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ON THE LAPLACIAN ENERGY OF SUNLET AND LADDER GRAPHS

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Abstract In this paper, we examine the graph G characterized by its n vertices and m edges, specifically focusing on its spectral properties by analyzing its Laplacian and Seidel Laplacian matrices. The Laplacian eigenvalues are denoted by $\mu_1, \mu_2, \ldots, \mu_n$, and the Seidel Laplacian eigenvalues are given as $\sigma_1, \sigma_2, \ldots, \sigma_n$. Our primary objective is to determine the upper and lower bounds of the Laplacian energy for sunlet and ladder graphs and to conduct a comparative analysis with their Seidel Laplacian energy counterparts. This investigation aims to deepen the understanding of how the structural attributes of these specific graph types influence their spectral properties.

1 Introduction

In this paper, we delve into the concept of the variants of the graph energy, initially proposed by Ivan Gutman et al. in 1978 [6]. While the comprehensive study of graph energy based on the adjacency matrix spectrum only gained momentum approximately 25 years after its introduction, the research landscape has since broadened significantly. A plethora of alternative graph energy measures have emerged, leveraging matrices other than the adjacency matrix, such as color energy, common-neighborhood energy, detour energy, domination energy, edge energy, Harary energy, Kirchhoff energy, matching energy, Laplacian energy, Seidel Laplacian energy, vertex energy, and α-distance energy [6]. Our focus in this article is specifically on Laplacian energy and Seidel Laplacian energy.

Laplacian eigenvalues are notably important in calculating centrality measures, facilitating network diffusion, and enhancing network synchronization. The Seidel matrix and its associated energy calculations have also been extensively investigated. Utilizing specific graph structures such as the sunlet graph and the ladder graph employed in electronics and wireless communication illustrates their applicability in understanding graph invariants and their functionality as examples of corona and cartesian products, respectively. The novelty of this work is to explore previously unexamined areas of these energy variants and establish comparative bounds.

The organization of the article is structured as follows: Section two revisits the essential definitions and core concepts required for our analysis; section three details the examination of Seidel Laplacian energy across a range of general graphs; section four articulates a general theorem regarding the bounds of Laplacian energy for graphs; the final section compares the outcomes associated with the two energy variants for the graphs under consideration.

Literature Review

Graph energy is a concept with broad applications across several scientific fields including chemistry of unsaturated conjugated molecules, crystallography, macromolecular theory, and protein sequence analysis and comparison. The utility of Laplacian energy, in particular, extends to the field of remote sensing where it is employed to enhance the resolution of satellite images [6].

In the realm of complex networks, the eigenvalues of the Laplacian matrix are critical to the analysis of small-world networks, which are often constructed using analytic recursive equations [9]. Moreover, Laplacian energy has been integrated into methodologies aimed at optimizing transportation management systems. These systems utilize T-spherical fuzzy graphs to model decision-making processes under uncertainty, effectively mirroring the preferences of decision makers[11].

Research on the Laplacian energy of various graph structures, including their variants, continues to expand. Notable studies have calculated the Seidel Laplacian energies of unitary Cayley graphs, fuzzy graphs, and non-complete extended p-sum (NEPS) graphs, contributing significantly to our understanding of these complex mathematical constructs.

2 Preliminaries

Definition 2.1. *[7], [4]* "Given a graph G characterized by n vertices and m edges, where $\mu_1, \mu_2, \ldots, \mu_n$ represent the eigenvalues of the Laplacian matrix of G, the Laplacian energy $LE(G)$ is defined as the sum of the absolute values of the differences between each eigenvalue and the average degree $\frac{2m}{n}$." Specifically, it is given by,

