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A STUDY ON LOWER DEG-CENTRIC GRAPHS

Timmy Tomy Thalavayalil and Sudev Naduvath

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Corresponding Author: Timmy Tomy Thalavayalil

Abstract *The lower deg-centric graph of a simple, connected graph G, denoted by G_{ld}, is a graph constructed from G such that $V(G_{ld}) = V(G)$ and $E(G_{ld}) = \{v_iv_j : d_G(v_i, v_j) < deg_G(v_i)\}$. For a connected graph G of order n, the lower deg-centric graph $G_{ld} \cong K_n$ if and only if $deg_G(v_i) > e_G(v_i)$, for all $v_i \in V(G)$. In this paper, the concepts of lower deg-centric graphs and iterated lower deg-centrication of a graph are introduced and discussed.*

1 Introduction

For a basic terminology of graph theory, we refer to [8]. For further topics on graph classes, (see[9, 11]. A graph is assumed to be a simple, connected, and undirected graph throughout this paper. The number of edges of a graph G is denoted by $\varepsilon(G)$. Recall that the distance between two distinct vertices v_i and v_j of G, denoted by $d_G(v_i, v_j)$, is the length of the shortest path joining them. The eccentricity of a vertex $v_i \in V(G)$, denoted by $e(v_i)$, is the farthest distance from v_i to some vertex of G. Vertices at a distance $e(v_i)$ from v_i are called the eccentric vertices of v_i. An eccentric graph *of a graph G, denoted by G_e, is obtained from the same set of vertices as G with two vertices v_i and v_j being adjacent in G_e if and only if v_j is an eccentric vertex of v_i or v_i is an eccentric vertex of v_j (see[1, 2]). The* iterated eccentric graph *of G, denoted by G_{e^k}, is defined in (see[3]) as the derived graph obtained by taking the eccentric graph successively k-times; that is, $G_{e^k} = ((Ge)e \ldots)e$, ($k$-times).*

Similarly, a particular type of newly derived graphs based on the vertex degrees and distances in graphs called deg-centric graphs*, have been introduced in (see[4]) as follows, The degree centric graph or deg-centric graph of G is the graph G_d with $V(G_d) = V(G)$ and $E(G_d) = \{v_iv_j : d_G(v_i, v_j) \le deg_G(v_i)\}$ (see[4]). Let G be a graph and G_d be the deg-centric graph of G. Then, the successive iteration deg-centric graph of G, denoted by G_{d^k}, is the derived graph obtained by taking the deg-centric graph successively k times; that is, $G_{d^k} = ((G_d)_d...)_d$, (k-times). This process is known as deg-centrication process (see[4]). The exact degree centric graph or exact deg-centric graph of a graph G, denoted by G_{ed}, is the graph with $V(G_{ed}) = V(G)$ and $E(G_{ed}) = \{v_iv_j : d_G(v_i, v_j) = deg_G(v_i)\}$. This graph transformation is called exact deg-centrication (see[5]). Let G be a graph and G_{ed} be the exact deg-centric graph of G. Then the iterated exact deg-centric graph of G, denoted by G_{ed^k}, is defined as the graph obtained by applying* exact deg-centrication *successively k-times; That is, $G_{ed^k} = ((G_{ed})_{ed}...)_{ed}$, ($k$-times) (see[5]).*

The upper degree centric graph or upper deg-centric graph of a graph G (assumed to be simple and connected) and denoted by G_{ud}, is the graph with $V(G_{ud}) = V(G)$ and $E(G_{ud}) = \{v_iv_j : d_G(v_i, v_j) \ge deg_G(v_i)\}$. This graph transformation is called upper deg-centrication (see[7]). Let G be a graph and G_{ud} be the upper deg-centric graph of G. Then the iterated upper deg-centric graph of G, denoted by G_{ud^k}, is defined as the graph obtained by applying upper deg-centrication *successively k-times; That is, $G_{ud^k} = ((G_{ud})_{ud}...)_{ud}$, ($k$-times) (see[7]).*

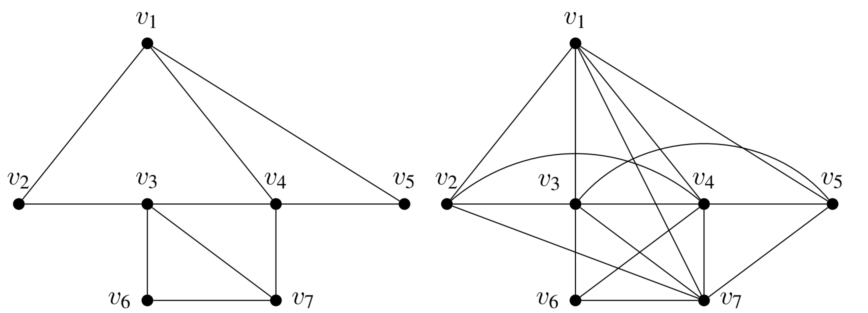
The coarse degree centric graph or coarse deg-centric graph of a graph G, denoted by G_{cd}, is the graph with $V(G_{cd}) = V(G)$ and $E(G_{cd}) = \{v_iv_j : d_G(v_i, v_j) > deg_G(v_i)\}$. Then the iterated coarse deg-centric graph of G, denoted by G_{cd^k}, is defined as the graph obtained by applying coarse deg-centrication *successively k-times; That is, $G_{cd^k} = ((G_{cd})_{cd}...)_{cd}$, ($k$-times) (see[6]).*

Motivated by the above-mentioned studies, in this paper, we introduce a new transformed graph called the lower deg-centric graph and investigate the properties and structural characteristics of this type of transformed graph concerned.

2 Lower Deg-centric Graphs

Definition 2.1. The *lower degree centric graph* or *lower deg-centric graph* of a graph G, denoted by G_{ld}, is the graph with $V(G_{ld}) = V(G)$ and $E(G_{ld}) = \{v_iv_j : d_G(v_i, v_j) < deg_G(v_i)\}$. This graph transformation is called *lower deg-centrication* of the graph. Note that this process is independent of the choice of v_i or v_j in the above sets.

