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STUDY ON PARTITION LABELING OF GRAPHS

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Abstract This article's goal is to examine the partition of integers. Specifically, the new labeling of a graph $G = (V, E)$ labels V vertex labels assigned by an appropriate vertex function and the edge was positioned E is an integer partition of the number's difference between two vertices. This approach is referred to as Graph G Partition Labeling, as it integrates the concept of partition integers with principles of graceful labeling. Partition integers involve combinatorial complexity, and to expedite calculations, we utilized Python, a widely popular high-level programming language. The partition labeling of the U-Windmill graph and the verification of the N-Star graph were demonstrated in this research work.

1 Introduction

Graph labeling is an important topic in graph theory, and there are many challenges and resources accessible for countless types of graphs. In 1967, Rosa who first proposed the concept of Graph Labeling [2, 7]. Subject to a few restrictions, it is the process of assigning well-defined integers to the graph's vertices and edges, respectively. A wide number of applications, including computer technology, satellite communication, coding theory, data mining, and image processing, benefit greatly from labeling graphs [2].

Number theory has an intriguing subfield called the theory of partitions of numbers. In the 18th century, Leonard Euler introduced the idea of divisions. Following Euler, many other well-known mathematicians, including Gauss, Jacobi, Schur, McMahon, Andrews, etc., researched and discussed the theory of partition. A partition of the non-negative integer 'n' is a series of natural integers, known as parts, whose summation is equal to n. A sum of positive integers can be used to express the number, in general [8].

The graph utilized in this research is an N-star graph $N(S_5)$ [5, 7] where each vertex of degree one ends with a five-star graph and another graph is U – windmill graph $U(W_3^4)$ [5, 7] whose vertex of one degree ends with four windmills.

2 Literature Survey

There is an enormous literature dealing with several kinds of labeling of graph. S. Ashok Kumar and S. Maragathavalli investigated some graph families which admit Prime Labeling. In 1982, Tout.A, Dabboucy .A.N and Howalla. K, revealed a new method called prime graph labelling [27]. In 1991, Deretsky, Lee, and Mitchen has together acquainted to establish a specific sector called 'Edge Labelling'. For a new type of graph, the vertex prime labelling method is considered by M.D.M.C.P Weerarathna, T.R.D.S.M thennakoon, K.D.E Dhanarjaya, A.A.I perara [12]. J. Lissy bennet, Dr. S. Chandrakumar have been influenced by some Applicable constraints of an assignment of a whole number to the points or lines or both, compliance with conditionally [8]. Meliana Pasaribu, Yundari, and Muhammad Iyer authored a comprehensive study that shed light on the patterns of graceful labeling and Skolem graceful labeling on graphs, contributing significantly to the field of graph theory [11]. Maheswari.V, Vinoth Kumar.L, Balaji proved that three-star graph is a mean cordial labeling [13]. Rekha. S and Maheswari. V proved that cycle C_n, P_n^2, C_3^t, graph is a difference modulo labeling [22]. The idea of skolem mean labeling was initially presented by A. Submanian, Ramesh D.S.T, and Balaji V., and subsequently expanded upon and motivated by V. Maheswari, A. Rupam Burman, Gurinder Singh and Ajit Singh demonstrated that $PDOt(n)$, which keep strack of all the tagged pieces across all of n's partitions. Gabriel, Manfred & Robert used the saddle-point method to derive a central limit theorem for the number of summands in partitions [9]. Nagahara, M., Azuma, provided a concise overview of Python programming with further references for more extensive study [20]. Al-Taie, M. Z., & Kadry, S. explains in detail how to build social networks from the ground up [19]. Farrelly, C. M., & Mutombo establishes the foundation topics in graph theory and Packages of Python [6]. Using these papers as a foundation, introduced the new graph using the concept of Partition of integers, Skolem Mean labeling admits Partition labeling of graph and made use of Python Programming for quick calculation.

3 Brief Review

In this part, we'll examine the meaning of the term partition set [28]. Additionally, we define some special graphs like U- windmill graphs, N- Star graphs [5], including the friendship graph, star graph, tripartite graph, and bistar graph [23, 12].

4 Definitions

Definition 4.1 (Partition of an Integer). Let D be a positive integer. A sequence of non-negative integers $d_1, \ldots, d_k$ where $d_1 \leq d_2 \leq \ldots \leq d_k$ such that $D = d_1 + d_2 + \ldots d_k$ is called a partition of D. It is symbolized by $P\,(D)$ [27, 28, 8].

