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EXPLICITING γ_{TCC} NUMBER FOR FRACTAL GRAPH AND SOME CLASSES OF GRAPHS

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Abstract A dominating set which is triple connected and certified (TCCD-set) in a graph G is a dominating set S where, for every vertex v in S, the number of neighbors of v in $(V - S)$ is either zero or at least k, where $k \geq 2$. Furthermore, any three vertices in S are connected by a path within the subgraph induced by S. The smallest possible size of such a set is known as Triple conected certified domination number(TCCD-number), denoted as $\gamma_{TCC}(G)$. This study explores the TCCD number for various types of graphs, such as the Harary graph, Circulant graph, Hypercube graph, and Sierpinski gasket.

1 Introduction

This article explores various types of graphs, encompassing finite, non-trivial, and simple ones. Paulraj Joseph et al. originally introduced the concept of triple connected graphs. [1], following that, the triple connected domination number was introduced [2], and more recently, M. Detlaff et al. proposed a parameter known as certified domination number [3]. Building upon this prior research, a new parameter is proposed[4]. γ_{TCC} valuse of the strong product of graphs was generalized in[5], which also provided γ_{TCC} values for Cartesian, corona, and lexicographic products of paths and cycles. Additionally, the γ_{TCC} number of power graphs of certain special graphs has been investigated in [6] and [7]. In the case where G is triple connected, $V(G)$ constitutes a TCCD-set, thus $3 \leq \gamma_{TCC}(G) \leq |V(G)|$, then $3 \leq \gamma_{TCC}(G) \leq |V(G)| - 2$ when G lacks triple connectivity.

Section 2 presents the precise values of the triple connected certified domination number for distance graphs. The Harary graph[8]-[9], denoted as $H_{m,n}$, is defined as follows: When m is even, represented as $2r$, $H_{2r,n}$ is constructed with vertices labeled from 0 to $n-1$. Let i and j be the vertices that are connected if their indices satisfy the condition $i - r \leq j \leq i + r$ (with addition performed modulo n). If m is odd and n is even, denoted as $m = 2r + 1$, $H_{2r+1,n}$ is formed by first drawing $H_{2r,n}$ and then adding edges that link vertex i to vertex $i + (n/2)$ for $1 \leq i \leq n/2$. For odd values of both m and n, where $m = 2r + 1$, $H_{2r+1,n}$ is created similarly to $H_{2r+1,n}$ but with additional edges connecting vertex 0 to the vertices at indices $(n-1)/2$ and $(n+1)/2$, and linking vertex i to vertex $i + (n+1)/2$ for $1 \leq i \leq (n-1)/2$. The hypercube graph Q_n is defined as the Cartesian product of a path of 2 vertices, repeated n times[10]-[11]. The triangular graph T_n is the line graph of K_n [12]. Section 3 deals with determining the γ_{TCC} values for the power graph of the family of cycle graphs, including peacock graphs, butterfly graphs[13], lollipop graphs[14], and sunlet graphs[15]. Section 4 provides the exact TCCD-number for the iterated graph, Sierpinski gasket [16], and d, r that is the diameter and radius also calculated for $< S >$ of the Sierpinski gasket where the maximum degree of the graph is denoted by Δ.

2 TCCD- number on distance graphs

The results derived from analyzing various distance graphs, such as the Harary graph, hypercube graph, and triangular graphs, are presented here. These graphs serve as important case studies, offering valuable insights into their distinct structural characteristics and distance-related properties.

Theorem 2.1. *For a Harary graph represented as $H_{m,n}$, where $m \geq 6$, m is even with $\frac{m}{2} = k$,*

$$\gamma_{TCC}(H_{m,n}) = \begin{cases} \lfloor \frac{n}{k} \rfloor - 1 & if\ n = kt \ or\ kt+1, \\ \lfloor \frac{n}{k} \rfloor & otherwise. \end{cases}$$

Proof. Let $V(H_{m,n}) = \{v_h : 0 \leq h \leq n-1\}$. Take $S_1 = \{v_i : i = kt, 0 \leq i \leq n-(m+2)\}$, $S_2 = \{v_{n-k}\}$. clearly $S = S_1 \cup S_2$ forms a dominating set that is triple connected and certified $H_{m,n}$ and hence

$$\gamma_{TCC}(H_{m,n}) \leq \mid S \mid = \begin{cases} \lfloor \frac{n}{k} \rfloor - 1 & if\ n = kt \ or\ kt+1, \\ \lfloor \frac{n}{k} \rfloor & otherwise. \end{cases}$$

