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ON LIGHTLIKE HYPERSURFACES OF SASAKI-LIKE STATISTICAL SPACE FORM

Pankaj Pandey, Ejaz Sabir Lone and Komal

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Corresponding Author: Ejaz Sabir Lone

Abstract The primary objective of this paper is to investigate the geometric properties of lightlike hypersurfaces $(E, \bar{h})$ within the framework of the Sasaki-like statistical space form $(\bar{E}(c))$, where c represents the constant curvature of the ambient space. Lightlike hypersurfaces, which are hypersurfaces with degenerate metrics, play a significant role in the study of Lorentzian and semi-Riemannian geometry, and their behavior within statistical manifolds is a subject of particular interest.

To this end, we derive and analyze the Gauss and Codazzi equations for lightlike hypersurfaces $(E, \bar{h})$ in the context of the Sasaki-like statistical manifold $(\bar{E}, h)$. These classical differential geometric formulae allow us to explore the intrinsic and extrinsic geometry of the hypersurfaces and provide insights into the curvature relations between the hypersurface and the ambient manifold. Furthermore, we demonstrate that it is not possible to construct a lightlike hypersurface of $(\bar{E}(c))$ that admits both a parallel screen distribution and a parallel second fundamental form. This result imposes a strong geometric constraint on the existence of such hypersurfaces, implying that certain desirable geometric structures cannot be realized under these conditions. Finally, we prove that the existence of a lightlike hypersurface in $(\bar{E}(c))$ is only possible when the curvature c takes the specific values $c = 1$ or $c = 5$. For any other values of c, lightlike hypersurfaces cannot exist, thereby establishing a critical link between the curvature of the ambient statistical space form and the geometric properties of lightlike hypersurfaces.

1 Introduction

In differential geometry, the theory of lightlike hypersurfaces in pseudo-Riemannian manifolds is a highly significant and intriguing area of research. This field has garnered considerable attention due to the unique and challenging nature of lightlike submanifolds compared to other types of submanifolds such as timelike, spacelike, and degenerate submanifolds. The classification of these submanifolds depends on the structure of the induced metric on the tangent space $(T_P\bar{E})$ of the ambient manifold.

In the case of lightlike hypersurfaces, the induced metric is degenerate, which makes their study substantially different from and more complex than the non-degenerate theory of semi-Riemannian manifolds. A crucial distinction between lightlike hypersurfaces and non-degenerate hypersurfaces is that, for lightlike hypersurfaces, the normal vector bundle and the tangent vector bundle have a non-trivial intersection. Moreover, in the case of a degenerate metric on a hypersurface, the tangent bundle contains the normal vector bundle, adding to the complexity of the geometry.

The theory of lightlike submanifolds has been extensively studied in the context of mathematical physics and general relativity, particularly due to its applications in the study of event horizons of black holes, such as the Kruskal and Kerr black holes. The concept of degenerate, particularly lightlike-geometry, in semi-Riemannian manifolds was introduced by Duggal and Bejancu in 1966 [9], where they presented an extrinsic approach to differential geometry. This breakthrough sparked widespread interest among researchers in exploring the lightlike geometry of semi-Riemannian manifolds.

Over time, several authors have contributed to the study of lightlike hypersurfaces in pseudo-Riemannian manifolds. For further reading, refer to [4], [18], and [11], among other references. Notably, N. Aktan [2] investigated the lightlike hypersurfaces of indefinite Sasakian space forms and indefinite Kenmotsu space forms [1], proving the non-existence of lightlike hypersurfaces in these specific manifolds.

On the other hand, the geometry of statistical manifolds lies at the intersection of several areas of research, including Information Geometry (IG), Affine Differential Geometry, and Hessian Geometry. This interdisciplinary nature of statistical manifold theory has opened new avenues for research, blending concepts from differential geometry with those from information theory and statistics. Deeper under-standings and geometrical approaches to families of statistical models are provided by IG and is related to study the statistical approach in this field differential geometry.

Information geometry (IG), found large number of applications in finance, chemistry, physics and biology. The main aim of IG is to use different types of geometrical tools, to acquire information from different types of statistical models. Event Horizon Telescope (EHT), in April (2019) by taking use of deep learning algorithms, released the shadow of black hole and is the direct resulting evidence of existence of theory of general relativity and black holes.

Amari in (1985), introduced the concept of statistical manifolds [3] and proved that in terms of geometrical properties of Riemannian manifolds, there exists statistical relationship between families of probability densities. This study shows intrinsic properties of Riemannian manifolds. In (1989), Vos [27] gave the notation and introduced the concept of statistical submanifolds in differential geometry. Oguzhan Bahadir and Mukut Mani Tripathi [6] conducted a significant study on the lightlike hypersurfaces of statistical manifolds, where they demonstrated that the tangent bundle $S(TE)$ possesses a canonical statistical structure. Their work revealed that a lightlike hypersurface of a statistical manifold does not inherit the statistical properties with respect to the induced connections, thereby presenting a notable divergence from the typical behavior of non-lightlike hypersurfaces in such settings.

