon equation is:

$$D_t^{\alpha}\sigma(x,y,z,\tau) = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} - \sigma^2 + x^2y^2z^2\tau^2$$

with initial conditions are:

$$\sigma(x,y,z,0) = 0, \quad \sigma_{\tau}(x,y,z,0) = xyz, \quad \text{where } 1 < \alpha \le 2$$

Note: It is known as Klein-Gordon equation when $\alpha = 2$.
Solution: From equation no. (3.2)

$$\sum_{n=0}^{\infty} p^n \sigma_n(\xi,\eta,z,\tau) = f_1(\xi,\eta,z) + \tau f_2(\xi,\eta,z)$$
$$+ \mathcal{F}^{-1}\left(\frac{u}{s}\right)^{\alpha} \mathcal{F}\left[\left(D_\xi^2 + D_\eta^2 + D_z^2\right)\sum_{n=0}^{\infty} p^n\sigma_n + a\sum_{n=0}^{\infty} p^n\sigma_n + b\sum_{n=0}^{\infty} p^n H_n\right] \quad (4.1)$$

This implies

$$\sum_{n=0}^{\infty} p^n \sigma_n = \xi\eta z\tau + p\mathcal{F}^{-1}\left(\frac{u}{s}\right)^{\alpha} \mathcal{F}\left[\left(D_\xi^2 + D_\eta^2 + D_z^2\right)\sum_{n=0}^{\infty} p^n\sigma_n - \sum_{n=0}^{\infty} p^n H_n + \xi^2\eta^2 z^2\tau^2\right],$$

Comparing the like powers of p on each side

$$p^0 : \sigma_0 = \xi\eta z\tau,$$
$$p^1 : \sigma_1 = \mathcal{F}^{-1}\left(\frac{u}{s}\right)^{\alpha}\mathcal{F}\left[\sigma_{\xi\xi} + \sigma_{\eta\eta} + \sigma_{zz} - H_0 + \xi^2\eta^2z^2\tau^2\right],$$
$$H_0 = \sigma_0^2 = \xi^2\eta^2z^2\tau^2.$$

p^1 : $\sigma_1 = 0$. and so on. Therefore, the solution is
$U(\xi,\eta,z,\tau) = \sigma_0 + \sigma_1 + \sigma_2 + \cdots$
$U(\xi,\eta,z,\tau) = \xi\eta z\tau$, which is exact solution.

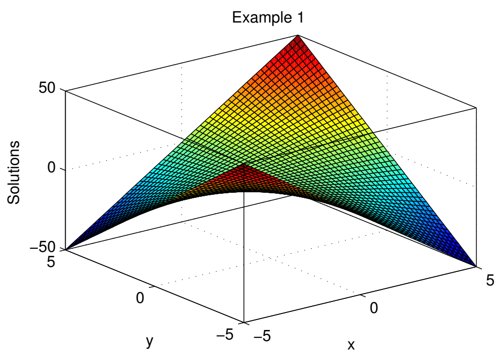

Figure 1. Physical behavior of solutions of Example 1 for $t = 1$ and $z = 2$.

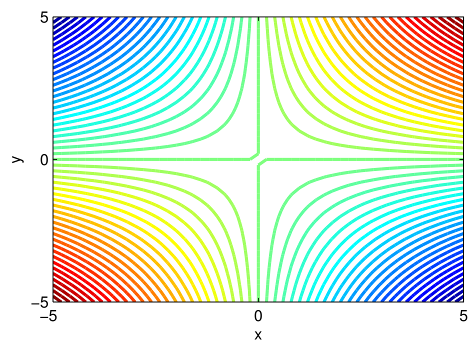

Figure 2. Contour diagram for solutions of Example 1 for $t = 1$ and $z = 2$.

Figure 1 and Figure 2 show the physical behavior of the solutions and the contour diagram of the solutions of Example 1 at $t = 1$ and $z = 2$ respectively.

Example 4.2. The 3D time fractional Klein-Gordon equation is:

$$D_t^\alpha \sigma(\xi,\eta,z,\tau) = \sigma_{\xi\xi} + \sigma_{\eta\eta} + \sigma_{zz} - \sigma + 2(\sin\xi + \sin\eta + \sin z)$$

with initial conditions are: $\sigma(\xi,\eta,z,0) = \sin\xi + \sin\eta + \sin z + 1, \quad \sigma_\tau(\xi,\eta,z,0) = 0$, where $1 < \alpha \leq 2$.