Let us consider the presence of a TCCD-set $D\ of\ H_{m,n}$ with a cardinality of no more than $d = \begin{cases} \lfloor \frac{n}{k} \rfloor - 2 & if\ n = kt \ or\ kt+1, \\ \lfloor \frac{n}{k} \rfloor - 1 & otherwise. \end{cases}$ Therefore, the induced subgraph $< D >$ lacks triple connectivity, implying that $\gamma_{TCC}(H_{m,n}) \geq d+1 = \begin{cases} \lfloor \frac{n}{k} \rfloor - 1 & if\ n = kt \ or\ kt+1, \\ \lfloor \frac{n}{k} \rfloor & otherwise. \end{cases}$

Accordingly, the conclusion follows.. □

Theorem 2.2. *For a Harary graph represented as $H_{m,n}$, where $m \geq 9$, m is odd with $\frac{m}{2} = k$, $\lfloor \frac{m}{2} \rfloor = p$,*

$$\gamma_{TCC}(H_{m,n}) = \begin{cases} \lfloor \frac{n}{k} \rfloor - 1 & if\ n = 2pt \ or\ 2pt+1 \ or\ 2pt+2, \\ \lfloor \frac{n}{p} \rfloor & if\ n = 2pt+4 \ or\ \ldots\ or\ 2pt+p-1, \\ \lfloor \frac{n}{k} \rfloor - 1 & otherwise. \end{cases}$$

Proof. Let $V(H_{m,n}) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$. Take $S_1 = \{v_i : i = pt, 0 \leq i \leq [\lfloor \frac{n+1}{2} \rfloor - (p+2)]\}$, $S_2 = \{v_{\lfloor \frac{n+1}{2} \rfloor}\}$, $S_3 = \{v_i : i = \lfloor \frac{n+1}{2} \rfloor + l, l = pt, l \geq p, \lfloor \frac{n+1}{2} \rfloor \leq i \leq n-p-2\}$.
Clearly $S = S_1 \cup S_2 \cup S_3$ forms a dominating set that is triple connected and certified $(H_{m,n})$ and hence

$$\gamma_{TCC}(H_{m,n}) \leq \mid S \mid = \begin{cases} \lfloor \frac{n}{k} \rfloor - 1 & if\ n = 2pt \ or\ 2pt+1 \ or\ 2pt+2, \\ \lfloor \frac{n}{p} \rfloor & if\ n = 2pt+4 \ or\ \ldots\ or\ 2pt+p-1, \\ \lfloor \frac{n}{k} \rfloor - 1 & otherwise. \end{cases}$$

Let us consider the presence of a TCCD-set $D\ of\ H_{m,n}$ with a cardinality of no more than $d = \begin{cases} \lfloor \frac{n}{k} \rfloor - 2 & if\ n = 2pt \ or\ 2pt+1 \ or\ 2pt+2, \\ \lfloor \frac{n}{p} \rfloor - 1 & if\ n = 2pt+4 \ or\ \ldots\ or\ 2pt+p-1, \\ \lfloor \frac{n}{k} \rfloor - 2 & otherwise. \end{cases}$ Therefore, the induced subgraph $< D >$ lacks triple connectivity, implying that $\gamma_{TCC}(H_{m,n}) \geq d+1$

$$= \begin{cases} \lfloor \frac{n}{k} \rfloor - 1 & if\ n = 2pt \ or\ 2pt+1 \ or\ 2pt+2, \\ \lfloor \frac{n}{p} \rfloor & if\ n = 2pt+4 \ or\ \ldots\ or\ 2pt+p-1, \\ \lfloor \frac{n}{k} \rfloor - 1 & otherwise. \end{cases}$$

Accordingly, the conclusion follows.. □

Theorem 2.3. *For a Hypercube graph represented as Q_n, where $n \geq 4$, $\gamma_{TCC}(Q_n) = 2^{n-3} + 2^{n-2}$.*

Proof. Let $V(Q_n) = \{u_1, u_2, \ldots, u_{2^n}\}$, $\mid V(Q_n) \mid = 2^n$ and for $n \geq 3$.
Let $S_1 = \{v_{i,1} : i = 8t+2 \ or\ 8t+7\}$, $S_2 = \{v_{i,2} : i = 4t \ or\ 4t+1, 4 \leq i \leq r-1\}$, $S_3 = \{v_{i,2} : i = 2, 3\}$. Clearly $S = S_1 \cup S_2 \cup S_3$ forms a dominating set that is triple connected and certified

Q_n and hence $\gamma_{TCC}(Q_n) \leq \mid S \mid = 2^{n-3} + 2^{n-2}$. Let us consider the presence of a TCCD-set $D\ of\ Q_n$ with a cardinality of no more than $d = 2^{n-3} + 2^{n-2} - 1$. Therefore, the induced subgraph $< D >$ lacks triple connectivity, implying that $\gamma_{TCC}(Q_n) \geq d + 1 = 2^{n-3} + 2^{n-2}$. Accordingly, the conclusion follows.. □

Example 2.4.

