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D-NEIGHBOURLY IRREGULAR GRAPHS (D-NI graphs)

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Abstract In this work, we are constructing a D-Neighbourly Irregular Graphs (G_{DNI}). Where D-Degree sequence. This is a new dimension of constructing Neighbourly Irregular Graphs. Also we have derived some theorems and its proofs.

1 Introduction

By Neighbourly Irregular Graphs (G_{DNI}) it means a graph in which each pair of adjacent vertices have distinct degrees by S.K Ayyasamy [4]. Aruldoss [2] et all found a positive method to construct a Neighbourly Irregular graph using the molecular Structures and named Neighbourly Irregular Chemical graphs (G_{NIC}). Also Aruldoss et all converted this (G_{NIC}) to Neighbourly Irregular Fuzzy Chemical graphs (G_{NIFC}).

Here, we are deriving Neighbourly Irregular Graphs apart from the known chemical graphs, namely G_D or (G_{DNI})(D–Neighbourly Irregular Graphs). Also some properties

2 PRELEMINARIES

Definition 2.1. [4, 8]

A graph G= (X,E) is called regular, if each vertex have positive same degree

i.e) $\delta(G) = \Delta(G) = r$,where $\delta(G)$ and $\Delta(G)$ are the minimum and maximum degrees of the vertices.

Definition 2.2. [4]

A graph is neighbourly irregular (G_{NI}) if any 2 adjacent vertices have different degrees.

i.e) $d(v_i) \neq d(v_{i+1})$.

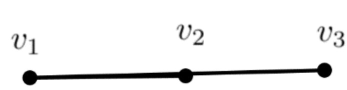

Figure 1. NI graph

Definition 2.3. [8]

Any graph G= (X,E) is called to be highly irregular (HI) if every vertex is neighbor vertex to the vertices of different degrees.

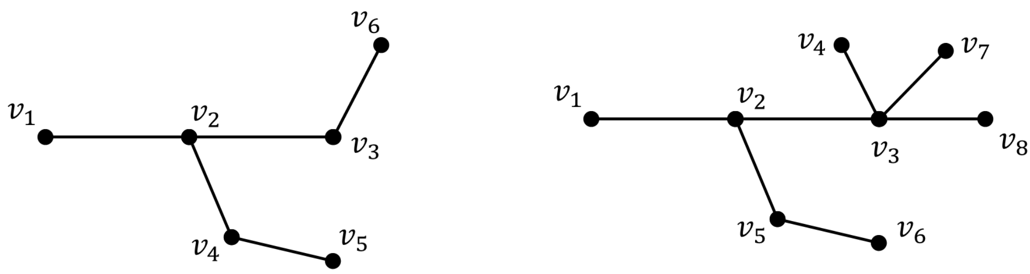

Figure 2. H.I graphs

3 D-NEIGHBOURING IRREGULAR GRAPHS (G_{DNI})

Definition 3.1. Any connected graph G=(V,X) is D-Neighbourly Irregular (G_{DNI}), then the degree of any two vertices must be $d(v_i) = n - 1$, and $d(v_j) = n - 2$, satisfying the following conditions,

(i) |V| = n; $n \geq 3$ except, n=4.

(ii) For the partition (or) realization of $p = (d_1, d_2, ..., d_n)$, where d_1 and d_2 or $d(v_1)$ and $d(v_2)$ must be $d(v_1) = n - 1$ and $d(v_2) = n - 2$, It is denoted as (G_{DNI})= (V,X).

This degree sequence of vertices is done for the proceeding vertices only.

Note:

1. Any G_{DNI} graphs need not be highly irregular.
2. G_{DNI} must be connected.
3. The degree of any two vertices must be (n-1, n-2)
4. This step is may be continued up to for $\frac{n}{2}$ vertices only.

Second partition of G_{DNI}

Any connected G_{DNI}, graph can be derived to second partition, if the degree of any two vertices are $d(v_i) = n - 2$ and $d(v_j) = n - 3$.

Note: In the above partition $P = (d_1, d_2, ..., d_n)$,

$P - m = (d_1 - 1, d_2 - 1, ..., d_{m1} - 1, d_m - 1, ..., d_n)$ where 2< m $\geq$ n is also G_{DNI} and it is denoted by G^m_{DNI} . The value of m is chosen in such a way the resultant graph should not have isolated vertex.

Example 3.2. Here we have given some G_{D-NI} graphs,

1. If |V|=3, then P=(2,1,1) and G_{DNI} = (3,2).

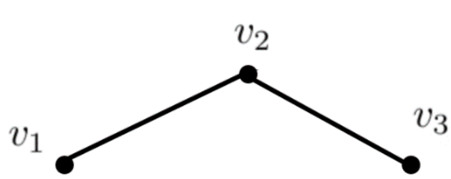

Figure 3.