Later, Oguzhan Bahadir [5] expanded on this research by exploring lightlike hypersurfaces in indefinite Sasakian statistical manifolds. He established certain relationships between the induced geometrical objects and dual connections on a lightlike hypersurface within an indefinite Sasakian manifold, offering new insights into the interaction between the statistical structure and the geometry of such hypersurfaces.

Vandana Rani and Jasleen Kaur [23] introduced the study of lightlike hypersurfaces in indefinite Kähler statistical manifolds, contributing to the growing body of work on the geometry of statistical manifolds. In a subsequent paper [24], the same authors focused on the lightlike geometry of indefinite Kähler statistical manifolds and analyzed the characteristics of these hypersurfaces in terms of their Cauchy-Riemannian structure. This work provided a deeper understanding of how the Cauchy-Riemann lightlike submanifold influences the overall geometry of the manifold.

The study of lightlike hypersurfaces, also referred to as lightlike hypersurfaces, is particularly challenging due to the degenerate nature of the induced metric. In pseudo-Riemannian manifolds, these hypersurfaces are a special class of submanifolds where the normal and tangent spaces intersect in a non-trivial way. The interplay between the ambient pseudo-Riemannian geometry and the induced geometry on the hypersurface introduces a number of complexities that distinguish this theory from the study of non-degenerate hypersurfaces.

A more detailed and thorough discussion of the fundamental ideas and techniques employed in this section can be found in the works of [11, 10, 25, 6, 27, 3, 20, 21, 26, 16, 19]. These references provide comprehensive insights into both the intrinsic and extrinsic geometry of Sasaki-like statistical manifolds and lightlike hypersurfaces, covering key results and methods that are essential for the understanding of this subject.

1.1 Sasaki-like Statistical Manifolds

Let $(\bar{E}, h)$ be the pseudo-Riemannian manifold, B^0 and B^1 be affine but torsion-free connections with torsion tensors T^{B^0} and T^{B^1} respectively. The pair (B^0, h) $((B^0, h))$ is said to be the statistical structure on a smooth manifold [27] if it satisfies following conditions

$$(B^o_G h)(H, I) - (B^o_H h)(G, I) = h(T^{B^0}(G, H), I) \tag{1.1}$$

$$T^{B^0} = 0 \tag{1.2}$$

$\forall G, H, I \in \Gamma(T\bar{E})$. Then the manifold $(\bar{E}, h)$ endowed with statistical structure (B^0, h), (respectively $((B^0, h))$) is supposedly a statistical manifold [27] if it satisfies

$$Ih(G, H) = h(B^0_I G, H) + h(G, B^1_I H) \tag{1.3}$$

$\forall G, H, I \in \Gamma(T\bar{E})$. In statistical manifolds we have [27]
1. The connections B^0 and B^1 are known to be as dual/ conjugate connections.
2. If structure (B^0,h) is statistical , so is (B^1,h) on $\bar{E}$.
3. For conjugate or dual connections B^0 and B^1, we have

$$B^{0*} = \frac{1}{2}(B^0 + B^1) \tag{1.4}$$

where B^{0*} denotes Levi-Civita connections on the Riemannian manifold $(\bar{E}, h)$.
4. U^0 and K^1 are curvature tensor fields of conjugate connections B^0 and B^1 respectively, satisfy the following condition [27]

$$h(K^1(G, H)I, O) = -h(I, U^0(G, H)O) \tag{1.5}$$

Let $(\bar{E})$ and (E) are $(2n + 1)$ and $n-$ dimensional Riemannian manifold and hypersurface respectively, then the Gauss formula [27] are as

$$B^0_G H = B_G H + \tau(G, H) \tag{1.6}$$

$$B^1_G H = B^*_G H + \tau^*(G, H). \tag{1.7}$$

Here"τ^*, τ represents bi-linear, symmetric and imbedding curvature tensors of submanifold $(E, \bar{h})$ in manifold $(\bar{E}, h)$. The corresponding Gauss equations, in relation to connections B^0 and B^1 respectively are [27]

$$\begin{aligned} h(U^0(G, H)I, O) =& h(R(G, H)I, O) + h(\tau(G, I), \tau^1(H, O)) - \\ & h(\tau^1(G, O), \tau(H, I)) \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} h(K^1(G, H)I, O) =& h(R(G, H)I, O) + h(\tau^*(G, I), \tau(H, O)) - \\ & h(\tau(G, O), \tau^*(H, I)) \end{aligned} \tag{1.9}$$

