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DYNAMICS OF GRAPH INDUCED SHIFT SPACES

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Abstract In this paper, we study the topological dynamics of a two-dimensional shift space X_G through matrices M and N; which are indexed with allowed triangular patterns of form ${}_{a}\,{}^{c}_{b}$, ${}^{x}_{y}\,{}^{z}$ respectively. We investigate how the characteristics of matrices M and N are related to one another. We prove that if $M_{IJ} \neq 0 \Leftrightarrow N_{I_1J_1} \neq 0$ holds for all E-pairs (I, I_1) and (J, J_1), then $X_G \neq \emptyset$ and contains periodic points. We establish that for such shift spaces, $(1, 1)$-mixing notions (transitivity, doubly transitivity and weak mixing) can be studied through the block presentation of matrix M. We prove that if $M_{IJ} \neq 0 \Leftrightarrow N_{I_1J_1} \neq 0$ holds, then $(1, 1)$-weak mixing and (r, s)-weak mixing (for $rs > 0$) are equivalent. In the process, we discuss (r, s)-transitivity and (r, s)-weak mixing of shift space X_G through appropriate triangular patterns. These results provide a deeper understanding of the dynamics of shift spaces, with implications for fields such as dynamical systems, control theory, computational biology, and information theory, where the study of periodicity, mixing, and transitivity is essential for optimizing algorithms and system behavior.

1 Introduction

Symbolic dynamics plays a crucial role in analyzing the behavior of various discrete dynamical systems. By representing any discrete system as a factor of a symbolic system, researchers gain insights into the qualitative aspects of its dynamics. This enables the visualization of the underlying properties and characteristics inherent in the system. The versatility of symbolic dynamics extends beyond theoretical realms, finding practical applications in diverse fields. For instance, it is utilized in automata theory, data transmission and storage, communication systems, study of quasi-crystals, boolean control networks, computational economics and biology [8, 12, 18, 6, 4, 21]. Prior to proceeding further, we'll provide an overview of essential preliminaries.

Let $A = \{a_1, a_2, \dots a_n\}$ be a finite set of symbols with discrete topology and $A^{\mathbb{Z}^2}$, the collection of all functions $x : \mathbb{Z}^2 \to A$ be endowed with product topology. The function $d : A^{\mathbb{Z}^2} \times A^{\mathbb{Z}^2} \to \mathbb{R}^+$ defined as $d(x, y) = \frac{1}{n+1}$ (where n is the least non-negative integer such that $x \neq y$ in $R_n = [-n, n]^2$) is a metric on $A^{\mathbb{Z}^2}$. For any $v \in \mathbb{Z}^2$, the map $\sigma^v : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ defined as $\sigma^v(y)|_u = y|_{u+v}$ (for all $u \in \mathbb{Z}^2$) is a 2-dimensional shift map and is a homeomorphism. Let $S \subset \mathbb{Z}^2$, then elements of A^S are called **patterns** (over S) and elements of $A^{\mathbb{Z}^2}$ are called **configurations**. A configuration $y \in A^{\mathbb{Z}^2}$ is referred to as **periodic** if for some $v \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, $\sigma^v(y)|_u = y|_u$ for all $u \in \mathbb{Z}^2$. A subset Y of $A^{\mathbb{Z}^2}$ is called **shift-invariant** if $\sigma^v(Y) \subseteq Y$ for all $v \in \mathbb{Z}^2$. A non-empty, closed, and shift-invariant subset Y of $A^{\mathbb{Z}^2}$ is called a **subshift** (of $A^{\mathbb{Z}^2}$). Let $\mathcal{F}$ be the collection of finite patterns and let $X_{\mathcal{F}} = \{x \in A^{\mathbb{Z}^2} : x|_P \notin \mathcal{F} \text{ for all } P \subset \mathbb{Z}^2\}$, then $X_{\mathcal{F}}$ is referred to as a **2-dimensional shift of finite type (SFT)**. A pattern is considered **allowed** (or **valid**) if it does not contain any elements from $\mathcal{F}$. For any SFT X, the set of all valid patterns is represented as $B(X)$. Refer to [3, 5, 8, 9, 12, 14, 19] for details.

A 2-dimensional SFT X is **transitive** if for every pair of non-empty open sets U_1 and U_2

in X, $\sigma^n(U_1) \cap U_2 \neq \emptyset$ for some $n \in \mathbb{Z}^2$. The SFT X is **r-transitive** ($r \in \mathbb{Z}^2$) if for any pair of non-empty open sets U_1 and U_2 in X, $\sigma^{nr}(U_1) \cap U_2 \neq \emptyset$ for some $n \in \mathbb{Z}$. The SFT X is referred to as **weakly mixing** if for any pair of non-empty open sets U_i and V_i in X ($i = 1, 2$), $\sigma^n(U_i) \cap V_i \neq \emptyset$ for some $n \in \mathbb{Z}^2$. The SFT X is **r-weakly mixing** if for any pair of non-empty open sets U_i and V_i in X ($i = 1, 2$), $\sigma^{kr}(U_i) \cap V_i \neq \emptyset$ for some $k \in \mathbb{Z}$. Let $A_{m\times m}$ be a binary matrix (with entries 0 and 1), then A is termed **irreducible** if for any pair of indices i, j, $\exists\, n \in \mathbb{N}$ such that $A^n_{ij} > 0$. The matrix A is said to be **primitive** if $A^n > 0$ for some $n \in \mathbb{N}$. A 1-dimensional SFT X is referred to as **doubly transitive** if for all non-empty open sets U_1 and U_2 in X, $\exists\, p, q \in \mathbb{N}$ such that $\sigma^p(U_1) \cap U_2 \neq \emptyset$ and $\sigma^{-q}(U_1) \cap U_2 \neq \emptyset$. Further, a 1-dimensional SFT X_A is doubly transitive $\Leftrightarrow$ A is irreducible. See [5, 7, 8, 12, 22] for details.

Let X be a 1-dimensional SFT, then $X \cong X_G$ for some graph G. Let $\mathcal{H}$ and $\mathcal{V}$ be graphs with alphabet A as a common set of vertices, and if graphs $\mathcal{H}$ and $\mathcal{V}$ respectively determine the horizontal and vertical compatibility of vertices; then we denote the tuple $G = (\mathcal{H}, \mathcal{V})$ as a 2-dimensional graph. For the sake of simplicity, we represent the **2-dimensional graph** as $G = (H, V)$, where H and V are the adjacency matrices of graphs $\mathcal{H}$ and $\mathcal{V}$ respectively. It is evident that if H and V don't have any zero rows (or columns), then the shift space satisfying the condition $(HV)_{kl} \neq 0 \Leftrightarrow (VH)_{kl} \neq 0 \quad$ for all $k, l \in V(G)$ (and its one-sided implications) is always non-empty and possesses periodic points. As the imposed condition forces every triangular pattern of the form $\begin{smallmatrix} & * \\ * & * \end{smallmatrix}$, $\begin{smallmatrix} * & * \\ * & \end{smallmatrix}$ to be extended to a valid 1×1 square pattern of X_G, the condition captures the extendability of the generated (valid) patterns for X_G. But in the absence of the imposed condition, a given pattern may or may not be extended to valid rectangles of arbitrary sizes. In such cases, let $A_1 = \{\begin{smallmatrix} & c \\ a & b \end{smallmatrix} : \exists\, d \in \mathcal{V}(G)$ such that $\begin{smallmatrix} d & c \\ a & b \end{smallmatrix} \in \mathcal{B}(X_G)\}$ and $A_2 = \{\begin{smallmatrix} y & z \\ x & \end{smallmatrix} : \exists\, w \in \mathcal{V}(G)$ such that $\begin{smallmatrix} y & z \\ x & w \end{smallmatrix} \in \mathcal{B}(X_G)\}$. Next, consider two matrices M and N that are indexed with the elements of A_1 and A_2 in the following way:

For $I = \begin{smallmatrix} & a_3 \\ a_1 & a_2 \end{smallmatrix}$, $J = \begin{smallmatrix} & a_6 \\ a_4 & a_5 \end{smallmatrix}$ $R = \begin{smallmatrix} b_2 & b_3 \\ b_1 & \end{smallmatrix}$, $S = \begin{smallmatrix} b_5 & b_6 \\ b_4 & \end{smallmatrix}$,

$$M_{IJ} = \begin{cases} 1, & \text{if } a_3 = a_4 \text{ and } \begin{smallmatrix} a_3 & a_5 \\ a_2 & \end{smallmatrix} \in A_2 \\ 0, & \text{otherwise} \end{cases} \text{ and } N_{RS} = \begin{cases} 1, & \text{if } b_3 = b_4 \text{ and } \begin{smallmatrix} & b_5 \\ b_2 & b_3 \end{smallmatrix} \in A_1 \\ 0, & \text{otherwise.} \end{cases}$$

Let $I = \begin{smallmatrix} & a_3 \\ a_1 & a_2 \end{smallmatrix}$ and $J = \begin{smallmatrix} a_4 & a_3 \\ a_1 & \end{smallmatrix}$ be valid triangular patterns such that $\begin{smallmatrix} a_4 & a_3 \\ a_1 & a_2 \end{smallmatrix} \in B(X_G)$, then (I, J) is referred to as an **E-pair** and I (or J) is called the **E-partner** of pattern J (or I). We say a shift space X_G has **unique E-pairs** if every index of M and N is uniquely extended to a valid 1×1 square pattern of X_G. If M and N consist of some zero rows (or columns), then the corresponding indices do not contribute to the generation of valid configurations, and removal of such indices does not affect the configurations of X_G. Hence, we assume that M and N do not consist of any zero rows or zero columns. Let P and Q be the sequences (patterns) generated by matrices M and N respectively. If they agree on diagonal corner points, then we say P is a **complementary pattern** of Q and vice versa. See [9, 12] for details.

In 1961, Wang proposed the concept of Wang tiles and investigated their characteristics. He questioned the possibility of covering a plane with a set of tiles according to a predetermined rule. He established that for any finite set of Wang tiles, if it is possible to tile the entire plane, then there is at least one periodic tiling of the plane [23]. However, the claim was proven to be false by Robert Berger in 1966. He showed that there exist sets of Wang tiles that can tile the entire plane in an aperiodic manner [1]. Consequently, two questions emerged: one regarding tiling a plane with specific tiles adhering to a prescribed tiling rule, and the other about the possibility of tiling the plane in a periodic manner. In 1967, Robinson utilized translation, reflection, and rotations of seven square tiles with notched edges to establish that the resulting 52 tiles can tile the entire plane in an aperiodic manner. Later, he discovered six polygonal tiles (a total of 32 tiles subjected to translation, reflection, and rotations) that can tile the plane aperiodically [17]. In multidimensional shift spaces, the determination of whether a local admissible block is globally admissible is undecidable. This uncertainty impedes the exploration of the underlying topological properties of a shift space. In [19], the authors characterized a multidimensional shift of finite type using an infinite matrix. In [16], the author established that on locally finite groups, every sofic shift is an SFT, every SFT is strongly irreducible, and every SFT is entropy minimal with a unique measure of maximal entropy. In [15], the author presented a criterion for establishing an upper bound on entropy perturbation. In [2], the author studied stable algebra of matrices in the context of symbolic dynamics, examining square matrices over a semiring S and their invariance under shift equivalence and strong shift equivalence. They also investigated module theory, algebraic K-theory, and contrasted problems in characterizing spectra of nonnegative real matrices, alongside a review of key developments in shift equivalence and automorphism groups.

In [20], the authors gave an algorithmic approach to address the non-emptiness problem for multidimensional shift spaces. In [9], the authors discussed the non-emptiness problem and the existence of periodic points for a 2-dimensional shift space generated by the graph $G = (H, V)$. The authors proved that if $(HV)_{kl} \neq 0 \Leftrightarrow (VH)_{kl} \neq 0$ holds for all $k, l \in V(X_G)$, then $X_G \neq \emptyset$ and has a non-empty set of periodic points. In [13], the authors established the transitivity and weak mixing behavior of the 2-dimensional shift space X_G under the conditions $(HV)_{kl} \neq 0 \Leftrightarrow (VH)_{kl} \neq 0$ and $(HV^T)_{kl} \neq 0 \Leftrightarrow (V^T H)_{kl} \neq 0$ for all $k, l \in V(X_G)$. In [7], the authors established several notions of transitivity for 2-dimensional Dot systems. In [11], the author investigated the dynamical structure of 2-dimensional SFT through graph products of reduced complexity. In [10], the authors showed that if $(HV)_{kl} \neq 0 \Leftrightarrow (VH)_{kl} \neq 0$ holds for all $k, l \in V(X_G)$, then (r, s)-directional mixing (for $rs > 0$) can be studied through the matrix $H^r V^s$. However, in the absence of imposed conditions on adjacency matrices, the study of (r, s)-mixing notions (for $rs > 0$) through $H^r V^s$ is not guaranteed. In this paper, we provide sufficient conditions for non-emptiness, the existence of periodic points, $(1, 1)$-transitivity, and $(1, 1)$-weak mixing of the shift space X_G through a pair of matrices M and N, where both matrices are indexed with valid E-pairs. In the process, we relate $(1, 1)$-weak mixing with (r, s)-weak mixing (under imposed conditions on M and N), and we discuss directional transitivity and directional weak mixing of the shift space X_G.

2 Results

We first investigate the relation between the algebraic structure of matrices M and N. Subsequently, we examine the structure of the shift space X_G using matrices M and N. In this paper, we proceed under the assumption that both M and N do not contain any rows or columns consisting entirely of zeros.

Proposition 2.1. *M is an irreducible matrix if and only if N is an irreducible matrix.*

Proof. Let M be an irreducible matrix and $I = \begin{array}{cc} a_2 & a_3 \\ a_1 & \end{array}$, $J = \begin{array}{cc} b_2 & b_3 \\ b_1 & \end{array}$ be any given indices of matrix N. Since the matrix N does not consist of any zero row (or column), let $P = \begin{array}{cccc} & & a_4 & a_5 \\ & a_2 & a_3 & \\ a_0 & a_1 & & \\ a_{-1} & & & \end{array}$ and $Q = \begin{array}{cccc} & & b_4 & b_5 \\ & b_2 & b_3 & \\ b_0 & b_1 & & \\ b_{-1} & & & \end{array}$ be the patterns of N containing I and J respectively. If $P^* = \begin{array}{ccc} & & a_4 \\ & a_2 & a_3 \\ a_0 & a_1 & \end{array}$ and $Q^* = \begin{array}{ccc} & & b_4 \\ & b_2 & b_3 \\ b_0 & b_1 & \end{array}$ are the patterns obtained by removing the first and last symbols in patterns P and Q respectively, then it can be seen that P^* and Q^* are generated by M. As M is an irreducible matrix, the patterns P^* and Q^* can be joined by a path of M. Consequently, indices I and J can be joined through a path of N, and hence, N is an irreducible matrix.