An example of the lower deg-centric graph is given in Figure 1.



(a) G (b) G_{ld}

Figure 1: A graph G and its lower deg-centric graph.

Observation 2.1. The lower deg-centric graph of a complete graph K_n of order $n \geq 3$ is always isomorphic to the complete graph K_n.

Definition 2.2. Let G be a graph and G_{ld} be the lower deg-centric graph of G. Then, the iterated *lower deg-centric graph* of G, denoted by G_{ld^k}, is defined as the graph obtained by applying *lower deg-centrication* successively k-times; That is,

$$G_{ld^k} = ((G_{ld})_{ld}...)_{ld}, \text{ (k-times)}.$$

An example of the lower deg-centrication process of a graph on seven vertices is given in Figure 2.

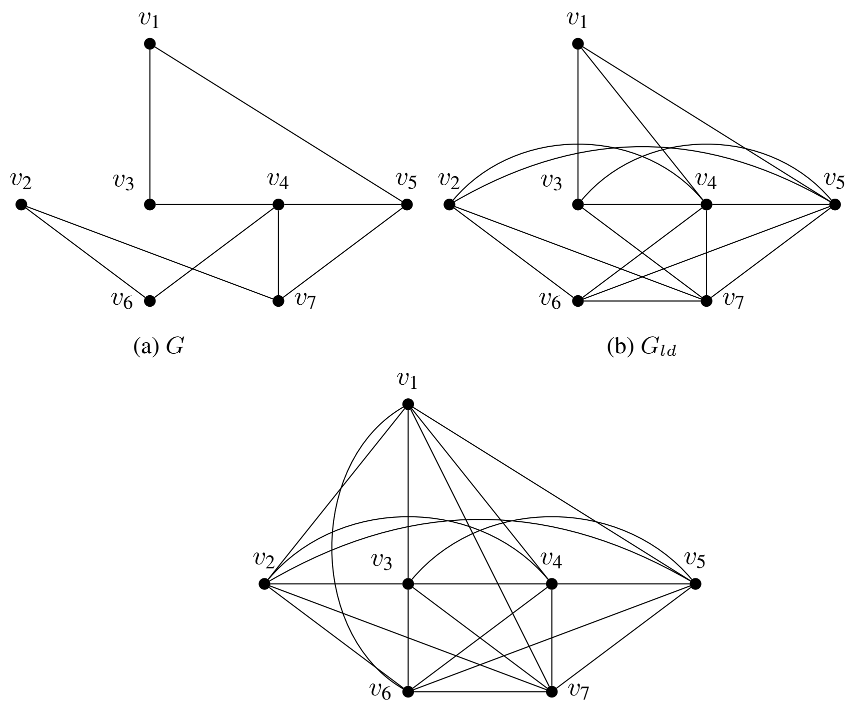
We say that a graph G is D-completable if, after a finite number of iterated lower deg-centrication, the resultant graph is complete. Note that a complete graph is inherently D-completable. An example of a D-completable graph is given in Figure 2.

Let $\varphi(G)$ denote the number of iterations required to transform a D-completable graph G to completion. By convention $\varphi(K_n) = 0$, $n \geq 1$ and $\varphi(K_{1,n}) = \infty$, $n \geq 2$.

Proposition 2.3. *For a connected graph G of order n, the lower deg-centric graph $G_{ld} \cong K_n$ if and only if $deg_G(v_i) > e_G(v_i)$, for all $v_i \in V(G)$.*

Proof. If $G_{ld} \cong K_n$ it implies that vertex $v_i \in V(G)$ contributed $n-1$ edge in G_{ld} thus, $deg_G(v_i) > e_G(v_i)$, for all $v_i \in V(G)$.
On the other hand, if $deg_G(v_i) > e_G(v_i)$, for all $v_i \in V(G)$, then the result that $G_{ld} \cong K_n$ is a direct consequence of the Definition 2.1. □

Proposition 2.4. *Consider a D-completable graph G. If $\delta(G) > diam(G)$, then G_{ld} is complete (or $\varphi(G) = 1$).*



(a) G (b) G_{ld}

(c) $G_{ld^2} = K_7$

Figure 2: Example of the lower deg-centrication process of G.

Proof. Consider any vertex v_i for which $deg_G(v_i) = \delta(G)$. Let

$$X(v_i) = \{v_j : d_G(v_i, v_j) < \delta(G), v_i \neq v_j\}.$$

Clearly, if $\delta(G) > diam(G)$ then, following lower deg-centrication in respect of v_i the resultant closed neighbourhood is $N_{G_{ld}}[v_i] = V(G)$. Finally, because $deg_G(v_j) \geq deg_G(v_i) = \delta(G) > diam(G)$ the resultant closed neighbourhood of each $v_j \in V(G) \backslash \{v_i\}$ is given by, $N_{G_d}[v_j] = V(G)$. Hence, G_{ld} is complete. □

Proposition 2.5. *For a D-completable graph G. Let $m = diam(G)$ then $\varphi(G) \leq \varphi(P_m)$.*

Proof. The proof follows similar reasoning as in Proposition 2.4. □

Theorem 2.6. *For two D-completable graphs G and H each of order n and $\varepsilon(G) < \varepsilon(H)$ it follows that, $\varphi(G) \geq \varphi(H)$.*

Proof. Let graphs G and H each of order n and let the number of edges in graph G be less than the number of edges in graph H. which means the total degree of graph G is less than the graph H. Since,

$$\sum_{v_i \in V(G)} deg_G(v_i) < \sum_{u_j \in V(H)} deg_H(u_j)$$

it implies that the number of new edges yielded through lower deg-centrication denoted by, $\gamma(G_{ld})$ and $\gamma(H_{ld})$ have the relation $\gamma(G_{ld}) < \gamma(H_{ld})$. Also,

$$\sum_{v_i \in V(G_{ld})} deg_{G_{ld}}(v_i) < \sum_{u_j \in V(H_{ld})} deg_{H_{ld}}(u_j).$$