Note: It is known as Klein-Gordon equation when $\alpha = 2$.

Solution: From equation no. (3.2)

$$\sum_{n=0}^{\infty} p^n \sigma_n(\chi,\upsilon,z,\tau) = f_1(\chi,\upsilon,z) + \tau f_2(\chi,\upsilon,z)$$
$$+ \mathcal{F}^{-1}\left(\frac{u}{s}\right)^{\alpha} \mathcal{F}\left[\left(D_\chi^2 + D_\upsilon^2 + D_z^2\right)\sum_{n=0}^{\infty} p^n\sigma_n + a\sum_{n=0}^{\infty} p^n\sigma_n + b\sum_{n=0}^{\infty} p^n H_n\right] \quad (4.2)$$

This implies

$$\sum_{n=0}^{\infty} p^n \sigma_n = \chi y z \tau$$
$$+ p\mathcal{F}^{-1}\left(\frac{u}{s}\right)^{\alpha} \mathcal{F}\left[(D_\chi^2 + D_\upsilon^2 + D_z^2)\sum_{n=0}^{\infty} p^n\sigma_n - \sum_{n=0}^{\infty} p^n\sigma_n + 2(\sin\chi + \sin\upsilon + \sin z)\right] \quad (4.3)$$

Comparing the like powers of p on each side:

$$p^0: \quad \sigma_0(\xi,\upsilon,z,\tau) = \sin\xi + \sin\upsilon + \sin z + 1,$$
$$p^1: \quad \sigma_1(\xi,\upsilon,z,\tau) = \mathcal{F}^{-1}\left(\frac{u}{s}\right)^{\alpha}\mathcal{F}\left[(\sigma_0)_{\xi\xi} + (\sigma_0)_{\upsilon\upsilon} + (\sigma_0)_{zz} - \sigma_0 + 2(\sin\xi + \sin\upsilon + \sin z)\right],$$
$$\sigma_1 = -\frac{\tau^{\alpha}}{\Gamma(\alpha+1)},$$
$$p^2: \quad \sigma_2(\xi,\upsilon,z,\tau) = \mathcal{F}^{-1}\left(\frac{u}{s}\right)^{\alpha}\mathcal{F}\left[(\sigma_1)_{\xi\xi} + (\sigma_1)_{\upsilon\upsilon} + (\sigma_1)_{zz} - \sigma_1\right],$$
$$\sigma_2 = \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$p^3: \quad \sigma_3(\xi,\upsilon,z,\tau) = \mathcal{F}^{-1}\left(\frac{u}{s}\right)^{\alpha}\mathcal{F}\left[(\sigma_2)_{\xi\xi} + (\sigma_2)_{\upsilon\upsilon} + (\sigma_2)_{zz} - \sigma_2\right],$$
$$\sigma_3 = -\frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)},$$
$$p^4: \quad \sigma_4(\xi,\upsilon,z,\tau) = \mathcal{F}^{-1}\left(\frac{u}{s}\right)^{\alpha}\mathcal{F}\left[(\sigma_3)_{\xi\xi} + (\sigma_3)_{\upsilon\upsilon} + (\sigma_3)_{zz} - \sigma_3\right],$$
$$\sigma_4 = -\frac{\tau^{4\alpha}}{\Gamma(4\alpha+1)},$$
$$\vdots$$

and so on. Therefore, the solution is

$$U(\xi,\upsilon,z,\tau) = \sigma_0 + \sigma_1 + \sigma_2 + \cdots,$$
$$U(\xi,\upsilon,z,\tau) = \sin\xi + \sin\upsilon + \sin z + \cos\tau.$$

which is exact solution.

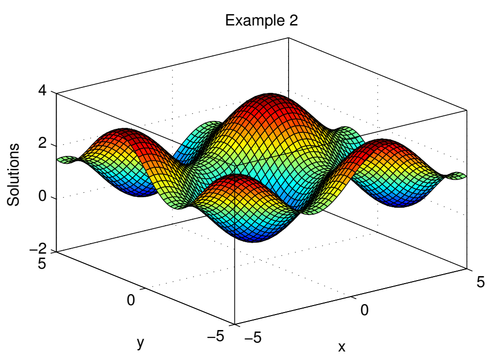

Figure 3. Physical behavior of solutions of Example 2 for $t = \pi/3$ and $z = \pi/2$.