Similarly, it can be shown that if N is an irreducible matrix, then M is also an irreducible matrix.

□

Remark 2.2. The Proposition 2.1 states that if M (or N) is an irreducible matrix, then N (or M) is also an irreducible matrix. The proof uses the fact that removal of first and last symbols from the finite sequences generated by M (or N) yields finite sequences of N (or M). It may be noted that a similar result to Proposition 2.1 holds when M is a primitive matrix. Further, if M (or N) is a permutation matrix, then N (or M) is also a permutation matrix. The proof is established by considering the fact that as M is a permutation matrix, fixing an index of M at the origin uniquely determines the rest of the indices in the sequences of M. Since the removal of first and last symbols in finite sequences of M yields finite sequences generated by N, every index of N has a unique incoming and outgoing entry. Thus, the subsequent corollaries are established.

Corollary 2.3. *M is a primitive matrix if and only if N is a primitive matrix.*

Proof. The proof is substantiated by the arguments presented in the Remark 2.2. □

Corollary 2.4. *M is a permutation matrix if and only if N is a permutation matrix. Further, M is an irreducible, permutation matrix if and only if N is an irreducible, permutation matrix.*

Proof. The proof is substantiated by the arguments presented in the Remark 2.2. □

Remark 2.5. The above corollary establishes that if M is an irreducible, permutation matrix, then N is also an irreducible, permutation matrix. It can be noted that if M and N are irreducible, permutation matrices, then two matrices have same size (or equivalently, generate 1-dimensional periodic points of same periods). The proof follows from the fact that an irreducible, permutation matrix corresponds to single periodic orbits (as 1-dimensional) and finite sequences of N (or M) can be visualized as truncation of finite sequences of M (or N). Therefore, M generates single periodic orbits of period m if and only if N generates single periodic orbits of period m.

It is worth mentioning that the matrices M and N always have the same size. This follows from the observation that every sequence of M (or N) can be interpreted as a sequence of N (or M) by simply removing the first and last symbols. Hence, the number of indices in M and N (which contribute in generation of one-dimensional bi-infinite sequences of M and N) are always equal. Furthermore, it can be noted that there might exist some SFT, where M and N initially do not appear to have the same size. However, by removing zero rows and columns (as it is assumed that M and N are void of any zero row or zero column throughout the paper), they can be reduced to matrices of the same size. We now provide an example to illustrate this claim.

Example 2.6. Let X_G be a SFT, where the generating matrices of graph $G = (H,V)$ are

$$H = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array} \qquad V = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array}$$

Then,

$$HV = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{array} \qquad VH = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{array}$$

It may be noted that there exist vertices k,l which do not satisfy the condition $(HV)_{kl} \neq 0 \iff (VH)_{kl} \neq 0$. As such vertices do not contribute towards the generation of valid configurations, setting such entries to 0 yields the following updated matrices:

$$HV = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{array} \qquad VH = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{array}$$

Next, categorize triangular patterns (which can be extended to valid 1×1 square patterns) as

$$A_1 = \left\{ \begin{smallmatrix} & 2 \\ 1 & 3 \end{smallmatrix}, \begin{smallmatrix} & 3 \\ 1 & 2 \end{smallmatrix}, \begin{smallmatrix} & 3 \\ 2 & 1 \end{smallmatrix}, \begin{smallmatrix} & 1 \\ 3 & 3 \end{smallmatrix}, \begin{smallmatrix} & 2 \\ 3 & 3 \end{smallmatrix}, \begin{smallmatrix} & 3 \\ 3 & 2 \end{smallmatrix} \right\}, \quad A_2 = \left\{ \begin{smallmatrix} 1 & 2 \\ 1 & \end{smallmatrix}, \begin{smallmatrix} 3 & 2 \\ 1 & \end{smallmatrix}, \begin{smallmatrix} 1 & 3 \\ 1 & \end{smallmatrix}, \begin{smallmatrix} 3 & 3 \\ 1 & \end{smallmatrix}, \begin{smallmatrix} 3 & 3 \\ 2 & \end{smallmatrix}, \begin{smallmatrix} 2 & 1 \\ 3 & \end{smallmatrix}, \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}, \begin{smallmatrix} 1 & 3 \\ 3 & \end{smallmatrix} \right\}$$

The corresponding matrices M and N with index set as A_1, A_2 respectively, are given as

$$M = \begin{array}{c|cccccc} & \begin{smallmatrix} & 2\\ 1 & 3\end{smallmatrix} & \begin{smallmatrix} & 3\\ 1 & 2\end{smallmatrix} & \begin{smallmatrix} & 3\\ 2 & 1\end{smallmatrix} & \begin{smallmatrix} & 1\\ 3 & 3\end{smallmatrix} & \begin{smallmatrix} & 2\\ 3 & 3\end{smallmatrix} & \begin{smallmatrix} & 3\\ 3 & 2\end{smallmatrix} \\ \hline \begin{smallmatrix} & 2\\ 1 & 3\end{smallmatrix} & 0 & 0 & 1 & 0 & 0 & 0 \\ \begin{smallmatrix} & 3\\ 1 & 2\end{smallmatrix} & 0 & 0 & 0 & 1 & 1 & 0 \\ \begin{smallmatrix} & 3\\ 2 & 1\end{smallmatrix} & 0 & 0 & 0 & 1 & 1 & 1 \\ \begin{smallmatrix} & 1\\ 3 & 3\end{smallmatrix} & 1 & 1 & 0 & 0 & 0 & 0 \\ \begin{smallmatrix} & 2\\ 3 & 3\end{smallmatrix} & 0 & 0 & 1 & 0 & 0 & 0 \\ \begin{smallmatrix} & 3\\ 3 & 2\end{smallmatrix} & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

$$N = \begin{array}{c|cccccccc} & \begin{smallmatrix} 1 & 2\\ 1 & \end{smallmatrix} & \begin{smallmatrix} 3 & 2\\ 1 & \end{smallmatrix} & \begin{smallmatrix} 1 & 3\\ 1 & \end{smallmatrix} & \begin{smallmatrix} 3 & 3\\ 1 & \end{smallmatrix} & \begin{smallmatrix} 3 & 3\\ 2 & \end{smallmatrix} & \begin{smallmatrix} 2 & 1\\ 3 & \end{smallmatrix} & \begin{smallmatrix} 1 & 2\\ 3 & \end{smallmatrix} & \begin{smallmatrix} 1 & 3\\ 3 & \end{smallmatrix} \\ \hline \begin{smallmatrix} 1 & 2\\ 1 & \end{smallmatrix} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \begin{smallmatrix} 3 & 2\\ 1 & \end{smallmatrix} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \begin{smallmatrix} 1 & 3\\ 1 & \end{smallmatrix} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \begin{smallmatrix} 3 & 3\\ 1 & \end{smallmatrix} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \begin{smallmatrix} 3 & 3\\ 2 & \end{smallmatrix} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \begin{smallmatrix} 2 & 1\\ 3 & \end{smallmatrix} & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \begin{smallmatrix} 1 & 2\\ 3 & \end{smallmatrix} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \begin{smallmatrix} 1 & 3\\ 3 & \end{smallmatrix} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

It can be observed that the first and third columns of the matrix N are zero. By removing the corresponding indices $\begin{smallmatrix} 1 & 2\\ 1 & \end{smallmatrix}$ and $\begin{smallmatrix} 1 & 3\\ 1 & \end{smallmatrix}$ (as such indices do not contribute in valid configuration of SFT), we obtain a 6×6 matrix. Furthermore, the matrix M and the updated matrix N are both of size 6×6. Thus, even if M and N initially had distinct sizes, they can be reduced to matrices of the same size through this process. This property will be utilized in the subsequent sections of this paper to extend a local pattern into a valid configuration of the SFT.