By iterative argument as above it implies that a greater or equal number of iterations are required for G to complete compared to the number of iterations required for H to complete. Hence, $\varphi(G) \geq \varphi(H)$. □

Theorem 2.7. *Any D-completable graph $G \neq K_1$ and which has at least two pendant vertices has $\varphi(G) \geq 2$.*

Proof. Assume G has at least two pendant vertices from the vertex set $v_1, v_2, v_3, \ldots, v_k$. Let X be the set of pendant vertices. At best on the 1^{st} iteration the induced subgraph $\langle V(G)\backslash X \rangle$ can be complete. However, then G_{ld} is a split graph. Which means, split graph is one in which the vertices of the graph can be partitioned into a clique and an independent vertex set. Therefore, $\varphi(G) \geq 2$. □

3 Lower Deg-centrication of Certain Graph Classes

This section will address bistar graphs, path graphs (or paths) and certain interesting graphs. For convenience, a path P_n is depicted on a horizontal line, and the vertices are labelled from left to right as v_1, v_2, v_3,..., v_n.

Proposition 3.1. *For $n \geq 3$, the lower deg-centric graph of a path graph P_n is isomorphic to P_n.*

Proof. Consider a path graph P_n, on a horizontal line with the vertices labelled from left to right as v_1, v_2, v_3,..., v_n. Then, the internal vertices $v_2, v_3, v_4, \ldots v_{n-1}$ have degree two. In view of Definition 2.1, these vertices form an edge to all adjacent vertices in $(P_n)_{ld}$. Both the pendant vertices v_1 and v_n have a degree one. By Definition 2.1, both vertices do not form any edges in $(P_n)_{ld}$, which implies that $(P_n)_{ld} \cong P_n$. □

Observation 3.1. For a path P_n, $n \geq 3$, $\varphi(P_n) = \infty$.

Proposition 3.2. *The lower deg-centric graph of a cycle C_n is isomorphic to C_n.*

Proof. Consider a cycle graph C_n, the vertices $v_1, v_2, v_3, \ldots v_n$ have degree two. In view of Definition 2.1, these vertices form an edge to all adjacent vertices in $(C_n)_{ld}$. Which implies that $(C_n)_{ld} \cong C_n$. □

Observation 3.2. For a cycle C_n, $n \geq 3$, $\varphi(C_n) = \infty$.

A star graph, *denoted by $k_{1,n}$, $n \geq 0$, is obtained by attaching n pendant vertices to a central vertex v_0. Note that, In view of Definition 2.1, the lower deg-centric graph of a star graph $k_{1,n}$, $n \geq 0$, is always isomorphic to the star graph. That is $\varphi(K_{1,n}) = \infty$, $n \geq 2$.*

A non-trivial bistar graph, *denoted by $S_{a,b}$, is a graph obtained by joining the centers of two non-trivial star graphs $K_{1,a}$, $a \geq 1$ and $K_{1,b}$, $b \geq 1$ with the edge v_0u_0.*

Proposition 3.3. *For a bistar graph $S_{a,b}$, $a, b > 1$, $\varepsilon((S_{a,b})_{ld}) = 2a + 2b + 1$.*

Proof. Consider a bistar graph $S_{a,b}$, $a, b > 1$. Let the pendant vertices of $K_{1,a}$ be the set $X = \{v_1, v_2, \ldots, v_a\}$ and let the pendant vertices of $K_{1,b}$ be the set $Y = \{u_1, u_2, \ldots, u_b\}$. Finally, let $W = \{v_0, u_0\}$ be center vertices. By Definition 2.1, it follows that both v_0, u_0 are adjacent with all other $a + b + 1$ vertices. Elements of sets X and Y are pendant vertices, then by Definition 2.1, no edges incident from these pendant vertices, which implies $a + b$ pendant vertices with degree two in $(S_{a,b})_{ld}$. Therefore,

$$\begin{aligned}
\varepsilon((S_{a,b})_{ld}) &= \frac{2(a+b+1) + 2(a+b)}{2} \\
&= 2a + 2b + 1.
\end{aligned}$$

□

Proposition 3.4. *For $S_{a,b}$, $a = 1$ and $b > 1$, $\varepsilon((S_{a,b})_{ld}) = b + 3$.*

Proof. Consider a bistar graph $S_{a,b}$, $a = 1$ and $b > 1$. Then $S_{1,b}$, is obtained by joining the centers of two non-trivial star graphs $k_{1,1}$ and $k_{1,b}$, $b > 1$ with the edge v_0u_0. Let u_0, v_0 be the center vertices of $k_{1,1}$ and $k_{1,b}$ respectively. In view of Definition 2.1, it follows that $deg(v_0) = b+2$ and $deg(u_0) = 2$ in $(S_{a,b})_{ld}$. All b pendant vertices have degree one, and the pendant vertex of star graphs $k_{1,1}$ have degree two in $(S_{a,b})_{ld}$. Therefore, after summation and simplification, it follows that,

$$\varepsilon((S_{a,b})_{ld}) = b + 3.$$

□

Proposition 3.5. *For a bistar graph $S_{a,b}$, $a, b \geq 1$, $\varphi(S_{a,b}) = \infty$.*

Proof. Consider a bistar graph $S_{a,b}$, $a, b \geq 1$. if $a, b = 1$ then $(S_{1,1})_{ld} \cong P_4$. By Observation 3.1, $\varphi(S_{1,1}) = \infty$. If $a, b > 1$, then by Proposition 3.3, all the $a + b$ pendant vertices have degree two in $(S_{a,b})_{ld}$. In view of Definition2.2, $(S_{a,b})_{ld} \cong (S_{a,b})_{ld^k}$. Therefore, $\varphi(S_{a,b}) = \infty$. □