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|.$$

Definition 2.2. *[5]* "A graph G of order n and size m possesses a Seidel matrix $S(G)$, which is defined by the following,

$$S(G) = \begin{cases} -1 & \text{if vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 1 & \text{if vertices } v_i \text{ and } v_j \text{ are not adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n$ are the Seidel eigenvalues of G, and they are ordered such that $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \ldots \geq \sigma_n$. The collection of these eigenvalues is known as the Seidel spectrum of G. The Seidel Laplacian matrix, $S_L(G)$, is defined as $S_L(G) = D_S(G) - S(G)$, where $D_S(G) = \text{diag}(n-1-2d_1, n-1-2d_2, \ldots, n-1-2d_n)$ and $S(G)$ is the Seidel matrix. The degree of vertex v_i is denoted d_i." The Seidel Laplacian energy $SLE(G)$ is defined as,

$$SLE(G) = \sum_{i=1}^{n} \left| \sigma_i - \frac{n(n-1)-4m}{n} \right|.$$

Definition 2.1. *[2]* "A simple graph in which each pair of distinct vertices is joined by an edge is called a complete graph K_n."

Definition 2.3. *[2]* "A complete bipartite graph $K_{m,n}$ is a simple bipartite graph that includes two disjoint vertex sets X and Y. Each vertex in set X is connected to every vertex in set Y by an edge."

Definition 2.4. *[3]* "The star graph $K_{1,n}$ is defined as a tree consisting of $n+1$ vertices, where one central vertex exhibits a degree n being connected to all other vertices and the remaining n vertices exhibiting a degree 1, joined only to the central vertex."

Definition 2.5. *[1]* "The n-sunlet graph is created by augmenting a cycle graph C_n, which comprises n vertices, with n additional pendant edges. Each pendant edge extends from one of the vertices of the cycle, resulting in a total of $2n$ vertices."

Definition 2.6. *[1]* "The ladder graph L_n for $n \geq 2$ is the Cartesian product of the path graphs P_n and P_2, resulting in a graph that includes $2n$ vertices and $3n-2$ edges."

Definition 2.2. *[10]* "A critical point occurs when a function's derivative is zero or undefined. The function's value at a critical point is known as the critical value."

Theorem 2.7. *[5] "*$SLE(G) \leq \sqrt{n\left(n(n-1)+4Z_1(G)-\frac{16m^2}{n}\right)}$ *where* $Z_1(G)$ *is the first Zagreb Index."*

Theorem 2.8. *[7] "*$LE(G) \leq \frac{2m}{n}+\sqrt{(n-1)\left[2M-(\frac{2n}{m})^2\right]}$ *where* $M = m+\frac{1}{2}\sum_{i=1}^{n}\left(d_i-\frac{2m}{n}\right)^2$*."*

3 Seidel Laplacian energy of graphs

This section comprises of the computation of $SLE(G)$ of a few graphs. The $LE(G)$ of complete, complete bipartite, star, friendship graphs are computed in [3]

Theorem 3.1. *The Seidel Laplacian energy of a complete graph* K_n *is* $2(n-1)$.

Proof. A complete graph K_n consists of n vertices and $m = \binom{n}{2}$ edges. The Seidel Laplacian matrix for K_n is given by:

$$\begin{pmatrix} n-1-2d_1 & 1 & 1 & \cdots & 1 \\ 1 & n-1-2d_2 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & 1 \\ 1 & 1 & \cdots & 1 & n-1-2d_n \end{pmatrix}$$

where each diagonal entry is $n-1-2d_i$ and each off-diagonal entry is 1, due to every pair of distinct vertices being adjacent. Therefore, the Seidel Laplacian spectrum for a complete graph is: $\{0, -n\}$ with multiplicities 1 and $n-1$ respectively.

$$SLE(G) = \sum_{i=1}^{n} |\sigma_i + (n-1)| = |0+(n-1)| + (n-1)|-n+(n-1)|.$$

This simplifies to $2(n-1)$, as asserted. □

Theorem 3.2. *The Seidel Laplacian energy of a complete bipartite graph* $K_{m,n}$ *is* $\frac{1}{m+n}\big[|-2n^2-2m^2+m+n|+|n^2+m^2-2mn-m-n|+min\{n-1,m-1\}|-|n-m|(n+m)-[n^2+m^2-m-n]|+max\{n-1,m-1\}\big||n-m|(n+m)-[n^2+m^2-m-n]\big|\big]$