On an odd-dimensional manifold $(\bar{E}, h)$, let us represent $(1, 1)$-tensor field as ϕ, ξ as the associated vector field and η as a 1-form, fulfilling the following properties

$$\eta(\xi) = 1$$

$$\phi^2 G = -G + \eta(G)\xi$$

for any $G \in \Gamma(T\bar{E})$ and $\eta(G) = h(G, \xi)$.
An odd-dimensional manifold $(\bar{E}, h)$ in contact geometry, is almost contact metric-like manifold [25] if it has an almost contact metric structure (ϕ, ξ, h) on $(\bar{E}, h)$, that satisfy

$$h(\phi G, H) + h(G, \phi^* H) = 0 \tag{1.10}$$

for any $G, H \in \Gamma(T\bar{E})$ and ϕ^* is also $(1,1)$-type tensor field.
An odd dimensional almost contact metric like manifold $(\bar{E}, h)$ satisfying, [25] the following conditions is known to be as the Sasaki-like statistical manifold $(\bar{E}, B^0, h, \phi, \xi)$.

$$B^0_G\xi = -\phi G$$
$$(B^0_G\phi)H = h(G,H)\xi - \eta(H)G, \tag{1.11}$$

where $G, H \in \Gamma(T\bar{E})$. The sectional curvature σ, in any space is the $\phi-$ sectional curvature. where σ denotes sectional curvature of a $\phi-$section Let $(\bar{E}, B^0, h, \phi, \xi)$ have constant ϕ - sectional curvature c, then by virtue of [25], $\bar{R}^0$ and $(\bar{R}^1)$ of $(\bar{E}, h)$ is given by see[25]

$$\begin{aligned} U^0(G,H)I =& \frac{c+3}{4}[h(H,I)G - h(G,I)H] \\ &+ \frac{c-1}{4}[h(\phi H, I)\phi G - h(\phi G, I)\phi H \\ &- h(\phi G, H)\phi I + h(G, \phi H)\phi I - h(H,\xi)h(I,\xi)G \\ &+ h(G,\xi)h(I,\xi)H + h(H,\xi)h(I,G)\xi - h(G,\xi)h(H,I)\xi] \end{aligned} \tag{1.12}$$

where $G, H, I \in T\bar{E}$. By interchanging ϕ for ϕ^* in the above equation we can obtain curvature tensor K^1 as given by

$$\begin{aligned} K^1(G,H)I =& \frac{c+3}{4}[h(H,I)G - h(G,I)H] \\ &+ \frac{c-1}{4}[h(\phi^* H, I)\phi^* G - h(\phi^* G, I)\phi^* H \\ &- h(\phi^* G, H)\phi^* I + h(G, \phi^* H)\phi^* I - h(H,\xi)h(I,\xi)G \\ &+ h(G,\xi)h(I,\xi)H + h(H,\xi)h(I,G)\xi - h(G,\xi)h(H,I)\xi]". \end{aligned}$$

2 Light-like Hypersurfaces

Let $(\bar{E}, h)$ be an $(n+2)$-dimensional pseudo-Riemannian differentiable manifold, where the index of h is $q \geq 1$. Let $(E, \bar{h})$ be a hypersurface of $(\bar{E}, h)$, with the induced metric $h = \bar{h}|_E$. If the induced metric $\bar{h}$ on $(E, \bar{h})$ is degenerate, the hypersurface $(E, \bar{h})$ is known as a lightlike hypersurface (see also [9, 11, 10, 8]). On such hypersurfaces, there exists a lightlike, non-zero vector field $\xi \neq 0$ such that

$$\bar{h}(\xi, G) = 0, \tag{2.1}$$

for all $G \in \Gamma(T\bar{E})$. In the context of the degenerate geometry of manifolds, the radical space (or null space) of the tangent space T_GE, denoted $RadT_GE$, is defined as follows for every $G \in (E, \bar{h})$ [9]:

$$RadT_GE = \{\xi \in RadT_GE : h_G(\xi, G) = 0, G \in \Gamma(T\bar{E})\}. \tag{2.2}$$

The nullity degree of the induced metric $\bar{h}$ is defined as the dimension of $RadT_GE$, and for lightlike hypersurfaces, this nullity degree is 1. The radical distribution $RadTE$ is spanned by the null vector field, while the screen bundle $S(TE)$ is the complementary vector bundle to $RadTE$ in TE.

It“is important to note that $S(TE)$ is non-degenerate, and its orthogonal complement, $S(TE)^\perp$, is a complementary vector bundle of rank 2, known as the screen transversal bundle. Since the radical distribution $RadTE$ is a null bundle, there exists a local section M of $S(TE)^\perp$, such that

$$h(G, M) = h(M, M) = 0, \quad h(N, M) = 1. \tag{2.3}$$

The pair (N, M) forms a local frame field of the screen transversal bundle $S(TE)^\perp$, with M being transversal to the degenerate hypersurface E. As a result, the tangent bundle of the manifold decomposes as follows"

$$T\bar{E} = TE \oplus ltr(TE) = S(TE) \oplus RadTE \oplus ltr(TE), \tag{2.4}$$

where $\oplus$ denotes a direct sum, though this sum is not orthogonal ([9, 11]).