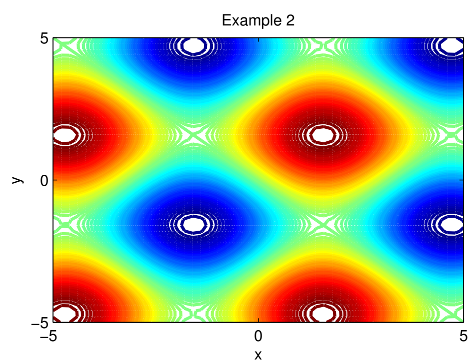

Figure 4. Contour diagram for solutions of Example 2 for $t = \pi/3$ and $z = \pi/2$.

Figure 3 and Figure 4 show the physical behavior of the solutions and the contour diagram of the solutions of Example 2 at $t = \pi/3$ and $z = \pi/2$ respectively.

Example 4.3. The 3D time fractional sine-Gordon equation is:

$$D_t^{\alpha}\sigma(\xi,\upsilon,z,\tau)+D_t^{\alpha-1}\sigma(\xi,\upsilon,z,\tau)=\sigma_{\xi\xi}+\sigma_{\upsilon\upsilon}+\sigma_{zz}-\sin\sigma$$
$$+3\pi^2e^{-\tau}\sin(\pi\xi)\sin(\pi\upsilon)\sin(\pi z)+\sin(e^{-\tau}\sin(\pi\xi)\sin(\pi\upsilon)\sin(\pi z)), \quad (4.4)$$

with initial conditions are:

$$\sigma(\xi,\upsilon,z,0)=\sin(\pi\xi)\sin(\pi\upsilon)\sin(\pi z),$$
$$\sigma_\tau(\xi,\upsilon,z,0)=-\sin(\pi\xi)\sin(\pi\upsilon)\sin(\pi z),$$

where $1<\alpha\le 2$. Note: It is known as sine-Gordon equation when $\alpha=1$

Solution: From equation no. (3.4)

$$\sum_{n=0}^{\infty}p^n\sigma_n=e^{-\tau}\left[f_3(\xi,\upsilon,z)+f_4(\xi,\upsilon,z)\right]$$
$$+p\mathcal{F}^{-1}\left(\frac{u}{s}\right)^{\alpha}\left(\frac{s}{s+u}\right)\mathcal{F}\left[\left(D_\xi^2+D_\upsilon^2+D_z^2\right)\sum_{n=0}^{\infty}p^n\sigma_n+\sum_{n=0}^{\infty}p^nH_n\right] \quad (4.5)$$

This implies

$$\sum_{n=0}^{\infty}p^nu_n=e^{-\tau}\sin(\pi\xi)\sin(\pi\upsilon)\sin(\pi z)$$
$$+p\mathcal{F}^{-1}\left(\frac{u}{s}\right)^{\alpha}\left(\frac{s}{s+u}\right)\mathcal{F}\Bigg[\left(D_\xi^2+D_\upsilon^2+D_z^2\right)\sum_{n=0}^{\infty}p^n\sigma_n-\sum_{n=0}^{\infty}p^nH_n$$
$$+3\pi^2e^{-\tau}\sin(\pi\xi)\sin(\pi\upsilon)\sin(\pi z)+\sin\left(e^{-\tau}\sin(\pi\xi)\sin(\pi\upsilon)\sin(\pi z)\right)\Bigg] \quad (4.6)$$

Comparing the like powers of p on each side:

$$p^0:\sigma_0=e^{-\tau}\sin(\pi\xi)\sin(\pi\upsilon)\sin(\pi z),$$
$$p^1:\sigma_1=\mathcal{F}^{-1}\left(\frac{u}{s}\right)^{\alpha}\left(\frac{s}{s+u}\right)\mathcal{F}\left[K_1-H_0+3\pi^2e^{-\tau}K_2+\sin\left(e^{-\tau}K_2\right)\right],$$