We now discuss the structure of a matrix that is not an irreducible matrix but generates a 1-dimensional transitive shift space. We say a matrix P is **semi-irreducible** if it contains precisely two irreducible, permutation sub-matrices P_1, P_2 such that for any $i \in P_1, j \in P_2$, the path connecting these indices is unique (which traverses through all the indices of $P \setminus (P_1 \cup P_2)$). We now provide an example of a semi-irreducible matrix:

Example 2.7. The matrix P is given by:

$$P = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 1 \\ 6 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Then, it can be seen that P is not an irreducible matrix (and hence X_P is not doubly transitive), but X_P is a 1-dimensional transitive shift space. Further, it can be seen that matrix P has only two irreducible, permutation sub-matrices P_1, P_2, where

$$P_1 = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & 1 \\ 2 & 1 & 0 \end{array} \qquad P_2 = \begin{array}{c|cc} & 5 & 6 \\ \hline 5 & 0 & 1 \\ 6 & 1 & 0 \end{array}$$

It is evident that any indices $k \in P_1$ and $l \in P_2$ can be connected by a unique path (which contains all the indices of $P \setminus (P_1 \cup P_2)$). Consequently, P is a semi-irreducible matrix.

Remark 2.8. The above discussion provides the characterization of a matrix that generates a 1-dimensional transitive (not doubly transitive) shift space. It can be noted that if M is a semi-irreducible matrix, then N is also a semi-irreducible matrix and vice versa. The proof relies on the observation that every finite sequence of M generates a finite sequence of N (and vice versa), thereby the sequences of M and N behave alike. We establish the result as following corollary.

Corollary 2.9. *M is a semi-irreducible matrix if and only if N is a semi-irreducible matrix.*

Proof. The proof is substantiated by the arguments presented in the Remark 2.8 □

Remark 2.10. The above discussion provides the relationship between the structure of matrices M and N. For any 2-dimensional shift space X_G, the condition $M_{IJ} \neq 0$ does not ensure that the corresponding pattern contributes to the generation of a valid configuration for X_G. Consequently, if there exist indices I, J of M such that the pattern corresponding to M_{IJ} cannot be extended to a valid 2×2 square, then we set $M_{IJ} = 0$. If the matrices M and N are processed in this manner, we refer to these updated matrices as $\mathfrak{M}$ and $\mathfrak{N}$ respectively.

Proposition 2.11. *Let $G = (H, V)$ be a graph and X_G be corresponding SFT. If the shift space X_G has unique E-pairs, then the non-emptiness problem for X_G is decidable. Further, if $X_G \neq \emptyset$, then X_G is doubly $(1,1)$-transitive if and only if $\mathfrak{M}$ is an irreducible matrix.*

Proof. Let the 2-dimensional shift space X_G has unique E-pairs and $\mathfrak{M}$, $\mathfrak{N}$ be the updated matrices. If $\mathfrak{M}$ and $\mathfrak{N}$ are zero matrices, then $X_G = \emptyset$; otherwise, every pattern of $\mathfrak{M}$ (or $\mathfrak{N}$) has its unique complementary pattern of $\mathfrak{N}$ (or $\mathfrak{M}$). Let z be a finite pattern generated by $\mathfrak{M}$ and z' be its complementary pattern of $\mathfrak{N}$, then (z, z') generates a valid pattern $P \in B(X_G)$. It can be seen that iteratively using finite sequences of $\mathfrak{M}$ and $\mathfrak{N}$, P can be extended to a valid square pattern of X_G and then to a valid configuration. Thus, the SFT X_G is non-empty and consequently, it is decidable whether $X_G \neq \emptyset$ or not.

Next, consider that X_G is doubly $(1,1)$-transitive shift space, and let I and J be any given indices of $\mathfrak{M}$. Let x, y be configurations where patterns I, J appear at the origin and U, V be the corresponding $\frac{1}{n}$-neighborhoods of configurations x, y respectively. Then, doubly $(1,1)$-transitivity of X_G ensures the existence of $k \in \mathbb{N}$ such that $\sigma^{(k,k)}(U) \cap V \neq \emptyset$ and equivalently, $\mathfrak{M}^p_{IJ} > 0$ for some $p \in \mathbb{N}$. Hence, $\mathfrak{M}$ is an irreducible matrix and the proof of the forward part is complete.

Conversely, let $\mathfrak{M}$ be an irreducible matrix, then it can be seen that $\mathfrak{N}$ is also an irreducible matrix. As the 2-dimensional shift space X_G has unique E-pairs, every finite pattern of $\mathfrak{M}$ (or $\mathfrak{N}$) can be uniquely extended to a valid square pattern of X_G. Let U and V be $\frac{1}{n}$-neighborhoods of configurations x, $y \in X_G$ respectively, and $x_n = \begin{matrix} x_{-n,n} & \cdots & x_{n,n} \\ \vdots & \vdots & \vdots \\ x_{-n,-n} & \cdots & x_{n,-n} \end{matrix}$ and $y_n = \begin{matrix} y_{-n,n} & \cdots & y_{n,n} \\ \vdots & \vdots & \vdots \\ y_{-n,-n} & \cdots & y_{n,-n} \end{matrix}$ be the central patterns of U and V respectively. If index I of $\mathfrak{M}$ appears in the top right corner of x_n and the index J of $\mathfrak{M}$ appears in the lower left corner of y_n, then the irreducibility of $\mathfrak{M}$ ensures that I and J can be connected along the $(1,1)$ direction. It can be seen that the resulting pattern P

$$\begin{array}{cccccccccc}
 & & & & & & & y_{-n,n} & \cdots & y_{n,n} \\
 & & & & & & & \vdots & \vdots & \vdots \\
 & & & & & & & y_{-n,-n} & \cdots & y_{n,-n} \\
 & & & & & & & \vdots & & \\
 & & & & & & \cdots & & & \\
 & & & & & \iddots & & & & \\
 & & & & \vdots & & & & & \in \mathcal{B}(X_G) \\
 & & & \cdots & & & & & & \\
 & & & \vdots & & & & & & \\
x_{-n,n} & \cdots & x_{n,n} & \cdots & & & & & & \\
\vdots & \vdots & \vdots & & & & & & & \\
x_{-n,-n} & \cdots & x_{n,-n} & & & & & & &
\end{array}$$

is valid for shift space X_G. Since E-pairs are unique and every sequence of $\mathfrak{M}$ and $\mathfrak{N}$ has its unique complementary sequence, pattern P can be extended to an allowed square pattern and then to a valid configuration $z \in X_G$ (if not, then there exists a pattern $\begin{smallmatrix} & & x_5 \\ y_1 & x_3 & x_4 \\ x_1 & x_2 & \end{smallmatrix}$ which cannot be extended to an allowed 2×2 square pattern; causing a contradiction to updated matrices $\mathfrak{M}$ and $\mathfrak{N}$). Since the central pattern of z is x_n and the central pattern of $\sigma^{(k,k)}(z)$ is y_n (for some $k \in \mathbb{N}$), it can be concluded that $\sigma^{(k,k)}(U) \cap V \neq \emptyset$. Consequently, X_G is doubly $(1,1)$-transitive. □