A wheel graph*, denoted by $W_{1,n}$, $n \geq 3$, is obtained by taking a cycle C_n, $n \geq 3$ (the rim with rim-vertices) and adding the central vertex v_0 with* spokes *namely, edges v_0v_i, $1 \leq i \leq n$.*

Proposition 3.6. *The lower deg-centric graph of a wheel graph $W_{1,n}$, $n \geq 3$ is always isomorphic to the complete graph K_{n+1}. That is, $\varphi(W_{1,n}) = 1$.*

Proof. For a wheel graph $W_{1,n}$, $n \geq 3$, note that, $deg(v_i) > e(v_i)$ in wheel graph, for all $v_i \in V(W_{1,n})$. In view of Definition 2.1, $(W_{1,n})_{ld}$ is isomorphic to K_{n+1}. □

A helm graph*, denoted by $H_{1,n,}$, $n \geq 3$, is a graph obtained from a wheel graph $W_{1,n}$ by attaching a pendant vertex u_i to the corresponding*rim vertex *v_i.*

Proposition 3.7. *For a helm graph $H_{1,n,}$, $n \geq 3$, the size of $(H_{1,n,})_{ld}$ is $\frac{3(n^2+n)}{2}$.*

Proof. For a helm graph $H_{1,n,}$, $n \geq 3$, clearly, the helm graph is of the order $2n + 1$. Let $V(H_{1,n,}) = \{v_0, v_1, v_2, \ldots, v_n, \underbrace{u_1, u_2, \ldots, u_n}_{\text{pendant vertices}}\}$. Since $deg(v_0) = deg(v_i) = n > e(v_0) = 2$, in $H_{1,n}$. Then by Definition 2.1, $2n$ edge incident at v_0 and v_i in $(H_{1,n})_{ld}$. However, since all u_i are pendant vertices, by Definition 2.1, no edge incident at u_i in $(H_{1,n})_{ld}$. Then we have

$$\begin{aligned}
\varepsilon((H_{1,n})_{ld}) &= \frac{\sum\limits_{w_i \in V((H_{1,n})_{ld})} deg(w_i)}{2} \\
&= \frac{2n(n+1) + n(n+1)}{2} \\
&= \frac{3(n^2+n)}{2}.
\end{aligned}$$

□

Proposition 3.8. *For a helm graph $H_{1,n,}$, $n \geq 3$. It follows that $\varphi(H_{1,n}) = 2$.*

Proof. Let the vertices of the wheel be $v_0, v_1, v_2, \ldots, v_n$. Let the numerically corresponding pendant vertices be $u_1, u_2, \ldots, u_n$. It is easy to verify that after the first deg-centrication iteration, the induced subgraph $\langle V(W_n) \rangle \cong K_{n+1}$. Furthermore, all edges u_iv_j, $1 \leq i \leq n$, $0 \leq j \leq n$ exist. Since $\delta((H_{1,n})_{ld}) \geq 4$ and $diam((H_{1,n})_{ld}) = 2$, the result follows by Proposition 2.3, Proposition 3.7 and Definition 2.2, which implies, $\varphi(H_{1,n}) = 2$. □

An illustration of proposition 3.7 is given in Figure 3.

A closed helm graph*, denoted by $CH_{1,n}$, $n \geq 3$, is the graph obtained from a helm graph $H_{1,n}$ by joining the pendant vertices, in order, forming a cycle, called the outer rim.*

Proposition 3.9. *For a closed helm graph $CH_{1,n}$, $n \geq 3$,*
(i)$\varphi(CH_{1,n}) = 1$, for $n < 6$.
(ii) $\varphi(CH_{1,n}) = 2$, for $n \geq 6$.

Proof. Consider a closed helm graph $CH_{1,n}$ $n \geq 3$. The closed helm graph is clearly of the order $2n + 1$. Let $V(CH_{1,n,}) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$.
(i) For all $CH_{1,n}$, $n < 6$, $\delta(CH_{1,n}) = 3$. For $n = 3, 4, 5$. In view of Definition 2.1 and Proposition 2.3, the lower deg-centric graph of a closed helm graph $CH_{1,n}$ of order $n < 6$, is the complete graph. Finally, $\varphi(CH_{1,n}) = 1$.
(ii) For $n \geq 6$ we have $\delta(CH_{1,n}) = 3$ and $diam(CH_{1,n}) = 4$. Thus, $\varphi(CH_{1,n}) > 1$. By Definition 2.2 and Proposition 2.3, $\varphi(CH_{1,n}) = 2$. Therefore, the result. □

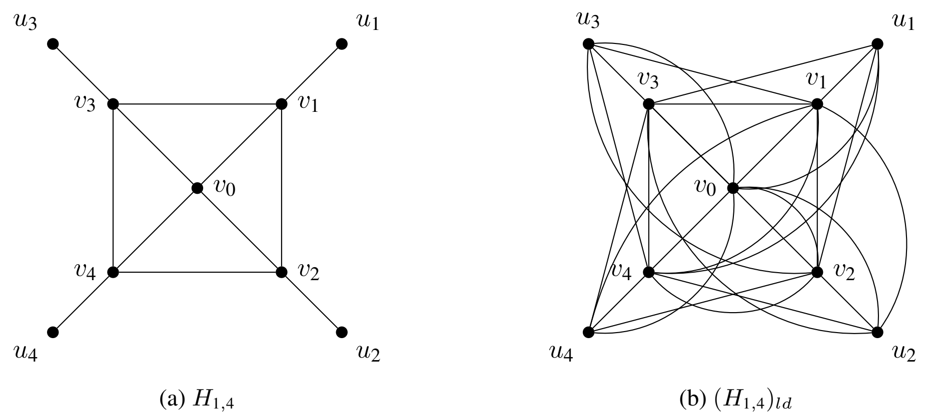


Figure 3: Lower deg-centric graph of $H_{1,4}$.