where

$$K_1=(\sigma_0)_{\xi\xi}+(\sigma_0)_{\upsilon\upsilon}+(\sigma_0)_{zz}, K_2=\sin(\pi\xi)\sin(\pi\upsilon)\sin(\pi z).$$
$$(\sigma_0)_{\xi\xi}+(\sigma_0)_{\upsilon\upsilon}+(\sigma_0)_{zz}=-3\pi^2e^{-\tau}\sin(\pi\xi)\sin(\pi\upsilon)\sin(\pi z),$$
$$H_0=\sin(\sigma_0)=\sin\left(e^{-\tau}\sin(\pi\xi)\sin(\pi\upsilon)\sin(\pi z)\right),$$
$$\sigma_1=0,$$
$$p^2:\sigma_2=\mathcal{F}^{-1}\left(\frac{u}{s}\right)^{\alpha}\left(\frac{s}{s+u}\right)\mathcal{F}\left[(\sigma_1)_{\xi\xi}+(\sigma_1)_{\upsilon\upsilon}+(\sigma_1)_{zz}-H_1\right],$$
$$(\sigma_1)_{\xi\xi}+(\sigma_1)_{\upsilon\upsilon}+(\sigma_1)_{zz}=0,$$
$$H_1=\sigma_1\cos(\sigma_0)=0,$$
$$\sigma_2=0,$$
$$p^3:\sigma_3=0,$$

and so on. Therefore, the solution is

$$U(\xi, \upsilon, z, \tau) = \sigma_0 + \sigma_1 + \sigma_2 + \cdots,$$
$$U(\xi, \upsilon, z, \tau) = e^{-\tau} \sin(\pi\xi) \sin(\pi\upsilon) \sin(\pi z).$$

which is exact solution.

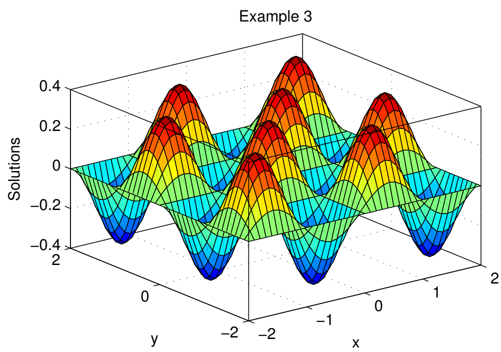

Figure 5. Physical behavior of solutions of Example 3 for $t = 1$ and $z = 1/2$.

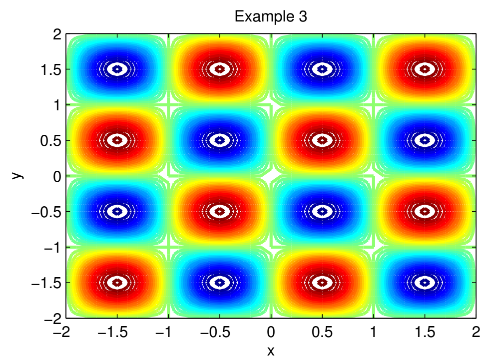

Figure 6. Contour diagram for solutions of Example 3 for $t = 1$ and $z = 1/2$.

Figure 5 and Figure 6 show the physical behavior of the solutions and the contour diagram of the solutions of Example 3 at $t = 1$ and $z = 1/2$ respectively.

Example 4.4. The 3D time-fractional Rosenau-Hyman equation is:

$$D_t^\alpha \sigma(\xi, \upsilon, z, \tau) + (\sigma^2)_x + (\sigma^2)_\upsilon + (\sigma^2)_z + (\sigma^2)_{\xi\xi\xi} + (\sigma^2)_{\upsilon\upsilon\upsilon} + (\sigma^2)_{zzz} = 0$$

Initial conditions:

$$\sigma(\xi, \upsilon, z, 0) = (\xi + \upsilon + z)$$

Where $0 < \alpha \leq 1$.

Note: It is known as Rosenau-Hyman equation when $\alpha = 1$.