2. If |V|=5, then P=(4,3,2,2,1) and G_{DNI} = (V,X)=(5,6).

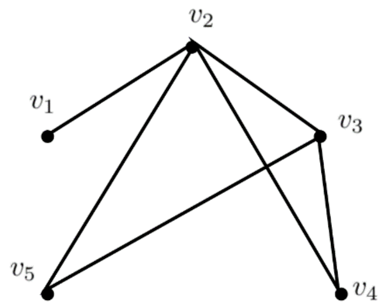

Figure 4.

And the second partition of G_{DNI} is P-2 =(3,2,1,1,1) also G^2_{DNI} = (5,4).

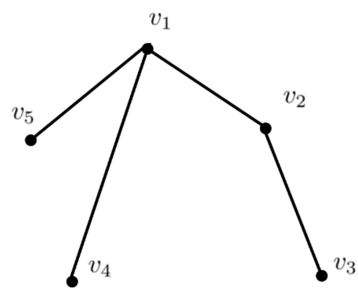

Figure 5.

3. If |V|=6, then P=(5,4,3,2,2,2) and G_{D-NI} = (V,X)=(6,9).

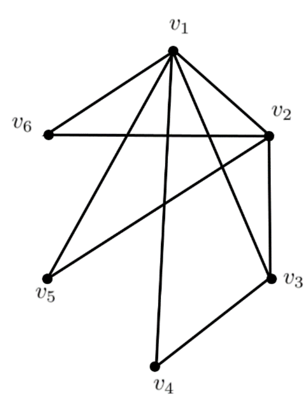

Figure 6.

And the second partition of G_{DNI} is G^2_{DNI} =(6,7), P-2 =(4,3,2,2,2,1).

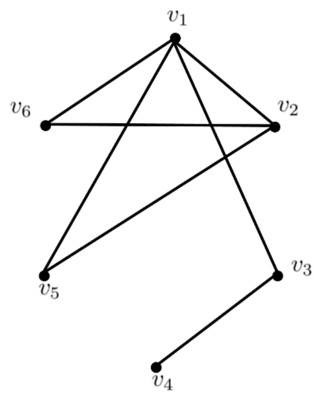

Figure 7.

Note: G_{DNI} graphs are need not be highly irregular.
Some more G_{DNI} graphs,
1. If n=9, then P = (8,7,6,5,4,3,3,2,2)

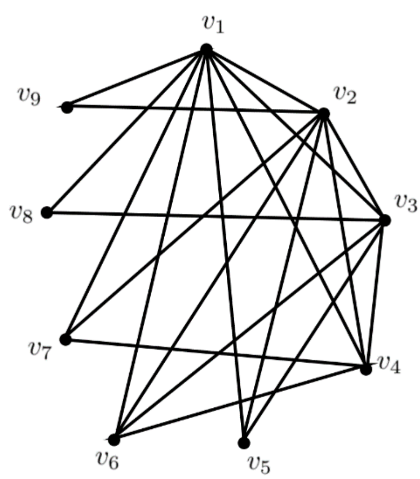

Figure 8.

2. If n=10, then P = (9,8,5,5,4,3,2,2,2).

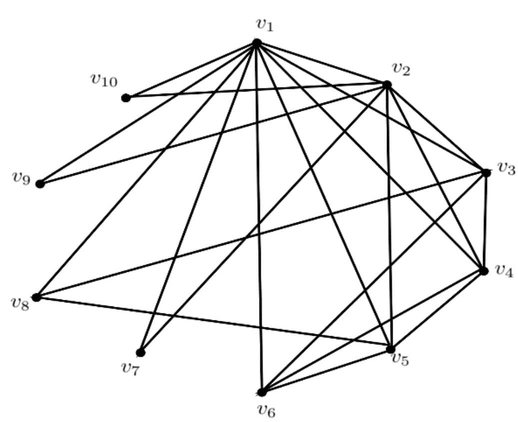

Figure 9.

Definition 3.3. Tree width of G_{DNI} graphs:
The tree width of a G_{DNI} graph is a tree.
i) $t_\omega(G_{DNI}) \iff (G_{DNI})$ is a forest.
ii) $t_\omega(G_{DNI}) = 2 \iff (G_{DNI})$ is a sub graph of series parallel graphs.
iii) $t_\omega(K_n) = n - 1$.

Definition 3.4. Subdivision graph of (G_{DNI})
A graph G_{DNI} is a subdivision graph $S(G_{DNI})$ by introducing a new vertex.