A djembe graph*, denoted by $D_{1,n}$, is obtained by joining the vertices $u_i's$; $1 \leq i \leq n$ of a closed helm graph $CH_{1,n}$ to its central vertex v_0.*

Proposition 3.10. *The lower deg-centric graph of a djembe graph $D_{1,n}$, $n \geq 3$, is D-completable ($\varphi = 1$).*

Proof. For a djembe graph $D_{1,n}$, $n \geq 3$, clearly, the djembe graph is of the order $2n+1$. Let $V(D_{1,n}) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. Since $deg(v_0) = 2n > e(v_0) = 1$ and $deg(v_i) = deg_G(u_i) = 4 > e(v_0) = 1$ in $D_{1,n}$, by Definition 2.1, $2n$ edge incident at all $2n+1$ vertices in $(D_{1,n})_{ld}$. That is, $(D_{1,n})_{ld} \cong K_{2n+1}$. □

If the edge v_1v_3 joins vertices v_1 and v_3, then the subdivision *of v_1v_3 replaces v_1v_3 by a new vertex v_2 and two new edges v_1v_2 and v_2v_3. A* gear graph*, denoted by G_n, $n \geq 3$, is a graph obtained by applying subdivision to each edge of the rim of a wheel graph $W_{1,n}$.*

Proposition 3.11. *For a gear graph G_n, $n \geq 3$, the size of $(G_n)_{ld}$ is $\frac{n^2+7n}{2}$.*

Proof. For a gear graph G_n, $n \geq 3$, clearly, the gear graph is of the order $2n+1$. Let $V(G_n) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. Since $deg(v_0) = n > e(v_0) = 2$ in G_n, by Definition 2.1, $2n$ edge incident at v_0 in $(G_n)_{ld}$. Vertices v_i are adjacent to the center vertex v_0 However, since $deg(v_i) = 3$ in G_n, by Definition 2.1, $n+2$ edge incident at v_i in $(G_n)_{ld}$. Since $deg(u_i) = 2$, in G_n, by Definition 2.1, distance one edge incident at u_i that is of degree three in $(G_n)_{ld}$. Then we have,

$$\begin{aligned}
\varepsilon((G_n)_{ld}) &= \frac{\sum\limits_{w_i \in V((G_n)_{ld})} deg(w_i)}{2} \\
&= \frac{2n + n(n+2) + 3n}{2} \\
&= \frac{n^2+7n}{2}.
\end{aligned}$$

□

Proposition 3.12. *For a gear graph G_n, $n \geq 3$, $\varphi(G_n) = 2$.*

Proof. For a gear graph G_n, $n \geq 3$. Clearly, the gear graph is of the order $2n+1$. Let $V(G_n) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. By Proposition 3.11, the size of $(G_n)_{ld}$ is $\frac{n^2+7n}{2}$. Then apply Definition 2.2, in $(G_n)_{ld}$. That is, $\varphi(G_n) = 2$. □

A double wheel *DW_n is obtained by taking two copies of a wheel $W_{1,n}$, $n \geq 3$, and* merging *the two central vertices.*

Proposition 3.13. *The lower deg-centric graph of a double wheel graph DW_n, $n \geq 3$ is always isomorphic to the complete graph K_{2n+1}, that is $\varphi = 1$.*

Proof. For a double wheel graph DW_n, $n \geq 3$, clearly, the double wheel graph is of the order $2n+1$. Let $V(DW_n) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. Since $deg(v_0) = 2n > e(v_0) = 1$ and $deg(v_i) = deg_G(u_i) = 3 > e(v_0) = 1$ in DW_n, by Definition 2.1, $2n$ edge incident from all $2n+1$ vertices in $(DW_n)_{ld}$. That is, $(DW_n)_{ld} \cong K_{2n+1}$. □

A flower graph, *$F_{1,n}$, $n \geq 3$, is a graph obtained from a helm graph $H_{1,n,}$ by joining each of its pendant vertices u_i's to its central vertex v_0.*

Proposition 3.14. *For a flower graph $F_{1,n,}$, $n \geq 3$, $\varepsilon(F_{1,n,})_{ld} = \frac{3(n^2+n)}{2}$.*

Proof. Consider a flower graph $F_{1,n,}$, $n \geq 3$. Clearly, the flower graph is of the order $2n+1$. Let $V(F_{1,n,}) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. Since $deg(v_0) = 2n > deg(v_i) = n > e(v_0) = 2$ in $F_{1,n,}$, by Definition 2.1, $2n$ edge incident at v_0 and v_i in $(F_{1,n,})_{ld}$. However, since $deg(u_i) = 2$ in $F_{1,n,}$, by Definition 2.1, distance one edge incident at u_i in lower deg-centric graph. Finally,

$$\begin{aligned}\varepsilon((F_{1,n})_{ld}) &= \frac{\sum\limits_{w_i \in V((F_{1,n})_{ld})} deg(w_i)}{2}\\ &= \frac{2n(n+1) + n(n+1)}{2}\\ &= \frac{3(n^2+n)}{2}.\end{aligned}$$

□

Proposition 3.15. *For a flower graph $F_{1,n}$, $n \geq 3$, $\varphi(F_{1,n}) = 2$.*

Proof. Consider a flower graph $F_{1,n,}$, $n \geq 3$, clearly, the flower graph is of the order $2n+1$. Let $V(F_{1,n,}) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. By Proposition 3.14, the size of $(F_{1,n})_{ld}$ is $\frac{3(n^2+n)}{2}$. In view of Definition 2.2, in $(F_{1,n})_{ld}$. That is, $\varphi(F_{1,n}) = 2$. □