Example. Let $D = 5$, and Different Forms of D will be,

(i) 5

(ii) 4+1

(iii) 3+2

(iv) 3+1+1

(v) 2+2+1

(vi) 2+1+1+1

(vii) 1+1+1+1+1

Then, $P(5) = 7$.

Definition 4.2 (Partition Labeling of Graph). Let a graph $G = (V(g), E(g))$ where $V(g)$ and $E(g)$ is the collection of vertex set and the edge set. Look at the vertex mapping $V(g)$, labeled with distinct counting $f : V(g) \to N$. Then for every edge $E(g)$ the induced mapping $f(uv) = P[|f(u) - f(v)|]$, for all u and v belong to $E(g)$, we call the graph as Partition Labeling of Graph.

Definition 4.3 (Star Graph). If $(n-1)$ vertices merge to form a single central vertex, the resulting graph is a star graph. The kind of graph named star of order n is termed as S_n.

Definition 4.4 (Bistar Graph). It is described as uniting two versions of a star $K_{1,n}$ [12] by their tip vertices.

Definition 4.5 (Windmill Graph). It is obtained in one point uniting m copies of the cycle K_m [12].

Definition 4.6 (Tripartite Complete graph). A Tripartite Complete graph is the $k = 3$ example of a full k-partite graph with the elements k, p, q, and r is known as a tripartite complete graph [12]. A set of graph vertices divided into three separate sets such that every vertex in each group is bordered by two more graph vertices.

Definition 4.7 (U-Windmill Star graph $U(W_3^4)$). A graph generated by merging a C_3 shaped U-graph with a 4-windmill graph (W_3^4) at the point of single point is called a U-Windmill star graph [11].

Definition 4.8 (N-Star graph $N(S_5)$). The N-star graph [11] whose vertex of degree one is the merger of S_5 shaped N – with a 5-star graph (S_5).

5 Partition Labeling of Some Special Graphs

Algorithm 5.1. • Draw the Special Graph.

- Name the vertices $u_1, u_2, \ldots$.
- Define the function for the vertices.
- Find the difference between the vertices say $j = |f(u_1) - f(u_2)|$.
- Apply partition integers for the resultant say $P(j) = h(u_1, u_2)$.

Theorem 5.2. *The U-Star graph $U(W_3^4)$ [11] describes a partition labeling graph.*

Proof. Consider the two copies of W_3^4 merge on the edges of degree one with vertices u_{1i}, u_{2i}, $(i = 1, 2, 3, 4)$ and v_{1i}, v_{2i}, $(i = 1, 2, 3, 4)$ respectively with u_1v_1 as the pivot vertex and every edge $u_1u_{1i},\ u_{1i}u_{2i},\ u_1u_{2i},\ v_1v_{1i},\ v_1v_{2i},\ v_{1i}v_{2i}$, for $1 \le i \le 4$.

Then the new vertex set is $V = \{u_0,\ u_1,\ u_{1i},\ u_{2i},\ v_0,\ v_1,\ v_{1i},\ v_{2i}\}$, for $1 \le i \le 4$.
Also, the new edge set is $E = \{u_0u_1,\ u_1u_{1i},\ u_1u_{2i},\ ,\ u_0v_0,\ v_0v_1,\ v_1v_{1i},\ v_1v_{2i}\}$, for $1 \le i \le 4$.

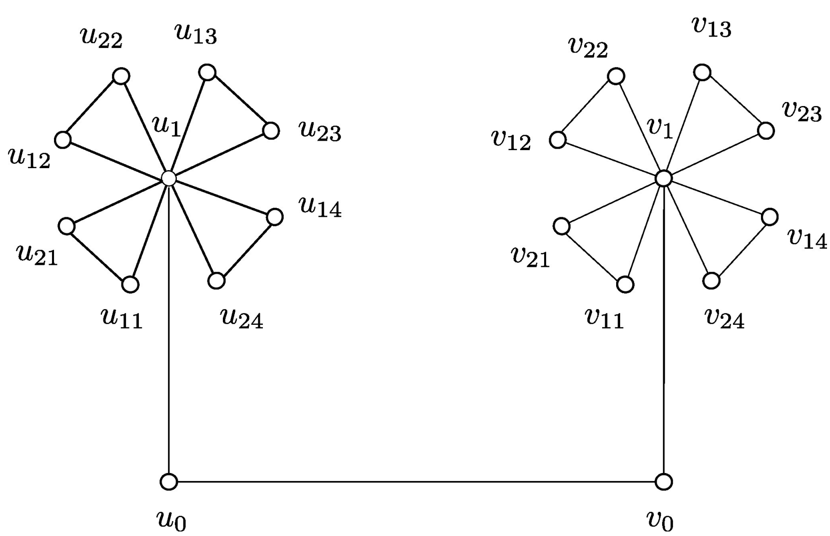

Figure 1.