Solution: From equation no. (3.6)

Apply homotopy perturbation method after applied initial condition and inverse formable transform,

$$\sum_{n=0}^{\infty} p^n \sigma_n = (\xi + \upsilon + z)$$
$$- pF^{-1}\left(\frac{u}{s}\right)^\alpha F\left[(D_\xi + D_\upsilon + D_z)\sum_{n=0}^{\infty} p^n H_n + (D_{\xi\xi\xi} + D_{\upsilon\upsilon\upsilon} + D_{zzz})\sum_{n=0}^{\infty} p^n H_n\right], \quad (4.7)$$

Comparing the like powers of p on each side

$$p^0 = \sigma_0 = (\xi + \upsilon + z),$$
$$p^1 = \sigma_1 = F^{-1}\left(\frac{u}{s}\right)^\alpha F\left[(H_0)_\xi + (H_0)_\upsilon + (H_0)_z + (H_0)_{\xi\xi\xi} + (H_0)_{\upsilon\upsilon\upsilon} + (H_0)_{zzz}\right],$$
$$H_0 = (\sigma_0)^2 = (\xi + \upsilon + z)^2,$$
$$p^1 = \sigma_1 = -6(\xi + \upsilon + z)\frac{\tau^\alpha}{\Gamma(\alpha + 1)},$$
$$p^2 = \sigma_2 = F^{-1}\left(\frac{u}{s}\right)^\alpha F\left[(H_1)_\xi + (H_1)_\upsilon + (H_1)_z + (H_1)_{\xi\xi\xi} + (H_1)_{\upsilon\upsilon\upsilon} + (H_1)_{zzz}\right],$$
$$H_1 = -12(\xi + \upsilon + z)^2\tau,$$

$$p^2 = \sigma_2 = 72(\xi + \upsilon + z)\frac{\tau^{\alpha+1}}{\Gamma(\alpha+2)},$$

$$p^3 = \sigma_3 = F^{-1}\left(\frac{u}{s}\right)^{\alpha} F\left[(H_2)_\xi + (H_2)_\upsilon + (H_2)_z + (H_2)_{\xi\xi\xi} + (H_2)_{\upsilon\upsilon\upsilon} + (H_2)_{zzz}\right],$$

$$H_2 = 108(\xi + \upsilon + z)^2\tau^2,$$

$$p^3 = \sigma_3 = -1296(\xi + \upsilon + z)\frac{\tau^{\alpha+2}}{\Gamma(\alpha+3)},$$

and so on then solution is

$$U(\xi, \upsilon, z, \tau) = \sigma_0 + \sigma_1 + \sigma_2 + \cdots,$$

$$U(\xi, \upsilon, z, \tau) = (\xi + \upsilon + z)(1 + 6\tau)^{-1}.$$

which is exact solution for $\alpha = 1$.

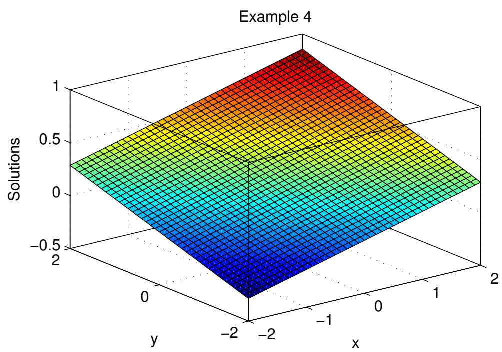

Figure 7. Physical behavior of solutions of Example 4 for $t = 1$ and $z = 2$.

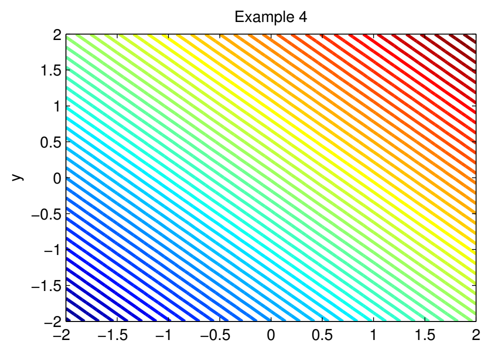

Figure 8. Contour diagram for solutions of Example 4 for $t = 1$ and $z = 2$.

Figure 7 and Figure 8 show the physical behavior of the solutions and the contour diagram of the solutions of Example 4 at $t = 1$ and $z = 2$ respectively.

5 Conclusion remarks

In this work, we utilized the Formable Transform-Homotopy Perturbation Method (STHPM) to address 3D fractional models. This technique successfully tackled the difficulties posed by time-fractional components, offering a reliable and efficient solution. By simplifying the computational process and ensuring convergence towards the exact solution, the method shows strong potential for wider use in fractional differential equations. Future research may extend its application to higher-dimensional fractional systems and more complex coupled models. In summary, using FTHPM for the 3D-TFKGE, 3D-TFSGE, and 3D-TFRHE confirms its effectiveness and paves the way for further applications in other fractional equations.

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