From equation (2.1), we have

$$B^0_G H = B_G H + h(G, H), \tag{2.5}$$

$$B^0_G M = -S_M G + B^t_G M. \tag{2.6}$$

where (3.3) and (3.3) are the Gauss and Weingartan formulae respectively. Substituting

$$P(G, H) = h(h(G, H), N), \tag{2.7}$$

and

$$\tau(G) = h(B^t_G M, N), \tag{2.8}$$

we get the following forms of the equations:

$$B^0_G H = B_G H + P(G, H)M, \tag{2.9}$$

$$B^0_G M = -S_M G + \tau(G)M, \tag{2.10}$$

for any $G, H \in \Gamma(TE)$, where $M \in \Gamma(ltr(TE))$, $-S_M G, B_G H \in \Gamma(TE)$, and $h(G, H), B^t_G M \in \Gamma(ltr(TE))$. Here, S_M and P represent the shape operator and second fundamental form, respectively, and B and B^t are linear connections on the lightlike hypersurface E and $ltr(TE)$, respectively [9].

From equation (2.6), B can be interpreted as an induced connection on the lightlike hypersurface E, such that

$$(B_G h)(H, I) = P(G, H)\eta(I) + P(G, I)\eta(H), \tag{2.11}$$

for all $G, H, I \in \Gamma(TE)$. Therefore, B is a non-metric connection induced on the smooth manifold E. Considering the projection P^0, the local Gauss and Weingarten formulas are given by:

$$\nabla_G P^0 H = \nabla^0_{P^0} H + C(G, P^0 H)\xi, \tag{2.12}$$

$$B_G \xi = -S^0_\tau G + \tau(G)\xi, \tag{2.13}$$

where $\nabla_G P^0 H$ and $S^0_\tau G \in S(TE)$. Here, S^0, C, and ∇^0 represent the local shape operator, second fundamental form, and induced connection on the screen bundle $S(TE)$, respectively.

The shape operators P and C on the degenerate hypersurface are related as follows:

$$\bar{h}(S_M G, P^0 H) = C(G, P^0 H), \quad \bar{h}(S_M G, M) = 0, \tag{2.14}$$

$$\bar{h}(S^0_\xi G, P^0 H) = P(G, P^0 H), \quad \bar{h}(S^0_\xi G, \xi) = 0, \tag{2.15}$$

for all $G, H \in \Gamma(TE)$. Iet U^0 and K^1 be the curvature tensors for the connections B^0 and B^1, respectively. We then have the following relations (see [9]):

$$\begin{aligned} U^0(G, H)I = {} & R(G, H)I + S_{h(G,I)}H - S_{h(H,I)}G \\ & + (B_G h)(H, I) - (B_H h)(G, I), \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} K^1(G, H)I = {} & R(G, H)I + S_{h(G,I)}H - S_{h(H,I)}G \\ & + (B_G h)(H, I) - (B_H h)(G, I), \end{aligned}$$

where B^0 and B^1 are linear connections on U^0 and K^1, respectively.

3 Geometric Properties of Null Hypersurfaces in Sasaki-like Statistical Manifolds

Let $(E,\ \bar{h},\ B,\ B^*)$ be lightlike-hypersurface of Sasaki-like statistical space form $\bar{E}(c)$. If $N \in Rad(TE)$ i,e. radical bundle, then $h(\phi N, N) = 0$. Hence we have the following equation

$$\begin{aligned} &h(\phi M, N) = -h(M, \phi N) = 0, \\ &h(\phi M, M) = 0, \\ &(\phi M, \phi N) = 1 \end{aligned} \tag{3.1}$$

P, the second fundamental form is self reliant on screen distribution, so we have

$$P(., N) = 0 \tag{3.2}$$

On screen bundle $S(TE)$ of degenerate hypersurface E, suppose P^0 be the projection and consider $Y = \phi M \in \phi(ltr(TE))$, as degenerate vector field, then for any $G \in \Gamma(TE)$ can be represented as

$$G = SG + QG, \quad Qx = Yu(G)$$
$$\phi G = \phi(SG) + \phi Y u(G)$$

put $\phi^* G = \phi(SG)$ in the previous equation, we obtain

$$\phi G = \phi^* G - Mu(G) \tag{3.3}$$

where ϕ and ϕ^* are $(1-1)-$ tensor fields on Sasaki-like statistical manifold $\bar{E}$ and lightlike hypersurface E respectively. Now again applying ϕ, we get

$$\phi^2 G = \phi\phi^* G - u(G)\phi M$$
$$\phi^2 G = -G + \eta(G)\xi - u(G)Y.$$

Let S represent the projection morphism of the tangent bundle TE onto the distribution G, and let Q represent the projection morphism of the tangent bundle TE onto the distribution G^*. Suppose u and v are differential 1-forms locally defined on the lightlike hypersurface $(E, \bar{h})$ as follows:

$$v(G) = h(G, V) \tag{3.4}$$
$$u(G) = h(G, M) \tag{3.5}$$

we note that

$$u(Y) = -1$$
$$u(H) = 0, \quad \forall H \in \Gamma(G)$$
$$\phi^2 M = \phi Y = -M \qquad \phi^2 N = \phi V = -N$$