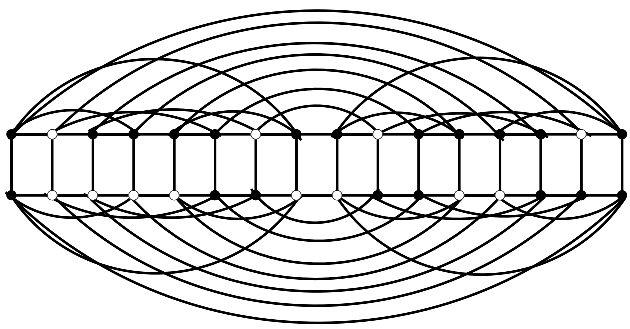

Figure 1. Hypercube graph Q_5

Illustration: Here the set of lightened vertices denote the TCCD set of Q_5.

Theorem 2.5. *For a triangular graph represented as T_n, wheren ≥ 5, $\gamma_{TCC}(T_n) = \frac{\Delta(T_n)}{2}$.*

Proof. Let $V(T_n) = \{v_1, v_2, \ldots, v_{\frac{n(n-1)}{2}}\}$ and $\mid V(T_n) \mid = \frac{n^2-n}{2}$. Since $\Delta(K_n) = n - 1$ we have $\Delta(T_n) = 2n - 4$. Clearly $S = \{v_i : 1 \leq i \leq n - 2\}$ forms a dominating set that is triple connected and certified T_n and hence $\gamma_{TCC}(T_n) \leq \mid S \mid = \frac{\Delta(T_n)}{2}$. Let us consider the presence of a TCCD-set $D\ of\ T_n$ with a cardinality of no more than $d = \frac{\Delta(T_n)}{2} - 1$. Therefore, the induced subgraph $< D >$ lacks triple connectivity, implying that $\gamma_{TCC}(T_n) \geq d + 1 = \frac{\Delta(T_n)}{2}$ Accordingly, the conclusion follows. □

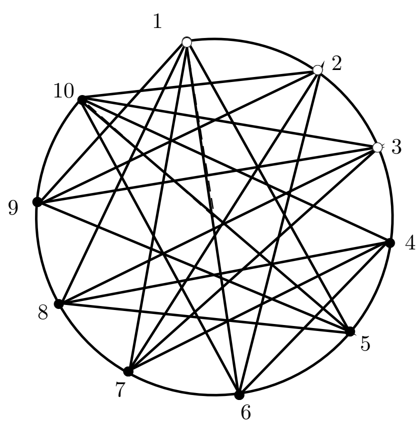

Figure 2. Triangular graph T_5

Example 2.6. Illustration: Here the set of lightened vertices denote the TCCD set of T_5.

3 TCCD number of power graphs on cycle family graphs

In this section, we present the results related to the power graphs of various cycle-based graph families. These include graphs that are characterized by distinct structural features and relationships between their vertices and edges. Specifically, we explore the power graphs for graphs such as the peacock head graph, the butterfly graph, lollipop graph, and the sunlet graph. The findings shed light on the underlying patterns and structural characteristics that define the behavior of power graphs for these particular families.

Observation

(i) For a peacock head graph $PH_{m,n}$, $n \geq 8$, then $\gamma_{TCC}(PH_{m,n})^2 = \lfloor \frac{n}{2} \rfloor - 1$.

(ii) If $k \geq 3$, and $5 \leq n \leq 3k+1$, then $\gamma_{TCC}(PH_{m,n})^k = 3$.