The sunflower graph, *denoted by $SF_{1,n}$, $n \geq 3$ is obtained from the wheel $W_{1,n}$ by attaching n vertices u_i, $1 \leq i \leq n$ such that each u_i is adjacent to v_i and v_{i+1} and count the suffix is taken modulo n.*

Proposition 3.16. *For be a sunflower graph $SF_{1,n,}$, $n \geq 3$, $\varepsilon((SF_{1,n,})_{ld}) = \frac{3(n^2+n)}{2}$.*

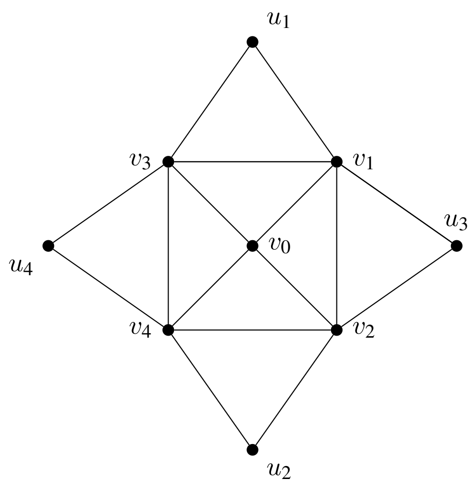
Proof. For a sun flower graph $SF_{1,n,}$, $n \geq 3$, the sunflower graph is of the order $2n+1$. Let $V(SF_{1,n,}) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. Since $deg(v_0) = n > e(v_0) = 2$ and $deg(v_i) = n+1 > e(v_i) = 2$ in $SF_{1,n}$. Then by Definition 2.1, $2n$ edge incident at v_0 and v_i in $(SF_{1,n,})_{ld}$. However, since $deg(u_i) = 2$ $SF_{1,n,}$, by Definition 2.1, distance one edges forms from u_i in $(SF_{1,n,})_{ld}$. Finally,

$$\begin{aligned}\varepsilon((SF_{1,n,})_{ld}) &= \frac{\sum\limits_{w_i \in V((SF_{1,n,})_{ld})} deg(w_i)}{2}\\ &= \frac{2n(n+1) + n(n+1)}{2}\\ &= \frac{3(n^2+n)}{2}.\end{aligned}$$

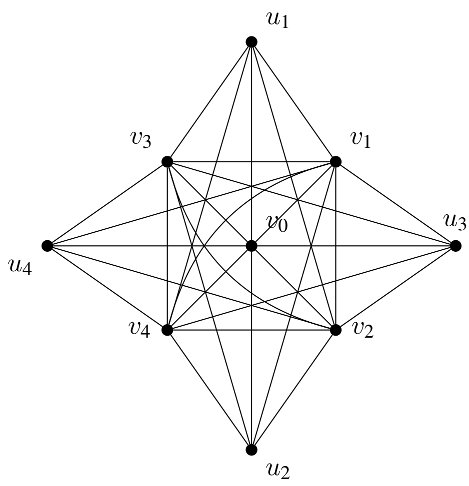
□

Proposition 3.17. *For a sun flower graph $SF_{1,n}$, $n \geq 3$, $\varphi(SF_{1,n}) = 2$.*

Proof. For a sun flower graph $SF_{1,n,}$, $n \geq 3$, the sunflower graph is of the order $2n+1$. Let $V(SF_{1,n,}) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. By Proposition 3.16, the size of $(SF_{1,n})_{ld}$ is $\frac{3(n^2+n)}{2}$. In view of Definition 2.2, in $(SF_{1,n})_{ld}$. That is, $\varphi(SF_{1,n}) = 2$. □



(a) $SF_{1,4}$



(b) $(SF_{1,4})_{ld}$

Figure 4: Lower deg-centric graph of $SF_{1,4}$.

An illustration of proposition 3.16 is given in Figure 4.

A closed sunflower graph *$CSF_{1,n}$ is obtained by adding the edge u_iu_{i+1} of the sunflower graph. In view of Definition 2.1, the lower deg-centric graph of a closed sunflower graph $CSF_{1,n}$, $n \geq 3$ is the complete graph K_{2n+1}.*

Proposition 3.18. *Let $G \cong CSF_{1,n}$ be a closed sunflower graph with $n \geq 3$. Then, G_{ld} is complete (or $\varphi(G) = 1$).*

Proof. Consider a closed sun flower graph $CSF_{1,n,}$, $n \geq 3$. The closed sunflower graph is of the order $2n+1$. Let $V(CSF_{1,n,}) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. By Proposition 2.3, G_{ld} is complete. □

A blossom graph*, denoted by $Bl_{1,n}$, is obtained by making each u_i adjacent to the central vertex of the closed sunflower graph. In view of Definition 2.1 and Proposition 2.3, the lower deg-centric graph of a blossom graph Bl_n, $n \geq 3$, is complete which implies $\varepsilon((Bl_n)_{ld}) = \varepsilon(K_{2n+1})$. That is, $\varphi(Bl_n) = 1$.*

Proposition 3.19. *For a complete bipartite graph $K_{2,b}$, $b \geq 3$. Then,*

$$\varepsilon((K_{2,b})_{ld}) = \varepsilon(K_{2,b}) + 1.$$

Proof. For a complete bipartite graph $K_{2,b}$, $b > 2$, clearly, $K_{2,b}$ is a graph whose vertex set can be partitioned into two independent sets X, $|X| = 2$ and Y, $|Y| = b$. Let $X = v_1, v_2$, and $Y = u_{1,2}, \ldots, v_b$. In view of Definition 2.1, construct $(K_{2,b})_{ld}$ as follows: since $deg(u_i) = 2$ in $K_{2,b}$ and all pairs of vertices u_i, v_j and have $d_(K_{2,b})(u_i, v_j) = 1$ set Y yields an edge with v_1 and v_2 that is complete bipartite graph $K_{2,b}$. Clearly, v_1 and v_2 have degree b, which are adjacent. Hence, $\varepsilon((K_{2,b})_{ld}) = \varepsilon(K_{2,b}) + 1$. □

Observation 3.3. For a complete bipartite graph $K_{a,b}$, $a, b \geq 3$. Then $(K_{a,b})_{ld}$ is complete which implies $\varepsilon((K_{a,b})_{ld}) = K_{a+b}$.