Remark 2.12. The Proposition 2.11 provides a necessary and sufficient condition for SFT X_G to exhibit doubly $(1,1)$-transitivity under imposed conditions. It is worth mentioning that a SFT can be infinite even if E-pairs are unique (Example 2.33 of [9]). Further, it is evident that X_G is $(1,1)$-transitive (but not doubly $(1,1)$-transitive) if and only if M is a semi-irreducible. The proof is established by considering the fact that the indices I, J of M get joined by a unique path, and the resulting finite sequence of M gets enclosed by a unique sequence (generated by N). Hence, any two allowed central blocks get connected by a unique path, and such a pattern gets extended uniquely to a valid configuration. Moreover, the converse is also ensured by the uniqueness of diagonal paths. Thus, the subsequent corollary is established.

Corollary 2.13. *Let $G = (H, V)$ be a graph and X_G be corresponding SFT, then the shift space X_G is $(1,1)$-transitive (but not doubly $(1,1)$-transitive) if and only if M is a semi-irreducible matrix.*

Proof. The proof is substantiated by the arguments presented in the Remark 2.12. □

We now provide one example of shift space X_G , whose dynamics can be examined through matrices M and N.

Example 2.14. Let X_G be a SFT, where the generating matrices of graph $G = (H, V)$ are

$$H = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad V = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Then,

$$HV = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{array} \qquad VH = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

It can be seen that there exist vertices k, l, where the condition $(HV)_{kl} \neq 0 \Leftrightarrow (VH)_{kl} \neq 0$ fails, setting such entries as 0 yields updated matrices:

$$HV = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array} \qquad VH = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 2 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Next, resulting triangular patterns (which can be extended to valid 1×1 square patterns) can be categorized as

$$A_1 = \{ \begin{smallmatrix} & 3 \\ 1 & 1 \end{smallmatrix}, \begin{smallmatrix} & 1 \\ 1 & 2 \end{smallmatrix}, \begin{smallmatrix} & 4 \\ 1 & 1 \end{smallmatrix}, \begin{smallmatrix} & 1 \\ 2 & 3 \end{smallmatrix}, \begin{smallmatrix} & 1 \\ 2 & 4 \end{smallmatrix}, \begin{smallmatrix} & 2 \\ 3 & 1 \end{smallmatrix}, \begin{smallmatrix} & 2 \\ 4 & 1 \end{smallmatrix} \}, \ A_2 = \{ \begin{smallmatrix} 2 & 3 \\ 1 & \end{smallmatrix}, \begin{smallmatrix} 3 & 1 \\ 1 & \end{smallmatrix}, \begin{smallmatrix} 4 & 1 \\ 1 & \end{smallmatrix}, \begin{smallmatrix} 2 & 4 \\ 1 & \end{smallmatrix}, \begin{smallmatrix} 1 & 1 \\ 2 & \end{smallmatrix}, \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}, \begin{smallmatrix} 1 & 2 \\ 4 & \end{smallmatrix} \}$$

The corresponding matrices M and N with index set as A_1, A_2 respectively, are given as

$$M = \begin{array}{c} \\ \begin{smallmatrix} & 3 \\ 1 & 1 \end{smallmatrix} \\ \begin{smallmatrix} & 1 \\ 1 & 2 \end{smallmatrix} \\ \begin{smallmatrix} & 4 \\ 1 & 1 \end{smallmatrix} \\ \begin{smallmatrix} & 1 \\ 2 & 3 \end{smallmatrix} \\ \begin{smallmatrix} & 1 \\ 2 & 4 \end{smallmatrix} \\ \begin{smallmatrix} & 2 \\ 3 & 1 \end{smallmatrix} \\ \begin{smallmatrix} & 2 \\ 4 & 1 \end{smallmatrix} \end{array} \begin{array}{c} \begin{array}{ccccccc} \begin{smallmatrix} & 3 \\ 1 & 1 \end{smallmatrix} & \begin{smallmatrix} & 1 \\ 1 & 2 \end{smallmatrix} & \begin{smallmatrix} & 4 \\ 1 & 1 \end{smallmatrix} & \begin{smallmatrix} & 1 \\ 2 & 3 \end{smallmatrix} & \begin{smallmatrix} & 1 \\ 2 & 4 \end{smallmatrix} & \begin{smallmatrix} & 2 \\ 3 & 1 \end{smallmatrix} & \begin{smallmatrix} & 2 \\ 4 & 1 \end{smallmatrix} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$$N = \begin{array}{c} \\ \begin{smallmatrix} 2 & 3 \\ 1 & \end{smallmatrix} \\ \begin{smallmatrix} 3 & 1 \\ 1 & \end{smallmatrix} \\ \begin{smallmatrix} 4 & 1 \\ 1 & \end{smallmatrix} \\ \begin{smallmatrix} 2 & 4 \\ 1 & \end{smallmatrix} \\ \begin{smallmatrix} 1 & 1 \\ 2 & \end{smallmatrix} \\ \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix} \\ \begin{smallmatrix} 1 & 2 \\ 4 & \end{smallmatrix} \end{array} \begin{array}{c} \begin{array}{ccccccc} \begin{smallmatrix} 2 & 3 \\ 1 & \end{smallmatrix} & \begin{smallmatrix} 3 & 1 \\ 1 & \end{smallmatrix} & \begin{smallmatrix} 4 & 1 \\ 1 & \end{smallmatrix} & \begin{smallmatrix} 2 & 4 \\ 1 & \end{smallmatrix} & \begin{smallmatrix} 1 & 1 \\ 2 & \end{smallmatrix} & \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix} & \begin{smallmatrix} 1 & 2 \\ 4 & \end{smallmatrix} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

It can be seen that E-pairs are not unique and both M, N are irreducible matrices. Every sequence of M has its complementary sequence of N and vice versa. Additionally, each sequence of M (or N) corresponds to the sequence of N (or M). Since M is an irreducible matrix (which is not an irreducible permutation matrix), M generates a 1-dimensional sequence that is not periodic, and characteristics of M and N ensure that such a sequence can be extended to a valid configuration of X_G. Hence, there exists $z \in X_G$ such that z is not $(1,1)$-periodic, and $O(z)$ (the orbit of z under shift map) is infinite. Consequently, $X_G \neq \emptyset$ and it is not a finite shift space. It is worth mentioning that every square pattern of size $n \times n$ (constructed with help of M and N) can be extended to form a valid configuration of X_G and every pair of $n \times n$ square patterns can be placed along $(1,1)$ direction in some valid configuration of X_G. Hence, X_G is doubly $(1,1)$-transitive. This proves our claim that even if the condition $(HV)_{kl} \neq 0 \Leftrightarrow (VH)_{kl} \neq 0$ fails to hold, the dynamics of SFT X_G such as non-emptiness problem, existence of periodic points and $(1,1)$ mixing notions can be studied through matrices of triangular patterns.