The Gauss and Weingarten formulas of $(E, \bar{h})$ are given by

$$B^0_G H = B_G H + P(G, H).M \tag{3.6}$$
$$B^0_G M = -S_M G + \tau(G)M \tag{3.7}$$
$$B^1_G H = B^*_G H + P^*(G, H)M \tag{3.8}$$
$$B^1_G M = -S^*_M G + \tau^*(G)M \tag{3.9}$$

$\forall\ M \in \Gamma ltr(TE),\ G,\ H\ \in \Gamma(TE)$, where $B_G H,\ B^*_G H,\ S_M G$ and $S^*_N G\ \in \Gamma(TE)$ and

$$P(G, H) = h(B^0_G H, \tau), \qquad \tau(G) = h(B_G M, \tau))$$
$$P^*(G, H) = h(B^1_G H, \tau), \qquad \tau^*(G) = h(B^1_G M, \tau))$$

Where S_M and S^*_M denote the Weingarten mappings, P and P^* correspond to the second fundamental forms, and B and B^* represent the induced connections associated with the conjugate connections B^0 and B^1, respectively. By using equation (1.1) and the Gauss formula, we get [6]

$$\begin{aligned} G\bar{h}(H, I) &= \bar{h}(B^0_G H, I) + \bar{h}(H, B^1_G I) \\ &= \bar{h}(B_G H, I) + \bar{h}(H, B^*_G I) + P(G, H)\eta(I) + P^*(G, I)\eta(H) \end{aligned} \tag{3.10}$$

where η is differential $1-$form defined on lightlike hypersurface E for any $G,\ H,\ I \in \Gamma(TE)$.

4 Main Results

Lemma 4.1. *Let $(E, \bar{h}, B, B^*)$ represent a lightlike hypersurface of $\bar{E}(c)$, then the Gauss and Codazzi formulae are given by"*

$$\begin{aligned} R(G,H)I = &\frac{c+3}{4}\left[h(H,I)G - h(G,I)H\right] + \frac{c-1}{4}\left[h(\phi H,I)\phi^* G - h(\phi G,I)\phi^* H\right. \\ &-h(\phi G,H)\phi^* I + h(G,\phi H)\phi^* I - \eta(H)\eta(I)G + \eta(G)\eta(I)H \\ &\left.+\eta(H)h(I,G)\xi - \eta(G)h(I,H)\xi\right] - P(G,I)S_M H + P(H,I)S_M G, \end{aligned} \tag{4.1}$$

and the Codazzi equation as:

$$\begin{aligned} (B_G h)(H,I) - (B_H h)(G,I) = &-\frac{c-1}{4}\left[h(\phi H,I)u(G) - h(\phi G,I)u(H)\right. \\ &\left.+h(\phi G,H)u(I) - h(G,\phi H)u(I)\right] M. \end{aligned} \tag{4.2}$$

Proof. Consider $(E, \bar{h})$ as a lightlike hypersurface of $(\bar{E}, h)$, where the ϕ-sectional curvature is constant, denoted as c. From (1.10) and (2.14), we have:

$$\begin{aligned} R(G,H)I = &\frac{c+3}{4}\left[h(H,I)G - h(G,I)H\right] \\ &+ \frac{c-1}{4}\left[h(\phi H,I)\phi G - h(\phi G,I)\phi H - h(\phi G,H)\phi I + h(G,\phi H)\phi I\right. \\ &\left.-h(H,\xi)h(I,\xi)G + h(G,\xi)h(I,\xi)H + h(H,\xi)h(I,G)\xi - h(G,\xi)h(H,I)\xi\right] \\ &- S_{h(G,I)}H + S_{h(H,I)}G - (B_G h)(H,I) + (B_H h)(G,I). \end{aligned} \tag{4.3}$$

Substituting equation (3.3) into equation (4.2) yields:

$$\begin{aligned} R(G,H)I = &\frac{c+3}{4}\left[h(H,I)G - h(G,I)H\right] \\ &+ \frac{c-1}{4}\left[h(\phi H,I)\phi^* G - h(\phi G,I)\phi^* H - h(\phi G,H)\phi^* I\right. \\ &+h(G,\phi H)\phi^* I - h(\phi H,I)u(G)M - h(\phi G,I)u(H)M \\ &-h(\phi G,H)u(I)M + h(G,\phi H)u(I)M - \eta(H)\eta(I)G + \eta(G)\eta(I)H \\ &\left.+\eta(H)h(I,G)\xi - \eta(G)h(H,I)\xi\right] - P(G,I)S_M H + P(H,I)S_M G\text{''}. \end{aligned} \tag{4.4}$$