The injective function $f : V \to \mathbb{N}$ such that,

$$
\begin{aligned}
f(u_0) &= 1 \\
f(u_1) &= 37 \\
f(u_{1i}) &= 4i + 5, && 1 \le i \le 4. \\
f(u_{2i}) &= 2(4i + 3), && 1 \le i \le 4. \\
f(v_0) &= 22 \\
f(v_1) &= 2 \\
f(v_{1i}) &= 3(3i + 2), && 1 \le i \le 4. \\
f(v_{2i}) &= 5(3i - 2), && 1 \le i \le 4.
\end{aligned}
$$

And an induced edge function $h : E \to \mathbb{N}$ such that,

$$h(u_0, v_0) = P(j)$$
$$\text{Where } j = |f(u_0) - f(v_0)|$$

$$h(u_0, u_1) = P(j_1)$$
$$\text{Where } j_1 = |f(u_0) - f(u_1)|$$

$$h(u_1, u_{1i}) = P(j_2)$$
$$\text{Where } j_2 = |f(u_1) - f(u_{1i})|$$

$$h(u_1, u_{2i}) = P(j_3)$$
$$\text{Whereæ}_3 = |f(u_1) - f(u_{2i})|$$

$$h(v_0, v_1) = P(j_4)$$
$$\text{Where } j_4 = |f(v_0) - f(v_1)|$$

$$h(v_1, v_{1i}) = P(j_5)$$
$$\text{Where } j_5 = |f(v_1) - f(v_{1i})|$$

$$h(v_1, v_{2i}) = P(j_6)$$
$$\text{Where } j_6 = |f(v_1) - f(v_{2i})|$$

Hence, in this labeling pattern, boundary of the U-Star graph $U(W_3^4)$ receive the partition of numbers.

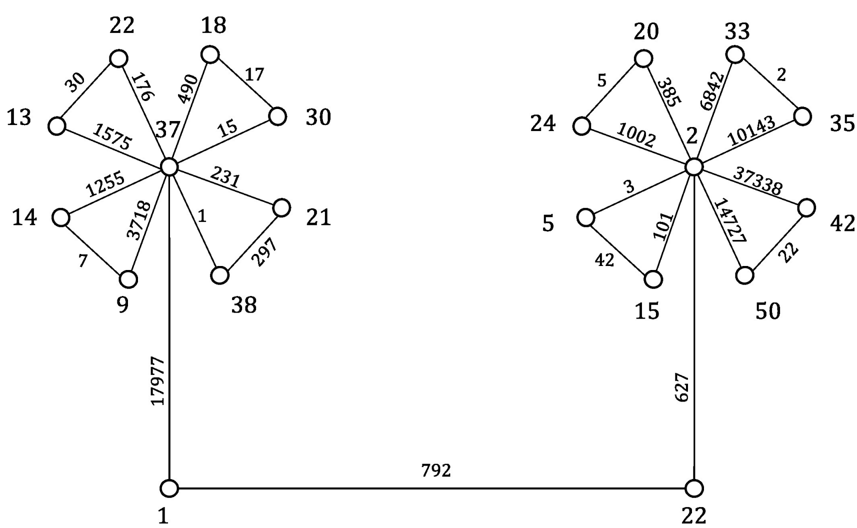

Figure 2.

□

Theorem 5.3. *The N-Star graph $N(S_5)$ [11] acknowledges the partition labeling graph.*

Proof. The N-star graph G is created by substituting a star graph S_n, an N-star partition graph with $n \geq 1$, for each edge of degree one.

Let the vertices S_n of two sets merge on one degree of a vertex, $u_{11}, \ldots, u_{1n}$ and $v_{11}, v_{12}, \ldots, v_{1n}$ respectively with u_1v_1 as the pivot vertex and every edge u_1u_{1i}, v_1v_{1i}, for $1 \leq i \leq n$, where $n \geq 1$.

The perfect vertex set is, $V = \{u_1, u_2, u_{1i}, v_1, v_2, v_{1i}\}$ for $1 \le i \le n,\ n \ge 1$.
Also, the new edge set, $E = \{u_1u_{1i},\ u_1u_2,\ u_2v_2,\ v_1u_{1i}\}$ for $1 \le i \le n$.