Proposition 2.15. *Let $G = (H,V)$ be a graph and X_G be the corresponding SFT. If M is an irreducible permutation matrix, then X_G is doubly $(1,1)$-transitive if and only if E-pairs are unique.*

Proof. Let X_G be a doubly $(1,1)$-transitive SFT and M be an irreducible permutation matrix. Then Corollary 2.4 forces N to be an irreducible permutation matrix. If possible, suppose that E-pairs are not unique and let $I_1 = \begin{smallmatrix} b_1 & a_3 \\ a_1 & \end{smallmatrix}$ and $I_2 = \begin{smallmatrix} b_2 & a_3 \\ a_1 & \end{smallmatrix}$ be E-partners of $I = \begin{smallmatrix} & a_3 \\ a_1 & a_2 \end{smallmatrix}$. As M and N are irreducible permutation matrices, M and N are of the same order (Remark 2.5). Let M and N be $k \times k$ matrices. Then $M^k_{RR} > 0 \iff N^k_{SS} > 0$ holds for every E-pair (R,S). Next, construct a pattern using a finite sequence corresponding to $M^k_{II} > 0$ such that

$$\begin{array}{ccccccccc}
 & & & & & & \beta & a_3 & \\
 & & & & & & a_1 & a_2 & \\
 & & & & & \iddots & & & \\
 & & & & a_9 & & & & \\
 & & & a_7 & a_8 & & & & \in \mathcal{B}(X_G) \\
 & & a_5 & a_6 & & & & & \\
\alpha & a_3 & a_4 & & & & & & \\
a_1 & a_2 & & & & & & &
\end{array}$$

It can be seen that if $\alpha = b_1$ is fixed, then $\beta = b_1$ (as N is an irreducible permutation matrix and generates single periodic orbits of period k). Consequently, 1×1 squares $\begin{smallmatrix} b_1 & a_3 \\ a_1 & a_2 \end{smallmatrix}$ and $\begin{smallmatrix} b_2 & a_3 \\ a_1 & a_2 \end{smallmatrix}$ cannot be placed along $(1,1)$ direction, but this contradicts doubly $(1,1)$-transitivity of shift space X_G. Hence, if M is an irreducible permutation matrix and X_G has doubly $(1,1)$-transitivity, then E-pairs are unique.

Conversely, let E-pairs be unique and M be an irreducible permutation matrix; then X_G has doubly $(1,1)$-transitivity (by Proposition 2.11). Consequently, if M is an irreducible permutation matrix, then X_G has doubly $(1,1)$-transitivity if and only if E-pairs are unique.

□

Proposition 2.16. *Let $G = (H,V)$ be a graph and X_G be the corresponding SFT. If $M_{IJ} \neq 0 \iff N_{I_1J_1} \neq 0$ holds for all E-pairs $(I,I_1),(J,J_1)$, then the shift space X_G has doubly $(1,1)$-transitivity if and only if M is an irreducible matrix.*

Proof. Let X_G be a doubly $(1,1)$-transitive SFT and $I = \begin{smallmatrix} & c \\ a & b \end{smallmatrix}$, $J = \begin{smallmatrix} & z \\ x & y \end{smallmatrix}$ be any given indices of matrix M, which appear at the origin in some valid configurations $u, v \in X_G$. Let U and V be $\frac{1}{n}$-neighborhoods of u and v respectively. Then, the doubly $(1,1)$-transitivity of X_G ensures the existence of $k \in \mathbb{N}$ such that $\sigma^{(k,k)}(U) \cap V \neq \emptyset$. Hence, indices I and J can be placed along the $(1,1)$ direction in some valid configuration of X_G, and $M^p_{IJ} > 0$ for some $p \in \mathbb{N}$. Consequently, M is an irreducible matrix, and the proof of the forward part is complete.

Conversely, let M be an irreducible matrix, and the condition $M_{IJ} \neq 0 \iff N_{I_1J_1} \neq 0$ holds for all E-pairs $(I,I_1),(J,J_1)$. Let U, V be $\frac{1}{n}$-neighborhoods of the configurations $x, y \in X_G$ respectively, and let $x_n = \begin{matrix} x_{-n,n} & \cdots & x_{n,n} \\ \vdots & \vdots & \vdots \\ x_{-n,-n} & \cdots & x_{n,-n} \end{matrix}$, $y_n = \begin{matrix} y_{-n,n} & \cdots & y_{n,n} \\ \vdots & \ddots & \vdots \\ y_{-n,-n} & \cdots & y_{n,-n} \end{matrix}$ be the central patterns of the open sets U and V respectively. If the index I appears in the top right corner of x_n and J appears in the bottom left corner of y_n, then the irreducibility of M ensures that I, J can be connected along the $(1,1)$ direction (through a pattern P of M). Clearly, the resulting pattern (obtained by concatenation of patterns x_n, P, and y_n)

$$\begin{array}{ccccccccccccc}
 & & & & & & & & & y_{-n,n} & \cdots & y_{n,n} \\
 & & & & & & & & & \vdots & \vdots & \vdots \\
 & & & & & & & & & y_{-n,-n} & \cdots & y_{n,-n} \\
 & & & & & & & & a_{k+2} & a_{k+3} & & \\
 & & & & & & & a_k & a_{k+1} & & & \\
 & & & & & & \iddots & & & & & \\
 & & & & & a_6 & & & & & & \\
 & & & & a_4 & a_5 & & & & & & \\
 & & & a_2 & a_3 & & & & & & & \\
x_{-n,n} & \cdots & x_{n,n} & a_1 & & & & & & & & \\
\vdots & \vdots & \vdots & & & & & & & & & \\
x_{-n,-n} & \cdots & x_{n,-n} & & & & & & & & &
\end{array}$$

is a valid pattern for X_G. As the imposed condition on M, N ensures that every sequence of M (or N) has its complementary sequence of N (or M), let Q be the complementary pattern of P. Then, it is evident that

$$\begin{array}{cccccccccccc}
 & & & & & & & & y_{-n,n} & \cdots & y_{n,n} \\
 & & & & & & & & \vdots & \vdots & \vdots \\
 & & & & & & & \alpha & y_{-n,-n} & \cdots & y_{n,-n} \\
 & & & & & & b_1 & a_{k+2} & a_{k+3} & & \\
 & & & & & & a_k & a_{k+1} & & & \\
 & & & & & \iddots & \iddots & & & & \\
 & & & & b_3 & a_6 & & & & & \\
 & & & b_2 & a_4 & a_5 & & & & & \\
 & & b_1 & a_2 & a_3 & & & & & & \\
x_{-n,n} & \cdots & x_{n,n} & a_1 & & & & & & & \\
\vdots & \vdots & \vdots & & & & & & & & \\
x_{-n,-n} & \cdots & x_{n,-n} & & & & & & & &
\end{array} \in \mathcal{B}(X_G)$$

Since $M_{IJ} \neq 0 \iff N_{I_1J_1} \neq 0$ holds all E-pairs $(I, I_1), (J, J_1)$; it can be seen that the resulting pattern $\begin{array}{ccc} & y_{-n,-n+1} & y_{-n+1,-n+1} \\ \alpha & y_{-n,-n} & y_{-n+1,-n} \\ a_{k+2} & a_{k+3} & \end{array}$ can be extended to an allowed square pattern of size 2×2 (for every permissible choice of α such that $\begin{smallmatrix} \alpha & y_{-n,-n} \\ a_{k+2} & a_{k+3} \end{smallmatrix} \in \mathcal{B}(X_G)$). Thus, iteratively, such a pattern can be extended to an allowed square pattern and eventually to a valid configuration $z \in X_G$. Consequently, $z \in \sigma^{(q,q)}(U) \cap V$ for some $q \in \mathbb{N}$ (as the central pattern of z is x_n and the central pattern of $\sigma^{(q,q)}(z)$ is y_n). Hence, X_G is doubly $(1, 1)$-transitive.