Proposition 3.5. *Let G_{DNI} be a DNI graph. The subdivision graph $S(G_{DNI})$ need not be a Neighbourly Irregular graph (G_{NI}) .*

Proof. G_{DNI} need not be a G_{NI} as some of the vertices of G_{DNI} graph has the possibility of having degree 2 [2]. □

Note: For n=5; S(G_{DNI}) is NI.

Proposition 3.6. *Let G(V,X) be a DNI graph. The SD graph $S(G_{DNI})$ is NI iff G_{DNI} has no vertex of deg 2.*

Proof. If G_{DNI} has no vertex of deg 2,
i.e) $d_{G_D}(u) \neq 2; \forall u \in V$
$\implies d_{G_D}(u) = d_{S(G_D)}(u)$
Since in G_{DNI}, every edge uv is inserted a new vertex w between u and v
Such that $d_{S(G_D)}(w) = 2$
Then $d(_{S(G_D)}(u)) \neq d(_{S(G_D)}(w)$.
This hold for both adjacent pair of w .
Hence is $S(G_D)$ is G_{NI} .
On the other hand,
If G_{DNI} has at least one vertex of degree 2, then evidently $S(G_{DNI})$ is G_{NI} .

□

Example 3.7.

1. Let us consider the graph for G_{DNI} for |V|=6, where P = (5,4,2,2,2,1) then
$S_{GD}(v_1) = 2 + 4 + 2 + 2 + 1 = 11$
$S_{GD}(v_2) = 5 + 2 + 2 + 2 = 11$
$S_{GD}(v_3) = 4 + 5 = 9$
$S_{GD}(v_4) = 5$
$S_{GD}(v_5) = 5 + 4 = 9$
$S_{GD}(v_6) = 5 + 4 = 9$
2. if n=8, then P=(7,6,4,4,3,2,2,2)

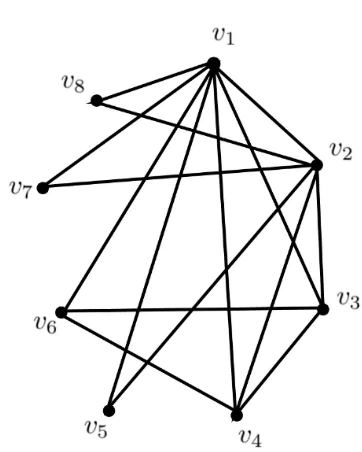

Figure 10. (G_{DNI}) graph

$S_D(v_1) = 25; S_D(v_2) = 23$
$S_D(v_3) = 23; S_D(v_4) = 21$
$S_D(v_5) = 18; S_D(v_6) = 16$
$S_D(v_7) = 13; S_D(v_8) = 13$
Here, since the graph has a vertex of degree 2, then the support of G_{DNI} is not NI.

Definition 3.8. The line graph $L(G_{DNI})$ of G_{DNI} is that, where the edges are changes to vertices and vertices are the lines of G_{DNI}, and two vertices of $L(G_{DNI})$ are neighbours iff respective lines are adjacent in G_{DNI}. Here $d_{L(G_D)}(e) = d_{G_D}(u) + d_{G_D}(v) - 2$.

Example 3.9. For n=5;

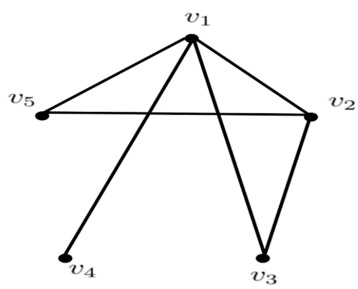

Figure 11. G_{DNI} graph

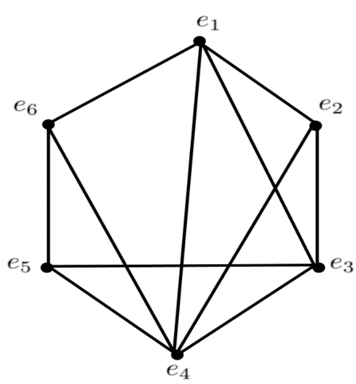

Figure 12. $L(G_D)$

Here,
$d(e_1) = d(v_1) + d(v_5) - 2$
$4 = 4 + 2 - 2 = 4$
Similarly we can verify for other edges.

Note: For the DNI graphs its line graph $L(G_D)$ cannot be G_{NI} .

Definition 3.10. Neighborhood of a vertex (N(u))
In G_D , N(u) represents the set of vertices, are neighbouring to u.