Theorem 3.1. *For a peacock head graph represented as $PH_{m,n}$, where $n \geq 3k+2, k \geq 3$,*

$$\gamma_{TCC}(PH_{m,n})^k = \begin{cases} \lfloor \frac{n}{k} \rfloor - 1 & if\ n = kt\ or\ kt+1, \\ \lfloor \frac{n}{k} \rfloor & otherwise. \end{cases}$$

Proof. Let $V(PH_{m,n}) = \{v_h : 1 \leq h \leq n, u_h : 1 \leq h \leq m\}$ and $E(PH_{m,n}) = \{v_i v_{i+1} : 1 \leq i \leq n-2,\} \cup \{v_n v_{n-1}\} \cup \{v_1 u_j : 1 \leq j \leq m\}$. Take $S_1 = \{v_i : i \equiv 1\ (mod\ k)\}$. Clearly

$$S = \begin{cases} S_1 - \{v_{n-k-1}\} & if\ n = kt, \\ S_1 - \{v_n, v_{n-k}\} & if\ n = kt+1, \\ S_1 - \{v_{n-i}\} & if\ n = kt + (i+1)\ where\ i = 1, 2, \ldots, k-2. \end{cases}$$

forms a dominating set that is triple connected and certified $(PH_{m,n})^k$ and hence

$$\gamma_{TCC}(PH_{m,n})^k \leq \mid S \mid = \begin{cases} \lfloor \frac{n}{k} \rfloor - 1 & if\ n = kt\ or\ kt+1, \\ \lfloor \frac{n}{k} \rfloor & otherwise. \end{cases}$$

Let us consider the presence of a TCCD-set $D\ of\ (PH_{m,n})^k$ with a cardinality of no more than $d = \begin{cases} \lfloor \frac{n}{k} \rfloor - 2 & if\ n = kt\ or\ kt+1, \\ \lfloor \frac{n}{k} \rfloor - 1 & otherwise. \end{cases}$ Therefore, the induced subgraph $< D >$ lacks triple connectivity, implying that $\gamma_{TCC}(PH_{m,n})^k \geq d+1 = \begin{cases} \lfloor \frac{n}{k} \rfloor - 1 & if\ n = kt\ or\ kt+1, \\ \lfloor \frac{n}{k} \rfloor & otherwise. \end{cases}$

Accordingly, the conclusion follows. □

Theorem 3.2. *For a Butterfly graph represented as $BF_{m,n}$, where$n \geq 2k+2$, k is even,* $\gamma_{TCC}(BF_{m,n})^k = 2(\lceil \frac{n}{k} \rceil - 2) + 1$.

Proof. Let $V(BF_{m,n}) = \{u_2, u_3, \ldots, u_n, v_2, v_3, \ldots, v_n\} \cup \{w_0, w_1, \ldots, w_m\}$ and $E(BF_{m,n}) = \{u_j u_{j+1} : 2 \leq j \leq n-2\} \cup \{v_i v_{i+1} : 2 \leq i \leq n-2\} \cup \{w_i w_0 : 1 \leq i \leq m\} \cup \{u_j w_0 : j = n-1, 2\} \cup \{v_i w_0 : i = n-1, 2\}$. Take $S_1 = \{u_j : j = kt+1, 2 \leq j \leq n-(k+1)\}$, $S_2 = \{v_i : i = kt+1, 2 \leq i \leq n-(k+1)\}$. Claerly $S = S_1 \cup S_2 \cup \{w_0\}$ forms a dominating set that is triple connected and certified $(BF_{m,n})^k$ and hence $\gamma_{TCC}(BF_{m,n})^k \leq \mid S \mid = 2(\lceil \frac{n}{k} \rceil - 2) + 1$. Let us consider the presence of a TCCD-set $D\ of\ (BF_{m,n})^k$ with a cardinality of no more than $d = 2(\lceil \frac{n}{k} \rceil - 2)$. Therefore, the induced subgraph $< D >$ lacks triple connectivity, implying that $\gamma_{TCC}(BF_{m,n})^k \geq d+1 = 2(\lceil \frac{n}{k} \rceil - 2) + 1$.

Accordingly, the conclusion follows. □

Theorem 3.3. *For a Butterfly graph represented as $BF_{m,n}$, where$n \geq 2k+2$, k is odd,*

$$\gamma_{TCC}(BF_{m,n})^k = \begin{cases} 2(\lceil \frac{n}{k} \rceil - 2) + 1 & if\ n = 2kt\ or 2kt+2\ or\ \ldots, or\ 2kt + (2k-2), \\ 2(\lfloor \frac{n}{k} \rfloor - 2) + 1 & if\ n = 2kt + (k+1). \end{cases}$$