□

Remark 2.17. The Proposition 2.16 provides a necessary and sufficient condition for doubly $(1, 1)$-transitivity of shift space X_G (under imposed conditions on M and N). The proof uses the fact that if the patterns of the form $\begin{array}{ccc} & \beta & a_5 \\ \alpha & a_3 & a_4 \\ a_1 & a_2 & \end{array}$ get extended to an allowed pattern of size 2×2 (for every permissible choice of α, β such that $\begin{smallmatrix} \alpha & a_3 \\ a_1 & a_2 \end{smallmatrix}, \begin{smallmatrix} \beta & a_5 \\ a_3 & a_4 \end{smallmatrix} \in \mathcal{B}(X_G)$), then irreducibility of M is equivalent to doubly $(1, 1)$-transitivity of shift space X_G. In fact, a shift space may have doubly $(1, 1)$-transitivity under a weaker condition. If every pattern generated by $M^2_{IJ} >$

0 has its complementary pattern corresponding to $N^2_{I_1J_1} > 0$ (and vice versa) for all E-pairs $(I, I_1), (J, J_1)$, then irreducibility of M is equivalent to doubly $(1, 1)$-transitivity of shift space X_G. The proof relies on the fact that if every pattern of the form

$$\begin{array}{cccc} & & \beta & a_7 \\ & & a_5 & a_6 \\ \alpha & a_3 & a_4 & \\ a_1 & a_2 & & \end{array}$$

gets extended to an allowed square pattern of size 3 × 3, then a similar approach (used in Proposition 2.16) yields the desired result. Further, this concept can be generalized for finite sequences of M and N. Thus, the subsequent corollary is established.

Corollary 2.18. *Let $G = (H, V)$ be a graph and X_G be the corresponding SFT. If there exists $k \in \mathbb{N}$ such that every pattern generated by $M^k_{IJ} > 0$ has its complementary pattern corresponding to $N^k_{I_1J_1} > 0$ (and vice versa) for all E-pairs $(I, I_1), (J, J_1)$; then X_G is doubly $(1, 1)$-transitive if and only if M is an irreducible matrix.*

Proof. The proof is substantiated by the arguments presented in the Remark 2.17. □

Remark 2.19. The above results establish $(1, 1)$-transitivity for shift spaces X_G under some imposed conditions on matrices M and N. It can be seen that it is decidable (in a finite number of iterations) whether $k \in \mathbb{N}$ can exist such that every pattern generated by $M^k_{IJ} > 0$ has its complementary pattern corresponding to $N^k_{I_1J_1} > 0$ (and vice versa) for all E-pairs $(I, I_1), (J, J_1)$. Further, it can be noted that under imposed conditions on M and N (as described in Proposition 2.16), X_G exhibits $(1, 1)$-weak mixing $\iff$ M is a primitive matrix. The proof is established by considering the fact that if M is a primitive matrix, then every index I, J of M can be placed along the $(1, 1)$ direction, and the imposed condition on M and N ensures the extension of such a pattern to a valid configuration of X_G. It can be seen that the result holds under weaker restrictions on M and N (as described in Corollary 2.18). It is evident that the condition $M^k_{IJ} \neq 0 \iff N^k_{I_1J_1} \neq 0$ for complementary sequences (for every choice of E-pairs $(I, I_1), (J, J_1)$) is a stronger version of imposed conditions on M, N defined in Proposition 2.27 of [9]. Therefore, if earlier imposed conditions hold, then shift space is non-empty and contains periodic configurations. Thus, the subsequent results are established.

Corollary 2.20. *Let $G = (H, V)$ be a graph and X_G be the corresponding SFT. If there exists $k \in \mathbb{N}$ such that every pattern generated by $M^k_{IJ} > 0$ has its complementary pattern generated by $N^k_{I_1J_1} > 0$ (and vice versa) for all E-pairs $(I, I_1), (J, J_1)$, then X_G has $(1, 1)$-weak mixing if and only if M is a primitive matrix.*

Proof. The proof is substantiated by the arguments presented in the Remark 2.19. □

Proposition 2.21. *Let $G = (H, V)$ be a graph and X_G be the corresponding SFT. If there exists $k \in \mathbb{N}$ such that every pattern generated by $M^k_{IJ} > 0$ has its complementary pattern generated by $N^k_{I_1J_1} > 0$ (and vice versa) for all E-pairs $(I, I_1), (J, J_1)$, then X_G is non-empty and possesses periodic points.*

Proof. If the imposed conditions on M and N hold, then the non-emptiness of X_G holds trivially. Let $X_G \neq \emptyset$, then there exists an index I of M such that $M^r_{II} > 0$ for some $r \in \mathbb{N}$. Let I_1 be the E-partner of I, then the imposed conditions on M and N ensure the existence of $k \in \mathbb{N}$ such that $M^k_{II} \neq 0 \iff N^k_{I_1I_1} \neq 0$ holds for complementary patterns. Next, construct a finite (but sufficiently large) pattern P, which is a concatenation of complementary patterns corresponding to $M^k_{II} \neq 0$ and $N^k_{I_1I_1} \neq 0$. Once again, as removal of the first and last symbol from a finite sequence of N (or M) yields a finite sequence of M (or N), repeating this process will generate a valid pattern Q, where an $n \times n$ square pattern is repeating along the $(1, 1)$ direction. As the square patterns of size n are finite, this process will generate a square pattern R of size n, which repeats horizontally as well. Let S be the horizontal strip of size $m \times n$, where pattern R appears at both ends of this strip, then this construction forces S to appear diagonally as well. Consequently, a configuration $z \in X_G$ can be constructed, which is horizontally periodic and (q, q) periodic for some $q \in \mathbb{N}$, and the proof is complete.