Ex: For n=5,
$N(w_1) = 4; N(w_2) = 3; N(w_3) = 2; N(w_4) = 1; N(w_5) = 2$

Lemma 3.11. *If u is a vertex with max deg of a G_{DNI}graph, then N(u) has at least two vertices with equal degree.*

Proof. Let the D-Neighbourly irregular graph (G_D) has n vertices such that,
$V(G) = v_1, v_2, ..., v_n$

i.e) |V|= n

In the construction process of D-Neighbourly irregular graph, we know that, among the n-vertices, n-1 vertices get degree 1 at first stage. And finally among n-2 vertices two vertices namely v_i and v_j get degrees n-1 and n-2 .

i.e) (n-2)-2 = n-4 .

Step 1:

Among n vertices, v_i gets, n-1 degree. Other n-1 vertices of degree 1.

Step 2:

Again among n vertices, v_2 gets degree n-2 and the remaining n-3 vertices degree 2 and one vertex remains degree 1.

proceeding this way, $\frac{n}{2}th$ vertex gets degree n-3

i.e.) deg$(v_{n_r}) = n - 3$

i.e.) $\{v_{n_{/2}} + 1, v_{n_{/2}} + 2, ..., v_n\}$ will get the degrees $\leq$ n-3, in which there will be at least two vertices with same degree.

□

Example 3.12.

1. n is odd,

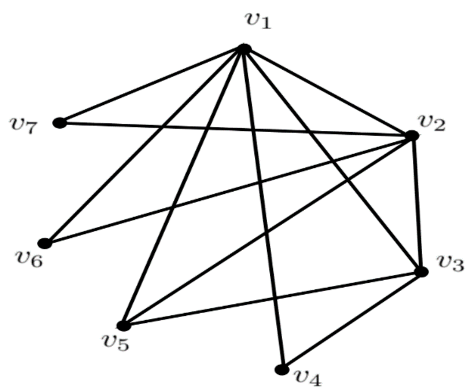

Figure 13. G_{DNI} graph

2. n is even,

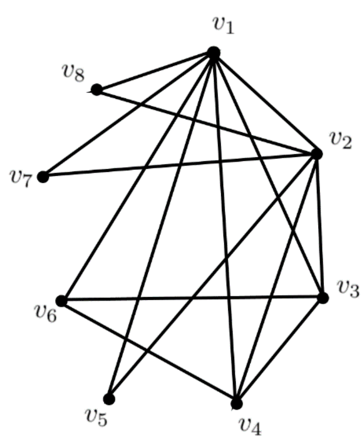

Figure 14. G_{DNI} graph

Proposition 3.13. *Any D-Neighbourly irregular graph* (G_{DNI})*,* $L(G_{DNI})$ *is not Neighborly Irregular graph.*

Proof. We suppose that, u is the vertex with max deg in G_{DNI} . By previous lemma, this N(u) has atleast 2 vertices of equal deg. Precisely, for v and w of equal deg.

i.e.) $e_1 = uv$ and $e_2 = uw$

Now,

$$\begin{aligned} \text{deg } LG_D(e_1) &= deg_{G_D}(u) + deg_{G_D}(v) - 2 \\ &= deg_{G_D}(u) + deg_{G_D}(w) - 2 \\ &= deg_L(G_D)(e_2) \end{aligned}$$

Hence $L(G_{DNI})$ is not NI graph.

□

Definition 3.14. In D-Neighbourly irregular graph (G_D),

$$|E(x)| = \begin{cases} (n-1) + (n-3) + ... + 1, & \text{n-even.} \\ (n-1) + (n-3) + ... + 0, & \text{n-odd.} \end{cases} \tag{3.1}$$

Note:

In D-Neighbourly Irregular graph (G_D) the maximum distance of a path is $\frac{n}{2}$.

Definition 3.15. [10]

In a graph G_{DNI}, the open neighbourhood is , for any vertex $w \in V$, its adjacent vertices are put in a set and it is said to be open neighbourhood of v, it is denoted by

$N(w) = \{u \in V/uw \in X(G(V, X))\}$

Definition 3.16. [10]

The closed neighbourhood is,

$N[w] = N(w) \cup \{w\}$

Note:

- 2 vertices v and w are neighbours if N[v] = N[w] and d(v) = d(w) is also called as pairable vertices.
- In DNI graph, it is that N[v] = N[w].

Definition 3.17. [2]**k-Regular Adjacency :-**

For $v \in V$ is said to be k-RA if all the vertices in N(v) has the degree k.

Note: In D-Neighbourly irregular graph only $K_{1,n}$ is k–RA.