By comparing the tangential and transversal components of (4.3), we derive the final Gauss and Codazzi equations. □

Lemma 4.2. *Let $(E, \bar{h})$ be a lightlike hypersurface of $\bar{E}(c)$, then the following relation holds:*

$$h(R(G,N)I,M) = -\frac{c+3}{4}h(G,I) - \frac{c-1}{4}\left[v(I)u(\phi G) - 2v(G)u(\phi I) - \eta(I)\eta(G)\right]. \tag{4.5}$$

Proof. The result follows directly from (4.1) by substituting N for H and taking the inner product with M. Thus, we have:

$$\begin{aligned} R(G,N)I = &\frac{c+3}{4}\left[h(N,I)G - h(G,I)N\right] \\ &+ \frac{c-1}{4}\left[h(\phi N,I)\phi^* G - h(\phi G,I)\phi^* N - 2h(\phi G,N)\phi^* I\right. \\ &\left.-\eta(N)\eta(I)G + \eta(G)\eta(I)N + \eta(N)h(I,G)\xi - \eta(G)h(N,I)\xi\right] \\ &- P(G,I)S_M N + P(N,I)S_M G. \end{aligned} \tag{4.6}$$

Taking the inner product of (4.6) with M gives:

$$\begin{aligned} h(R(G,N)I,M) = & \frac{c+3}{4}\left[h(N,I)h(G,M) - h(G,I)h(N,M)\right] \\ & + \frac{c-1}{4}\left[h(\phi N,I)h(\phi^* G,M) - h(\phi G,I)h(\phi^* N,M)\right. \\ & -2h(\phi G,N)h(\phi^* I,M) - \eta(N)\eta(I)h(G,M) \\ & \left.+\eta(G)\eta(I)h(N,M) + \eta(N)h(I,G)h(\xi,M) - \eta(G)h(N,I)h(\xi,M)\right] \\ & - P(G,I)h(S_M N,M) + P(N,I)h(S_M G,M), \end{aligned} \tag{4.7}$$

which simplifies to:

$$h(R(G,N)I,M) = -\frac{c+3}{4}h(G,I) - \frac{c-1}{4}\left[v(I)u(\phi G) - 2v(G)u(\phi I) - \eta(G)\eta(I)\right]. \tag{4.8}$$

□

Lemma 4.3. *For a lightlike hypersurface $(E,\bar{h})$ in the Sasaki-like statistical manifold $(\bar{E},h)$ with constant ϕ-sectional curvature c, the second fundamental form satisfies:*

$$P(H,Y) = C(H,V), \tag{4.9}$$

for any $H \in \Gamma(TE)$.

Proof. Using the properties of the second fundamental form P, we derive:

$$\begin{aligned} P(H,\phi M) &= h(h(H,\phi M),N) \\ &= h(B^0_H \phi M,N) \\ &= -h(B^0_H M,\phi N) + h((B^0_H \phi)M,N). \end{aligned}$$

From the relations (1.9) and (2.12), we find:

$$P(H,\phi M) = -h(B^0_H M,\phi N),$$

which further simplifies to:

$$P(H,\phi M) = -h(S_M H,\phi N),$$

and consequently:

$$P(H,\phi M) = C(H,\phi N),$$

where $U = \phi M$. This concludes the proof. □

Theorem 4.4. *Let $(E,\bar{h})$ be a lightlike hypersurface of $(\bar{E},h)$ with constant ϕ-sectional curvature c. If the second fundamental form P is parallel on $(E,\bar{h})$, then we can not obtain a lightlike hypersurface for $c \neq 1$.*

Proof. Assume that a lightlike hypersurface exists in the Sasaki-like statistical manifold with $c \neq 1$. Substituting $H = N$ and $I = \phi M$ into equation (4.2), we obtain:

$$\begin{aligned} (B_G h)(N,\phi M) - (B_N h)(G,\phi M) = & -\frac{c-1}{4}\left[h(\phi N,\phi M)u(G) - h(\phi G,\phi M)u(N)\right. \\ & \left.-h(\phi G,N)u(\phi M) + h(G,\phi N)u(\phi M)\right] M. \end{aligned} \tag{4.10}$$

Since the second fundamental form is parallel, we conclude that $c = 1$. This contradicts our assumption, so there cannot exist a lightlike hypersurface for $c \neq 1$ in the Sasaki-like statistical manifold. □

Theorem 4.5. *There is no lightlike hypersurface $(E,\bar{h})$ with parallel screen distribution in the Sasaki-like statistical manifold $(\bar{E}(c))$ for $c \neq 5$.*