A tree *denoted by T_n, $n \geq 1$ is a connected acyclic graph. It is known that a tree T_n has $n-1$ edges. In views of Definition 2.2, the lower deg-centric graph of a tree T_n,$n \geq 3$ and $\Delta(T_n) \geq 3$, then, $(T_n)_{ld^k}$ is complete.*

Observation 3.4. If $n \geq 3$ and $\Delta(T_n) \geq 3$, then lower deg-centric graph of tree is $D-$completable.

A sunlet graph *, denoted by Sl_n, $n \geq 3$, is a graph obtained by attaching a pendant vertex to every vertex of a cycle graph c_n, $n \geq 3$. In other words, a sunlet graph on $2n$ vertices is obtained by taking the corona product $C_n \circ K_1$.*

Proposition 3.20. *For a sunlet graph Sl_n, $n \geq 3$,*

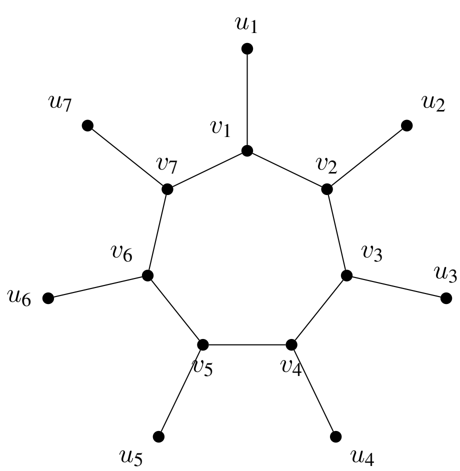
$$\varepsilon((Sl_n)_{ld}) = \begin{cases} \frac{n(2n+2)}{2} & \text{if } n = 3 \\ \frac{n(2n+1)}{2} & \text{if } n = 4 \\ \frac{7n+3n}{2} & \text{if } n = 5, 6, 7, \ldots. \end{cases}$$

Proof. Consider a sunlet graph Sl_n, $n \geq 3$. The sunlet graph is of the order $2n$. Let $V(Sl_n) = \{v_1, v_2, \ldots, v_n, \underbrace{u_1, u_2, \ldots, u_n}_{\text{pendant vertices}}\}$.

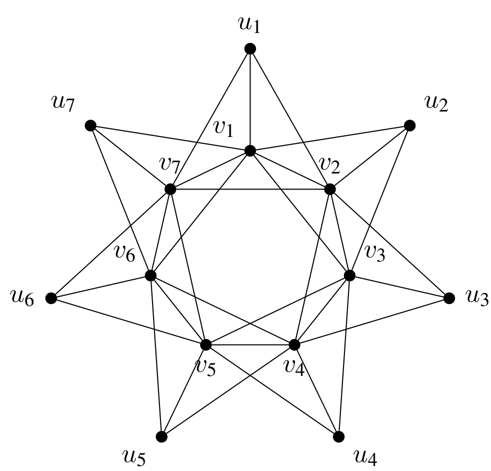
(i) For Sl_3, $deg(v_i) = 3 > e(v_i) = 2$ in Sl_n, as per Definition 2.1, then all v_i vertices are adjacent with other $2n-1$ vertices in lower deg-centric graph. However, since all u_i are pendant vertices, by Definition 2.1, there is no edge incident at u_i. Then number of edges equals $\frac{(3\times5)+(3\times3)}{2} = 12$. Then we have, $\varepsilon((Sl_3)_{ld}) = \frac{n(2n+2)}{2} = \frac{3(2\times3+2)}{2} = 12$.
(ii) For Sl_4, by Definition 2.1, all v_i vertices are adjacent with other $2n-2$ vertices. However, since all u_i are pendant vertices, by Definition 2.1, there is no edge incident at u_i. Then number of edges equals $\frac{(4\times6)+(4\times3)}{2} = 18$. Finally $\varepsilon((Sl_4)_{ld}) = \frac{n(2n+1)}{2} = \frac{4((2\times4)+1)}{2} = 18$.
(iii) Consider $n \geq 5$. By Definition 2.1, all v_i vertices are adjacent with seven vertices. However, since all u_i are pendant vertices, no edge incident at u_i in $(Sl_n)_{ld}$. Then, all u_i have degree three. Then we have $\varepsilon((Sl_n)_{ld}) = \frac{7n+3n}{2}$.

□

An illustration to proposition 3.20 is given in Figure 5.



(a) Sl_7



(b) $(Sl_7)_{ld}$

Figure 5: Lower deg-centric graph of Sl_7.

An antiprism graph, *denoted by A_n, $n \geq 3$ is a graph obtained two cycles C_n and C'_n of order n with vertex sets $V = v_1, v_2, v_3, \ldots, v_n$ and $U = u_1, u_2, u_3, \ldots, u_n$ respectively. Join the vertices u_iv_i and u_iv_{i+1} to form the additional edges.*

Proposition 3.21. *For an antiprism graph A_n, $n \geq 3$,*

$$\varepsilon((A_n)_{ld}) = \begin{cases} \varepsilon(K_{2n}) & \text{if } 3 \leq n \leq 6 \\ 12n & \text{if } n \geq 7. \end{cases}$$

Proof. Consider an antiprism graph A_n, $n \geq 3$. The antiprism graph is of the order $2n$. Let $V(A_n) = v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$.