Proof. Let $V(BF_{m,n}) = \{u_2, u_3, \ldots, u_n, v_2, v_3, \ldots, v_n\} \cup \{w_0, w_1, \ldots, w_m\}$ and $E(BF_{m,n}) = \{u_z u_{z+1} : 2 \leq z \leq n-2\} \cup \{v_h v_{h+1} : 2 \leq h \leq n-2\} \cup \{w_h w_0 : 1 \leq h \leq m\} \cup \{u_z w_0 : z = n-1, 2\} \cup \{v_h w_0 : h = n-1, 2\}$. Take $S_1 = \{u_z : z \equiv 1 (mod\ k), 2 \leq z \leq n-(k+1)\}$, $S_2 = \{v_h : h \equiv 1\ (mod\ k), 2 \leq h \leq n-(k+1)\}$. Clearly $S = S_1 \cup S_2 \cup \{w_0\}$ forms a dominating set that is triple connected and certified $(BF_{m,n})^k$ and hence

$$\gamma_{TCC}(BF_{m,n})^k \leq \mid S \mid = \begin{cases} 2(\lceil \frac{n}{k} \rceil - 2) + 1 & if\ n = 2kt\ or 2kt+2\ or\ \ldots, or\ 2kt + (2k-2), \\ 2(\lfloor \frac{n}{k} \rfloor - 2) + 1 & if\ n = 2kt + (k+1). \end{cases}$$

Let us consider the presence of a TCCD-set $D\ of\ (BF_{m,n})^k$ with a cardinality of no more than $d = \begin{cases} 2(\lceil \frac{n}{k} \rceil - 2) & if\ n = 2kt\ or 2kt+2\ or\ \ldots, or\ 2kt+(2k-2), \\ 2(\lfloor \frac{n}{k} \rfloor - 2) & if\ n = 2kt+(k+1), \end{cases}$ Therefore, the induced subgraph $< D >$ lacks triple connectivity, implying that $\gamma_{TCC}(BF_{m,n})^k \geq d+1$ $= \begin{cases} 2(\lceil \frac{n}{k} \rceil - 2)+1 & if\ n = 2kt\ or 2kt+2\ or\ \ldots, or\ 2kt+(2k-2), \\ 2(\lfloor \frac{n}{k} \rfloor - 2)+1 & if\ n = 2kt+(k+1). \end{cases}$

Accordingly, the conclusion follows. □

Theorem 3.4. *For a lollipop graph represented as $L_{n,m}$, where $m \geq 3k+1$,*

$$\gamma_{TCC}(L_{n,m})^k = \begin{cases} \frac{m}{k} - 1 & if\ m = kt, \\ \lfloor \frac{m}{k} \rfloor & otherwise. \end{cases}$$

Proof. Let $V(L_{n,m}) = \{u_h : 2 \leq h \leq n, v_z : 2 \leq z \leq m\}$ and $E(L_{n,m}) = \{u_h u_z : 2 \leq h \leq n-1,\ h+1 \leq z \leq n\} \cup \{v_h v_{h+1} : 2 \leq h \leq m-1\} \cup \{u_z w_0 : 2 \leq z \leq n\} \cup \{v_2 w_0\}$.
Take $S_1 = \{v_h : h = kt\}$. Clearly

$$S = \begin{cases} S_1 - \{v_m\} & if\ m = kt, \\ S_1 & if\ m = kt+1\ or\ kt+2\ or \ldots\ or\ kt+(k-1). \end{cases}$$

forms a dominating set that is triple connected and certified $(L_{n,m})^k$ and hence

$$\gamma_{TCC}(L_{n,m})^k \leq \mid S \mid = \begin{cases} \frac{m}{k} - 1 & if\ m = kt, \\ \lfloor \frac{m}{k} \rfloor & otherwise. \end{cases}$$

Let us consider the presence of a TCCD-set $D\ of\ (L_{n,m})^k$ with a cardinality of no more than d $= \begin{cases} \frac{m}{k} - 2 & if\ m = kt, \\ \lfloor \frac{m}{k} \rfloor - 1 & otherwise. \end{cases}$ Therefore, the induced subgraph $< D >$ lacks triple connectivity, implying that $\gamma_{TCC}(L_{n,m})^k \geq d+1 = \begin{cases} \frac{m}{k} - 1 & if\ m = kt, \\ \lfloor \frac{m}{k} \rfloor & otherwise. \end{cases}$

Accordingly, the conclusion follows. □

Example 3.5.

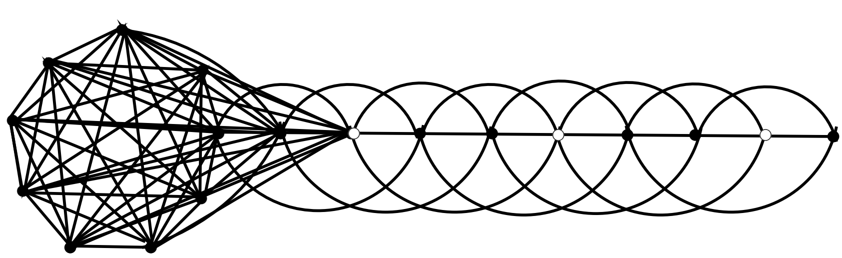

Figure 3. Cube of lollipop graph

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Illustration: Here the set of lightened vertices denote the TCCD set of $L^3_{10,10}$.