Proof. Let us assume that E is a lightlike hypersurface in $(\bar{E}(c))$ with parallel screen distribution. Substituting $G = N$, $H = \phi M$, and $I = \phi N$ in the curvature relation, we derive the following:

$$\begin{aligned} U^0(N, \phi M)\phi N = & \frac{c+3}{4}\left[h(\phi M, \phi N)N - h(N, \phi N)\phi M\right] \\ & + \frac{c-1}{4}\left[h(\phi^2 M, \phi N)\phi N - h(\phi N, \phi N)\phi^2 O\right. \\ & \left. - h(\phi N, \phi M)\phi^2 N + h(N, \phi^2 O)\phi^2 E\right] \\ & + \frac{c-5}{4} h(N, \xi)h(\phi N, \xi)\phi M. \end{aligned} \tag{4.11}$$

Contracting equation (4.11) with M gives:

$$h(U^0(N, \phi M)\phi N, M) = \frac{c-5}{4}. \tag{4.12}$$

From a previous result [10], we have:

$$h(U^0(G, H)P^0 I, M) = 0. \tag{4.13}$$

Using (4.13), it follows that $h(U^0(N, \phi M)\phi N, M) = 0$. Hence, we must have $c = 5$, which contradicts our assumption, proving the result. □

Lemma 4.6. *Let $(E, \bar{h})$ be a lightlike hypersurface of a Sasaki-like statistical manifold and V is the principal vector field, then the following relation holds:*

$$P(V, Y) = 0, \quad C(V, V) = 0, \tag{4.14}$$

Proof. Using equations (1.11) and (2.9), we get:

$$B^0_G Y = B^0_G \phi M,$$

and

$$B_G Y + P(G, Y)M = \phi B^0_G M + (B^0_G \phi)M.$$

Substituting equation (3.3), we derive:

$$B_G Y + P(G, Y)M = -\phi^* S_M G + u(S_M G)M + \tau(G)\phi M + h(G, M)\xi.$$

Comparing tangential and transversal components, we get (4.14):

$$B_G Y = \tau(G)\phi M + h(G, M)\xi - \phi^* S_M G,$$

and

$$P(G, U) = u(S_M G) + h(S_M G, \phi N) = C(G, V).$$

□

Lemma 4.7. *Let $(E, \bar{h})$ be a lightlike hypersurface of a Sasaki-like statistical manifold, then the Codazzi equation is:*

$$\begin{aligned} (B_G S_M)H - (B_H S_M)G = & \frac{c+3}{4}[u(H)G - u(G)H] \\ & + \frac{c-1}{4}\left[h(G, Y)\phi H - h(H, Y)\phi G\right. \\ & \left. + h(\phi G, H)Y - h(G, \phi H)Y + \eta(G)u(G)\xi\right] \\ & + \tau(G)S_M H - \tau(H)S_M G. \end{aligned} \tag{4.15}$$

Proof. Substituting $I = M$ in Lemma 1 and performing straightforward calculations, we obtain the desired result. □

Now, let $\{a_1, \ldots, a_{n-2}, \ldots, a_{2n-4}, \xi, N, \phi M, \phi N\}$ be an orthonormal basis of the tangent bundle $\Gamma(TE)$ such that:

$$\phi a_i = a_{n-2+i}, \quad \phi a_{n-2+i} = -a_i,$$

and

$$\phi\xi = 0$$

for each $i = 1, 2, \ldots, m-2$.

Lemma 4.8. *Let $(E, \bar{h})$ be a lightlike hypersurface of a Sasaki-like statistical manifold $(\bar{E}, h)$. Then we have:*

$$S_M Y = \sum_{i=1}^{2n-4} \frac{C(Y, a_i)}{e_i} a_i + C(Y, \xi)\xi + C(Y, Y)V + C(Y, V)Y, \tag{4.16}$$

and

$$S_M N = \sum_{i=1}^{2n-4} \frac{C(N, a_i)}{e_i} a_i + C(N, \xi)\xi + C(N, Y)V, \tag{4.17}$$

where (e_i) represents the signature of the basis (a_i).

Proof. From the definitions and properties of lightlike hypersurfaces in Sasaki-like statistical manifolds, we can write:

$$S_M Y = \sum_{i=1}^{2n-4} \gamma_i a_i + \lambda\xi + \beta_1 N + \beta_\phi N + \beta_3 \phi M.$$

Using equation (2.14), we get $\gamma_i = \frac{C(N, a_i)}{e_i}$, $\lambda = C(Y, \xi)$, $\beta_1 = 0$, $\beta_2 = C(Y, Y)$, and $\beta_3 = -C(Y, V)$. Thus, equation (4.16) follows, and a similar process gives equation (4.17). □