□

Proposition 2.22. *Let $G = (H, V)$ be a graph and X_G be the corresponding SFT, and the condition $M_{IJ} \neq 0 \iff N_{I_1 J_1} \neq 0$ holds for all E-pairs $(I, I_1), (J, J_1)$. Then, X_G has $(1,1)$-weak mixing if and only if X_G has (r,s)-weak mixing ($rs > 0$).*

Proof. Let the condition $M_{IJ} \neq 0 \iff N_{I_1 J_1} \neq 0$ hold for all E-pairs $(I, I_1), (J, J_1)$, and let X_G be a $(1,1)$-weak mixing SFT. Let $x, y, z, w \in X_G$ and U, V, Z, W be $\frac{1}{n}$-neighborhoods of configurations x, y, z, w respectively. Let x_n, y_n, z_n, w_n be the central patterns of size $(2n+1)$ of U, V, Z, W respectively. Since X_G has $(1,1)$-weak mixing, M is a primitive matrix (Corollary 2.20), then $\exists\, q \in \mathbb{N}$ such that $M^p > 0$ for all $p \geq q$ and for all indices I, J of M. Let (R, S) be a point in (r,s) direction such that $(R-n) > q$ and $(S-n) > q$. Let P be the pattern obtained by placing the lower-left corner of x_n at the origin and placing the lower-left corner of y_n at (R, S). Similarly, let Q be the pattern obtained by placing the lower-left corner of z_n at the origin and placing the lower-left corner of w_n. Since y_n, w_n appear in configurations y, w respectively; the patterns y_n, w_n can be extended to y_n^*, w_n^* (downwards) such that the lower-left corner of y_n^*, w_n^* falls on the $(1,1)$ direction. Let I, J be indices of M appearing at the top right corner of x_n, z_n and K, L be indices of M appearing at the lower-left corner of y_n^*, w_n^* respectively. Then, the primitiveness of M ensures that $M^u_{IK} > 0$ and $M^u_{JL} > 0$ for some $u > q$. Further, the imposed condition on M and N ensures that the resulting patterns P and Q can be extended to valid configurations, say $a, b \in X_G$. Finally, note that $a \in U, b \in Z$ (as the central blocks of a and b are x_n and z_n respectively), $\sigma^{c(r,s)}(a) \in V$ and $\sigma^{c(r,s)}(b) \in W$ (as the central blocks of $\sigma^{c(r,s)}(a)$ and $\sigma^{c(r,s)}(b)$ are y_n and w_n respectively), where $c \in \mathbb{N}$. Consequently, $\sigma^{c(r,s)}(U) \cap V \neq \emptyset$ and $\sigma^{c(r,s)}(Z) \cap W \neq \emptyset$, and X_G exhibits (r,s)-weak mixing behavior.

Conversely, let X_G have (r,s)-weak mixing (for $rs > 0$), then for any n pairs of open sets (U_i, V_i), there exists $k \in \mathbb{N}$ such that $\sigma^{k(r,s)}(U_i) \cap V_i \neq \emptyset$ for $i = 1, 2, \ldots, n$. Equivalently, if the central patterns of U_i and V_i are separated by kr length horizontally and ks length vertically, then such patterns can be placed along (r,s) direction in valid configurations of shift space X_G. Let P_i be 1×1 patterns containing all indices of M and Q_i be $1 \times m$ patterns (for some $m \in \mathbb{N}$), then (r,s)-weak mixing ensures that such patterns can be placed along (r,s) direction. If m was chosen in such a way that patterns Q_i (placed at some point (R, S) in the direction (r,s)) hits $(1,1)$ direction, then the resulting patterns can be extended to valid configurations of X_G. It can be seen that every index of M interacts with others (along $(1,1)$ direction) in such configurations, and there exists $p \in \mathbb{N}$ such that $M^p > 0$. Consequently, M is primitive and Corollary 2.20 ensures that X_G has $(1,1)$-weak mixing.

□

Remark 2.23. Proposition 2.22 establishes that $(1,1)$-weak mixing of 2-dimensional SFT X_G is equivalent to (r,s) weak mixing (for $rs > 0$) under the imposed condition. It can be noted that this result holds under a weaker condition on generating matrices M and N (as described in Corollary 2.20). Further, for such shift spaces X_G, X_G has $(1,1)$-weak mixing if and only if M is a primitive matrix (Corollary 2.20). Thus, the subsequent corollary is established.

Corollary 2.24. *Let $G = (H, V)$ be a graph and X_G be the corresponding SFT. If there exists $k \in \mathbb{N}$ such that every pattern generated by $M^k_{IJ} > 0$ has its complementary pattern corresponding to $N^k_{I_1 J_1} > 0$ (and vice versa) for all E-pairs $(I, I_1), (J, J_1)$; then X_G has (r,s) weak mixing (for $rs > 0$) if and only if M is a primitive matrix.*

Proof. The proof is substantiated by the arguments presented in the Remark 2.23. □

Remark 2.25. The above results establish mixing notions (transitivity, doubly transitivity, and weak mixing) in the $(1,1)$ direction through matrices M and N. It can be noted that the mixing notions in the $(1,-1)$ direction can be studied through matrices M^* and N^* (indexed with triangular patterns of the form $\begin{smallmatrix} x & \\ y & z \end{smallmatrix}$ and $\begin{smallmatrix} a & b \\ & c \end{smallmatrix}$, respectively). It can be seen that if the imposed condition on M^* and N^* holds, then $(1,-1)$-weak mixing and (r,s)-weak mixing (for $rs < 0$) are equivalent. Consequently, (r,s)-weak mixing (for $rs \neq 0$) can be investigated through the primitiveness of matrices M and M^* (under imposed conditions). Additionally, the mixing notions in the direction (r,s) (for $rs \neq 0$) can be studied (using matrices indexed with appropriate triangular patterns of length r and height s) in a manner akin to the mixing notions in the $(1,1)$ direction.

3 Conclusion

This paper explores the topological dynamics of the shift spaces X_G, introducing a novel method to investigate their structural properties. Our approach extends beyond the constraints of condition $(HV)_{kl} \neq 0 \iff (VH)_{kl} \neq 0$ for all $k, l \in V(G)$, allowing for a comprehensive analysis. By employing matrices indexed with valid triangular patterns, we have partially addressed problems such as the non-emptiness problem, the existence of periodic configurations, and various mixing notions. In particular, we investigated $(1,1)$-mixing notions for 2-dimensional shift space X_G through matrices M and N; where M and N are indexed with allowed triangular patterns of the form ${}_{a}\,{}^{c}_{b}$ and ${}^{x}_{y}\,{}^{z}$ respectively. We also established the relation between the algebraic structure of matrices M and N. We have demonstrated that adherence to the condition $M_{IJ} \neq 0 \iff N_{I_1 J_1} \neq 0$ for all E-pairs $(I, I_1), (J, J_1)$ ensures non-emptiness and the presence of periodic configurations within the SFT X_G. Moreover, we proved that for such SFT, $(1,1)$-weak mixing and (r, s)-weak mixing (for $rs > 0$) are equivalent.

The findings of this work not only advance the theoretical understanding of multidimensional shift spaces but also offer potential for diverse real-world applications. For example, this analysis of periodicity and mixing properties is crucial in coding theory, where such structural dynamics can help to design robust error-correcting codes for data transmission. Further, in the study of complex networks such as transportation systems and neural networks, the understanding of mixing properties and periodic behaviors can lead to improved reliability and functionality. Furthermore, in image and signal processing, the principles derived from symbolic dynamics can enhance pattern recognition and compression algorithms. Similarly, in computational biology, they can be used to explore multi-dimensional interaction networks, such as protein-protein interactions or gene regulatory systems, improving insights into biological processes. The study of topological dynamics of shift spaces bridges the symbolic dynamics to the practical challenges of science, fostering interdisciplinary exploration.

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