Theorem 3.6. *For a Sunlet graph represented as S_n, where $n \geq 3k$,*

$$\gamma_{TCC}(S_n) = \begin{cases} \lceil \frac{n}{k} \rceil & if\ n = kt, \\ \lfloor \frac{n}{k} \rfloor & otherwise. \end{cases}$$

Proof. Let $V(S_n) = \{v_1v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and $E(S_n) = \{v_hv_{h+1} : 1 \leq h \leq n-1\} \cup \{v_nv_{n-1}\} \cup \{v_hu_h : 1 \leq h \leq n\}$. Take $S_1 = \{v_h : h = kt+1\}$. Claerly

$$S = \begin{cases} S_1 & if\ n = kt, \\ S_1 - \{v_{n-(h-1)}\} & if\ n = kt+h, 1 \leq h \leq k-1. \end{cases}$$

forms a dominating set that is triple connected and certified $(S_n)^k$ and hence

$$\gamma_{TCC}(S_n)^k \leq | S | = \begin{cases} \lceil \frac{n}{k} \rceil & if\ n = kt, \\ \lfloor \frac{n}{k} \rfloor & otherwise. \end{cases}$$

Let us consider the presence of a TCCD-set $D\ of\ (S_n)^k$ with a cardinality of no more than $d = \begin{cases} \lceil \frac{n}{k} \rceil - 1 & if\ n = kt, \\ \lfloor \frac{n}{k} \rfloor - 1 & otherwise. \end{cases}$ Therefore, the induced subgraph $< D >$ lacks triple connectivity, implying that $\gamma_{TCC}(S_n)^k \geq d+1 = \begin{cases} \lceil \frac{n}{k} \rceil & if\ n = kt, \\ \lfloor \frac{n}{k} \rfloor & otherwise. \end{cases}$

Accordingly, the conclusion follows. □

Example 3.7.

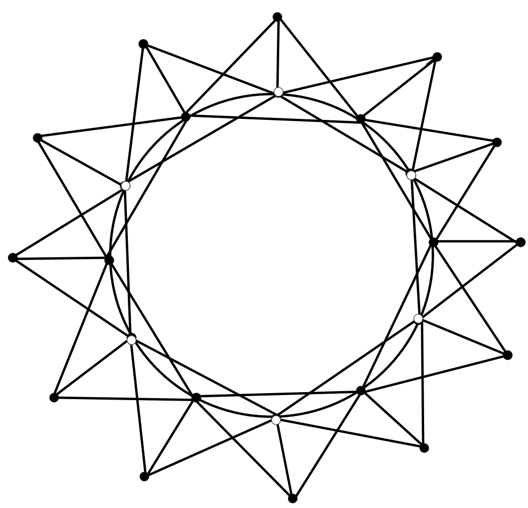

Figure 4. Square of sunlet graph

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Illustration: Here the set of lightened vertices denote the TCCD set of S_{12}^2.

4 TCCD number of Sierpinski Gasket

Construction of Sierpinski graph SG_n:
Step 1: Starting with acycle $C_3 = (v_1, v_2, v_3, v_1)$, place v_1 at top and v_2 in left and v_3 in right, let us denote this graph as SG_1.
Step 2: Make three copies of SG_1 namely SG_1^1, SG_1^2 and SG_1^3 and assume $V(SG_1^i) = \{v_1^i, v_2^i, v_3^i\}$ with v_1^i placed at top, v_2^i placed at left, v_3^i placed at right.
Step 3: Merge v_1^2 and v_2^1 (called a), v_3^2 and v_2^3 (called b), v_1^3 and v_3^1 (called c), then we get the graph SG_2.
Step 4: Repeat the steps 1,2,3 to get SG_{n+1}, where $n \geq 2$.

Example 4.1.

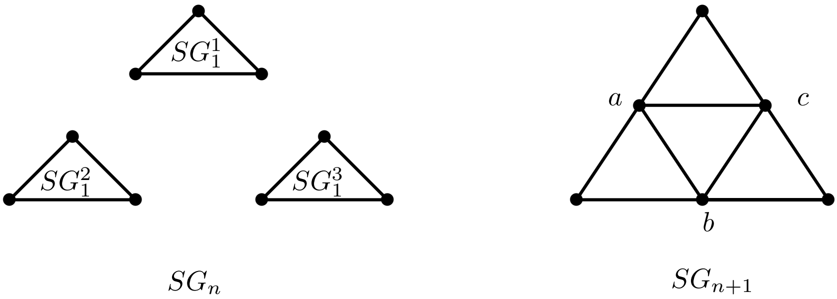

Figure 5.