Theorem 4.9. *There does not exist a lightlike hypersurface $(E, \bar{h})$ of a Sasaki-like statistical manifold $(\bar{E}, h)$ with $c \neq 5$ such that:*

$$P(Y, Y) = 0$$

and

$$\bar{h}((B_N S_M)Y, V) = \bar{h}((B_Y S_M)N, V).$$

Proof. Substituting $H = U$ and $G = N$ into equation (4.15), we have:

$$(B_N S_N)Y - (B_Y S_N)N = \frac{c-5}{4} Y + \tau(Y) S_M N - \tau(N) S_M Y.$$

Using equations (4.16) and (4.17), we get:

$$\begin{aligned}(B_N S_M)Y - (B_Y S_N)N = \frac{c-5}{4} Y + \tau(Y) \left[\frac{C(N, a_i)}{e_i} a_i + C(N, \xi)\xi + C(N, Y)V\right] \\ - \tau(N) \left[\frac{C(Y, a_i)}{e_i} a_i + C(Y, \xi)\xi + C(Y, Y)V + C(Y, V)Y\right].\end{aligned} \tag{4.18}$$

Taking the inner product of equation (4.18) with V and using Lemma (3.4), we conclude:

$$\bar{h}((B_N S_M)Y - (B_Y S_M)N, V) = \frac{c-5}{4} - \tau(N) P(Y, Y).$$

This completes the proof. □

5 Novelty of Lightlike Hypersurfaces of Sasaki-like Statistical Spaces

The concept of lightlike hypersurfaces in a statistical space form is an interesting and relatively novel area of research that combines aspects of differential geometry, relativity, and information geometry. Here's a brief look at the novelty of this field:

(i) **Intersection of Differential Geometry and Statistics:** Statistical space forms are spaces endowed with both a Riemannian metric and a dual connection, typically arising in the study of information geometry (which involves statistical manifolds like Fisher information metrics). Investigating lightlike hypersurfaces (also known as null hypersurfaces in relativity theory) within these forms bridges two distinct fields: differential geometry and information geometry/statistical manifolds.

(ii) **Extension of Classical Lightlike Geometry:** In classical geometry, lightlike hypersurfaces arise naturally in general relativity as the surfaces where the induced metric becomes degenerate. Extending this concept to statistical manifolds opens up new avenues for understanding the geometric structure of statistical models. The novelty here is studying the behavior of such hypersurfaces in a statistical context, where connections and metrics are influenced by probabilistic interpretations.

(iii) **Application in Generalized Relativity or Information Theory:** Lightlike hypersurfaces in a statistical space form could provide new insights into generalized theories of gravity or even in quantum information theory. For example, statistical manifolds often describe families of probability distributions, and understanding lightlike geometry within such contexts may yield new tools for studying entropic properties or information propagation.

(iv) **Geometrical Properties and Curvatures:** The study of the curvature and geometric properties of lightlike hypersurfaces in statistical space forms presents a new challenge. The interaction between the induced degenerate metric and the statistical structure (such as the Fisher metric and its dual connections) may lead to new invariants, types of curvature, and interesting classifications of these hypersurfaces.

(v) **Potential Applications in Machine Learning:** Statistical manifolds are often used in machine learning, especially in the study of information-theoretic measures. Investigating lightlike hypersurfaces could provide novel insights into optimization techniques, such as gradient flow on statistical manifolds, or new ways of understanding data manifolds where certain degeneracies (lightlike directions) emerge naturally.

In summary, the study of lightlike hypersurfaces in statistical space forms is a novel endeavor because it fuses geometric concepts from relativity (null hypersurfaces) with statistical structures. This could lead to new mathematical tools and theories in both information geometry and theoretical physics.

6 Conclusion

In this paper, we have explored the geometric properties of lightlike hypersurfaces $(E, \bar{h})$ within the Sasaki-like statistical space form $(\bar{E}(c))$, focusing on the curvature-driven behavior of these hypersurfaces. By deriving and analyzing the Gauss and Codazzi equations, we have gained valuable insights into the intrinsic and extrinsic geometry of these hypersurfaces and their relationship with the ambient manifold.

A key result of our investigation is the geometric constraint on the existence of lightlike hypersurfaces in $(\bar{E}(c))$ that simultaneously admit both a parallel screen distribution and a parallel second fundamental form. We demonstrated that such hypersurfaces cannot exist, revealing significant restrictions on the types of geometric structures possible in this context.

Furthermore, our analysis has shown that lightlike hypersurfaces in $(\bar{E}(c))$ can only exist when the curvature of the ambient space form takes the specific values $c = 1$ or $c = 5$. This establishes a critical link between the ambient curvature and the geometric properties of lightlike hypersurfaces, contributing to the broader theory of hypersurfaces in statistical manifolds.

Overall, the results presented in this work deepen our understanding of the interplay between curvature and hypersurface geometry in Sasaki-like statistical manifolds, offering new avenues for future research in differential geometry and its applications to Lorentzian and semi-Riemannian structures.

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Author information

Pankaj Pandey, Department of Mathematics, Lovely Professional University, India.
E-mail: pankaj.anvarat@gmail.com

Ejaz Sabir Lone, Department of Mathematics, Geeta University Panipat, India.
E-mail: ejazsabirlone1@gmail.com

Komal, Department of Mathematics, Lovely Professional University, India.
E-mail: kamakshisharma82@gmail.com