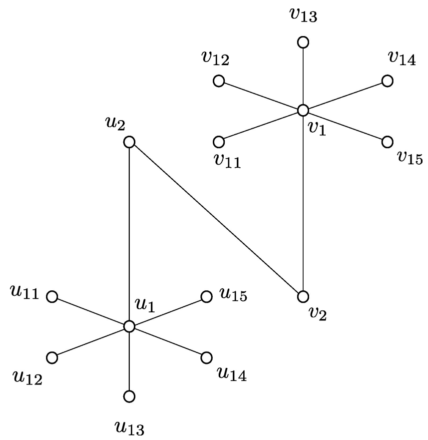

Figure 3.

State a one-one function $f : V \to \mathbb{N}$, such that,

$$
\begin{aligned}
f(u_1) &= 3\\
f(v_1) &= 4\\
f(u_2) &= 15\\
f(v_2) &= 31\\
f(u_{1i}) &= 2(8+i),\ \text{for } i = 1,2,3,4,5\\
f(v_{1i}) &= 2(i+2),\ \text{for } i = 1,2,3,4,5.
\end{aligned}
$$

Also, an induced edge function $h : E \to \mathbb{N}$ such that,

$$
\begin{aligned}
h(u_1, u_{1i}) &= P(k_1)\\
&\qquad \text{Where } k_1 = |f(u_1) - f(u_{1i})|\\
h(v_1, v_{1i}) &= P(k_2)\\
&\qquad \text{Where } k_2 = |f(v_1) - f(v_{1i})|\\
h(u_1, u_2) &= P(k_3)\\
&\qquad \text{Where } k_3 = |f(u_1) - f(u_2)|\\
h(u_2, v_2) &= P(k_4)\\
&\qquad \text{Where } k_4 = |f(u_2) - f(v_2)|\\
h(v_1, v_2) &= P(k_5)\\
&\qquad \text{Where } k_5 = |f(v_1) - f(v_2)|
\end{aligned}
$$

Hence in this way of the labeling pattern all the links of the graph N-Star $N(S_5)$ receive the partition of numbers.

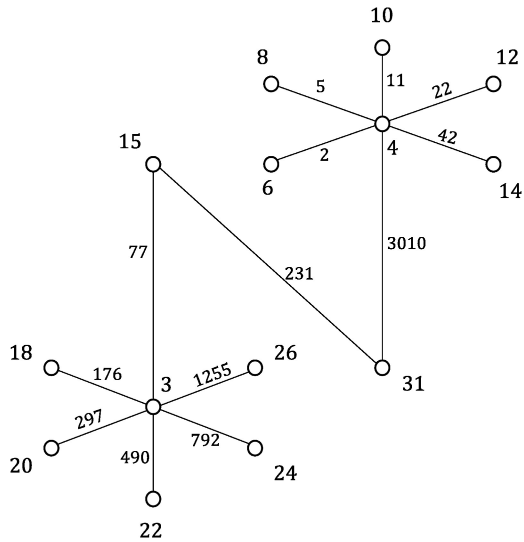

Figure 4. The N-Star Graph $N(S_5)$

Since integer partition is a combinatorial complexity, we can use Jerome Kelleher's dynamic Python programming to reduce time and space complexity. The Python program link to generate an integer partition is given below.

Source: http://jeromekelleher.net/tag/integer-partitions.html

Screen Grab of the Program:

```
def rule_asc(n):
    a = [0 for i in range(n + 1)]
    k = 1
    a[1] = n
    while k != 0:
        x = a[k - 1] + 1
        y = a[k] - 1
        k -= 1
        while x <= y:
            a[k] = x
            y -= x
            k += 1
        a[k] = x + y
        yield a[:k + 1]
```

Figure 5. Python Code

□

6 Application

Partition labeling can be applied in communication networks, especially in secure communication. The U-Windmill graph and N-Star graph can be used for cryptography, which involves the application of any type of cipher text to encrypt and decrypt the message or information.

7 Conclusion remarks

The research has proved that the partition labeling for the graph such as U-Star graph $U(W_3^4)$ and the N-Star graph $N(S5)$ admitted partition labeling. Bi-star graph $B_{5,5}$, Tripartite Complete graph $K_{1,4,2}$ and $K_{1,3,2}$, Friendship graph F_5 also admits partition labeling. In Tree graph, Wheel graph, Helm graph, Diamond graph, Triangular snake graph, and path graph this analogous work partition labeling graph cannot be carried out.

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