Illustration: Formation of SG_2 from SG_1.

Remark 4.1. $| V(SG_{n+1}) |= \sum_{p=1}^{n-1} 3^p + 3,\ n \geq 2.$

Remark 4.2. Let $L(SG_n)$ be the number of levels in S_n, then $L(SG_n) = 2^{n-2} + 1$.

Remark 4.3. Let S be the γ_{TCC}-set of SG_n. Then

(i) The diameter of the $< S >$ is $d(< S >) = \gamma_{TCC}(SG_n) - 1$.

(ii) The radius of the $< S >$ is $r(< S >) = \lceil \frac{d(<S>)}{2} \rceil$.

(iii) $d(< S >) = | S_n | - \Delta(< S >) + 1$.

(iv) The periphery of $< S >$ is isomorphic to $\bar{K}_2$.

Example 4.2.

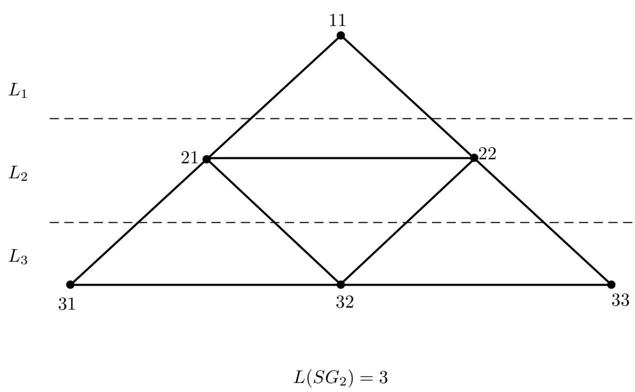

Figure 6. SG_2

Illustration: Here the vetices $21, 22, 32$ are the TCCD - set and $\gamma_{TCC}(SG_2) = L(SG_2) = 3$.

Lemma 4.3. $\gamma_{TCC}(SG_3) = 7$.

Proof. Let $V(SG_3) = \{v_{11}, v_{21}, v_{22}, v_{31}, v_{32}, v_{33}, v_{41}, v_{42}, v_{43}, v_{44}, v_{51}, v_{52}, v_{53}, v_{54}, v_{55}\}$ and $E(SG_3) = \{v_{ij}v_{(i+1)j}, v_{ij}v_{(i+1)(j+1)}, v_{(i+1)j}v_{(i+1)(j+1)} : 1 \leq i \leq 4, 1 \leq j \leq i\} - \{v_{32}v_{42}, v_{42}v_{43}, v_{32}v_{43}\}$. Then $S = \{v_{41}, v_{31}, v_{21}, v_{22}, v_{33}, v_{43}, v_{54}\}$ forms a dominating set which is triple connected and certified SG_3 and hence $\gamma_{TCC}(SG_3) \leq | S |= 7$. Let X be a dominating set which is triple connected and certified of SG_3. Let's consider the presence of a dominating set D with a cardinality of no more than 6. Therefore, the induced subgraph $< D >$ lacks triple connectivity, implying that $| X | \geq 6 + 1 = 7$.
Accordingly, the conclusion follows. □

Example 4.4.

Illustration: Here A^1, A^2, A^3 are the three copies of SG_2

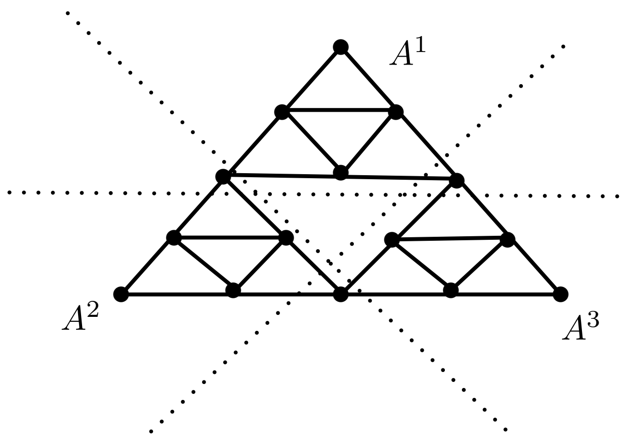

Figure 7. SG_3

Lemma 4.5. $\gamma_{TCC}(SG_4) = 22$.