(i)If $3 \leq n \leq 6$, $deg(v_i) = deg(u_i) = 4 > e(v_i) = e(u_i)$ in A_n then by Theorem 2.3, $(A_n)_{ld} \cong K_{2n}$.
(ii) If $n \geq 7$, $deg(v_i) = deg(u_i) = 4$ in A_n then by Definition 2.1, $deg(v_i) = deg(u_i) = 12$ in $(A_n)_{ld}$. That implies $\varepsilon((A_n)_{ld}) = 12n$.

□

Proposition 3.22. *For an antiprism graph A_n, $n \geq 3$, $(A_n)_{ld^k}$ is complete.*

Proof. The result is a direct consequence of Definition 2.2. □

Consider a complete graph K_n with the vertex set $V = v_1, v_2, v_3, \ldots, v_n$. Let $U = u_1, u_2, u_3, \ldots, u_n$ be a copy of $V(G)$ such that u_i corresponds to v_i. The sun graph*, denoted by S_n, is a graph with vertex set $V \cup U$ and two vertices x and y are adjacent in S_n if $x \sim y$ in K_n and $x = u_i$, $y \in v_i, v_{i+1}$.*

Recall that the sequence of pentagonal numbers is generated by:

$$q_n = \frac{3n^2-n}{2},\ n = 0, 1, 2, \ldots.$$

In expanded form it is:

$$1, 5, 12, 22, 35, 51, 70, 92, 117, 145, 176, 210, 247, \ldots$$

The relation between the size of the lower deg-centric sun graphs and the pentagon numbers follows immediately as a proposition.

Proposition 3.23. *For a sun graph S_n, $n \geq 3$, $\varepsilon((S_n)_{ld}) = q_n, n \geq 3$.*

Proof. For a sun graph $S_n, n \geq 3$, the sun graph is of the order $2n$. Let $V(S_n) = v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. Since $deg(v_i) = n+1 > e(v_i) = 2$ in S_n, by Definition 2.1, $2n-1$ edge incident at v_i in $(S_n)_{ld}$. However, since $deg(u_i) = 2$ in S_n, then by Definition 2.1, distance one edge incident at u_i in the lower deg-centric graph. Then we have

$$\begin{aligned}
\varepsilon((S_n)_{ld}) &= \frac{\sum\limits_{w_i \in V((S_n)_{ld})} deg(w_i)}{2} \\
&= \frac{n(2n-1) + (n)(n)}{2} \\
&= \frac{3n^2 - n}{2}.
\end{aligned}$$

□

A closed sun graph *CS_n is the graph obtained from adding the edges u_iu_{i+1} in the sun graph. In view of Definition 2.1, the lower deg-centric graph of a closed sun graph CS_n, $n \geq 3$, is complete which implies $\varepsilon((CS_n)_{ld}) = \varepsilon(K_{2n})$. That is, $\varphi = 1$.*

The ladder graph*, L_n, $n \geq 1$ is obtained by taking two copies of a path P_n with respective vertices say, $v_1, v_2, v_3, \ldots, v_n$ and $u_1, u_2, u_3, \ldots, u_n$ and adding the edges v_iu_i, $1 \leq i \leq n$. Note that $L_n \cong P_n \Box K_2$ where $\Box$ denotes the Cartesian product.*

Proposition 3.24. *For a ladder $G = L_n$, $n \geq 1$ it follows that:*

$$\begin{gathered}
\varepsilon(L_{1_{ld}}) = 0, \\
\varepsilon(L_{2_{ld}}) = 4, \\
\varepsilon(L_{3_{ld}}) = 11, \\
\varepsilon(L_{4_{ld}}) = 20, \\
\varepsilon(G_{ld}) = \varepsilon(H_{ld}) + 7 \text{ where } H = L_{n-1} \text{ and } n \geq 5.
\end{gathered}$$

Proof. By applying Definition 2.1, it easily follows that $\varepsilon(L_{1_{ld}}) = 0$, $\varepsilon(L_{2_{ld}}) = 4$, $\varepsilon(L_{3_{ld}}) = 11$ and $\varepsilon(L_{4_{ld}}) = 20$. Now, besides the claimed result, it is valid that for any $n \geq 5$ and $H = L_{n-1}$ the size of H_{ld}, that is, $\varepsilon(H_{ld})$ can be determined by applying Definition 2.1. Consider $H = L_{n-1}$ and assume that both H_{ld} and $\varepsilon(H_{ld})$ has been determined. Now consider the extension from H to $G = L_n$. Some subgraph (or altered graph) of H_{ld} is a subgraph of G_{ld}. Note that in G the degree of respectively v_{n-1}, u_{n-1} has increased to 3. Therefore, in G_{ld} the seven edges $v_nu_n, v_nv_{n-1}, v_nu_{n-1}, v_nv_{n-2}, u_nv_{n-1}, u_nu_{n-2}$ and u_nu_{n-1} added to H_{ld}. Hence,

$$\varepsilon(G_{ld}) = \varepsilon(H_{ld}) + 7.$$

Finally, since an initial value, that is, $\varepsilon(L_{4_{ld}}) = 20$, is known, the result for $n \geq 5$ follows through mathematical induction. □

A friendship graph*, denoted by $F_n, n \geq 1$, is obtained by joining n copies of the complete graph K_3 with a common vertex. Note that, In view of Definition 2.1, the lower deg-centric graph of a friendship graph F_n, $n \geq 1$, is always isomorphic to the friendship graph F_n. Which implies, $\varphi(F_n) = \infty$.*

4 Conclusion

The graph transformation called lower deg-centrication has been introduced. Various exploratory results have been presented to establish some foundation for further research. As a scope of the study, the researchers can extend the study on graph theoretical parameters to lower deg-centric graphs of various classes of graphs and obtain fruitful results.

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Author information

Timmy Tomy Thalavayalil, Department of Mathematics, Christ University, Bangalore-560029, India.
E-mail: `timmy.thalavayalil@res.christuniversity.in`

Sudev Naduvath, Department of Mathematics, Christ University, Bangalore-560029, India.
E-mail: `sudev.nk@christuniversity.in`