4 Support of a D-Neighbourly Irregular Graph :

Definition 4.1. For $v \in V(G_D)$, Support of v is the sum of degrees of its adjacent points.

i.e.)$SP(G_D)(v) = \sum_{u \in N(v)} d(u)$

Example 4.2. If |v| = 6,

$SP(u_1) = 1, SP(u_2) = 11, SP(u_3) = 5, SP(u_4) = 9, SP(u_5) = 9$

Definition 4.3. Degree of a D-NI graph (G_D) **:**

In a DNI graph, the degree of graph is $|v_1|$ such that the number of vertices which are of equal degree, with the max no.of vertices.

i.e.)$d(G_D) = \{|v_1|/d(v_i) = d(v_j)$; where $v_i, v_j \in V_1$

Here $V_1 \subseteq V$

Definition 4.4. Maximum degree of a D-NI Graph

In a D-NI graph, the max degree of a graph is $|V_{max}|$, such that the no.of vertices which are of same degree with max degree.

$d_{max}(G_D) = \{|V_{max}|/d(v_i) = d(v_j) \forall v_i, v_j \in |V_{max}|$

Facts:
1. For a regular graph, $|V_1| = |V|$.
2. In a DNI graph, $|V_1| < |V|$.
3. For a bipartite graph DNI graph $K_{1,n}$
   $|V_1| = n$ and $|V_{max}| = 1$
4. For a complete bipartite graph $K_{m,n}, |V_1| = |V_{max}|$.

Example 4.5.

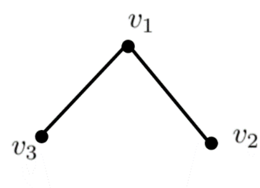

Figure 15. $K_{1,2}$

Definition 4.6. Total Support (B – support) of a D – Neighbourly Irregular Graph
In a G_{DNI} graph, for any vertex $u \in V$ the B-support of DNI graph $BS_{DNI}(u)$ is sum of deg(u) and its neighbors.
$BS_{DNI}(u) = \sum_{u \in N(u)} d(v) + d(u)$

Definition 4.7. Support Neighbourly Irregular Graphs
In a graph G(V,X), for any two neighbouring vertices have distinct supports, then G(V,X) is SNI graph.

Example 4.8.

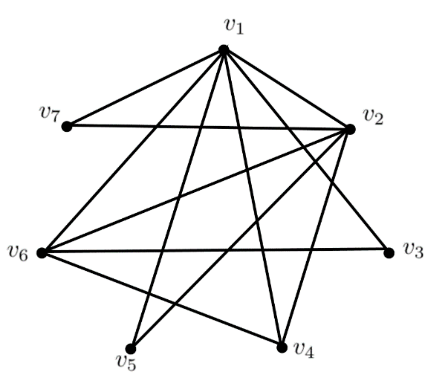

Figure 16. SNI graph

$SP(v_1) = 5 + 4 + 3 + 3 + 3 + 2 = 20$
$SP(v_2) = 6 + 3 + 3 + 3 + 2 = 17$
$SP(v_3) = 6 + 3 + 3 + 3 = 15$
$SP(v_4) = 6 + 5 + 4 = 15$
$SP(v_5) = 6 + 5 + 4 = 15$
$SP(v_6) = 6 + 5 + 3 + 4 = 18$
$SP(v_7) = 6 + 5 + 3 = 14$

Proposition 4.9. *All DNI graph can be drawn to SNI graph. It is named as D - Support NI graph.*

Proof. Let G be a given DNI graph and if |V| = n then the deg seq of the graph is arranged in such a way that $P = (d_1, d_2, d_3, ..., d_n)$
= (n – 1, n – 2, . . . 2 or 1)
then support v_i for $i = 1$ to n is calculated as
$SP(v_i) = \sum d(N(v))$
Since D-NI is NI, the degrees of any two adjacent points are different.
Let us suppose that DNI is not SNI,
Such that $SP(v_i) = SP(v_j)$ for $i \neq j$
i.e.) sum of adjacent vertices of v_i equals the sum of adjacent vertices
$SP(v_i) = SP(v_j) = k$
In each vertex v_i and v_j the integer k can be arranged in $d(v_i)$ distinct ways.
In that $d(v_1)$ is fixed as n-1. The remaining $d(v_1) - 1$ ways are left.
The suppose that v_i^{th} vertex is calculated for D - NI graphs as
$SP(v_i) = (n - 1) + \sum_{i=2}^{n} d(v_i)$
$= d(v_1) + \sum_{i=2}^{n} d(v_i)$
Since the degrees of adjacent vertices of D – NI can be arranged a descending order, the support of DNI is also NI.

□

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