Proof. Let three copies of SG_3 be SG_3^{θ}, $1 \leq \theta \leq 3$, if $v_{ij} \in V(SG_3)$ then, $v_{ij}^{\theta} \in SG_3^{\theta}$ is a vertex corresponding to v_{ij},$1 \leq \theta \leq 3$. The graph SG_4 is obtained from $SG_3^1 \cup SG_3^2 \cup SG_3^3$ by merging the vertices v_{11}^2 and v_{51}^1,v_{11}^3 and v_{55}^1, v_{55}^2 and v_{51}^3 and we label these vertices as a, b, c respectively. $A^1 = \{v_{21}, v_{22}, v_{31}, v_{33}, v_{41}, v_{43}, v_{54}\}$, $A^2 = \{v_{22}, v_{31}, v_{32}, v_{41}, v_{52}, v_{53}, v_{54}\}$ and $A^3 = \{v_{22}, v_{33}, v_{42}, v_{44}, v_{53}, v_{54}\}, X = \{a, c\}$. It is clear that $D = A^1 \cup A^2 \cup A^3 \cup X$ forms a dominating set that is triple connected and certified S_4 and hence $\gamma_{TCC}(SG_4) \leq \mid D \mid = 7{+}7{+}6{+}2 = 22$. Let X be a dominating set which is triple connected and certified of SG_4. Let's consider the presence of a dominating set D with a cardinality of no more than 21 Therefore, the induced subgraph $< D >$ lacks triple connectivity, implying that $\mid X \mid \geq 21 + 1 = 22$.
Accordingly, the conclusion follows. □

Theorem 4.6. *For* $n \geq 4$, *then* γ_{TCC} $(SG_n) = 3^{n-1} - (\sum_{p=n-3}^{1} 3^p + 2)$.

Proof. Let three copies of SG_{n-1} be SG_{n-1}^1, SG_{n-1}^2 and SG_{n-1}^3. The graph SG_n is obtained from $SG_{n-1}^1 \cup SG_{n-1}^2 \cup SG_{n-1}^3$ by merging the end vertices $v_{11}^{(2)}$ and $v_{L(SG_n)1}^{(1)}$, $v_{11}^{(3)}$ and $v_{L(SG_n)L(SG_n)}^{(1)}$, $v_{L(SG_n)L(SG_n)}^{(2)}$ and $v_{L(SG_n)1}^{(3)}$ and we label these vertices as a, b, c as given in *lemma 4.2.* . Let A^i be the γ_{TCC}-set of SG_{n-1} and it is clear that $\mid A^i \mid = 3^{n-1} - (\sum_{p=n-3}^{1} 3^p + 2)$ where $1 \leq i \leq 3$. Fix B^1 where $B^1 = A^1 \cup \{b\}$ and fix B^2, where $\mid B^2 \mid = \mid A^2 \mid = 3^{n-2} - (\sum_{p=n-4}^{1} 3^p + 2)$ such that $< B^1 \cup B^2 >$ is a path and $B^1 \cup B^2$forms a dominating set that is triple connected and certified $SG_{n-1}^1 \cup SG_{n-1}^2$. Then fix B^3, where $\mid B^3 \mid = 3^{n-2} - (\sum_{p=n-4}^{1} 3^p + 3$ with $< B^1 \cup B^2 \cup B^3 \cup \{c\} >$ is a path. It is clear that $D = B^1 \cup B^2 \cup B^3 \cup \{c\}$forms a dominating set that is triple connected and certified SG_n and hence $\gamma_{TCC}(SG_n) \leq \mid D \mid = 3^{n-1} - (\sum_{p=n-3}^{1} 3^p + 2)$. Let X be a dominating set which is triple connected and certified of SG_n. Let's consider the presence of a dominating set D with a cardinality of no more than $3^{n-1} - (\sum_{p=n-3}^{1} 3^p + 2) - 1$ Therefore, the induced subgraph $< D >$ lacks triple connectivity, implying that $\mid X \mid \geq 3^{n-1} - (\sum_{p=n-3}^{1} 3^p + 2) - 1 + 1 = 3^{n-1} - (\sum_{p=n-3}^{1} 3^p + 2)$.
Accordingly, the conclusion follows. □

5 Conclusion remarks

Throughout this article we have determined the TCCD-number for various graph models such as distance graphs, power graphs, iterated graphs. In upcoming discussions, this parameter will be explored in relation to certain graph operations and compare the results to other graph theoretical parameters.

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