Proof. Consider a complete bipartite graph $K_{m,n}$ consisting of $m+n$ vertices and mn edges. The Seidel Laplacian energy spectrum for $K_{m,n}$ is represented as:

$$\begin{Bmatrix} -(n+m) & 0 & -|n-m| & |n-m| \\ 1 & 1 & \min\{n-1,m-1\} & \max\{n-1,m-1\} \end{Bmatrix}$$

$$SLE(G) = \sum_{i=1}^{m+n} \left|\sigma_i - \frac{(n+m)(n+m-1)-4mn}{m+n}\right|$$

Let $r = \dfrac{n^2+m^2-2mn-m-n}{n+m}$.

$$SLE(G) = |-(n+m)-r|+|-r|+\min\{m-1,n-1\}\,|-|n-m|-r|+\max\{m-1,n-1\}\,||n-m|-r|$$

This formula calculates the total Seidel Laplacian energy by evaluating the absolute differences between the eigenvalues and the specific value r. □

Corollary 3.3. *If* $n = m$ *in* $K_{m,n}$*, the Seidel Laplacian energy is* $4n-2$.

Proof. Consider the graph $K_{n,n}$ whose spectrum is characterized by the Seidel Laplacian eigenvalues as

$$\begin{Bmatrix} -(2n) & 0 \\ 1 & 2n-1 \end{Bmatrix}.$$

From Theorem 3.2, we compute the average vertex degree r adjusted by the total number of vertices, given by

$$r = \frac{-2n}{2n} = -1.$$

Using this value,

$$SLE(G) = |-(2n)+1| + (2n-1) = 4n-2.$$

Therefore, when $n = m$ in $K_{m,n}$, $SLE(G)$ simplifies to $4n-2$, thereby confirming the assertion. □

Corollary 3.4. *The Seidel Laplacian energy of a star graph $K_{1,n}$ is* $\dfrac{(6n-2)(n-1)}{1+n}$.

Proof. Consider the Seidel Laplacian spectrum of the star graph $K_{1,n}$, which consists of

$$\begin{Bmatrix} -(n+1) & 0 & (n-1) \\ 1 & 1 & n-1 \end{Bmatrix}$$

From Theorem 3.2, the average vertex degree adjusted by the total number of vertices, r, is calculated as

$$r = \frac{n(n-3)}{1+n}$$

$$SLE(G) = \frac{1}{1+n}[6n^2 - 8n + 2]$$

Upon simplification, this expression can be factored and rewritten as

$$SLE(G) = \frac{(6n-2)(n-1)}{1+n}$$

Thus, for the star graph $K_{1,n}$, the Seidel Laplacian energy is exactly $\dfrac{(6n-2)(n-1)}{1+n}$, as asserted. □

4 Laplacian energy bounds in sunlet and ladder graphs

In this section, we have obtained the lower and upper bound of simple connected and finite graphs. We know that $2\sqrt{M} \leq LE(G) \leq 2M$ is one of the bounds of $LE(G)$ where $M = m + \frac{1}{2}\sum_{i=1}^{n}(d_i - \frac{2m}{n})^2$ [7].

Theorem 4.1. *If G is a simple, connected, and finite (n, m) graph, then $2\sqrt{M} \leq M, \forall M \geq 4$ and $M \leq LE(G), \forall M$.*

Proof. To demonstrate $M \leq LE(G)$, assume that:

$$\sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right| < M.$$

Since 0 is a Laplacian eigenvalue of G, we have

$$n\left|0 - \frac{2m}{n}\right| < M,$$

which implies $2m < M$. Moreover, considering the following inequality:

$$2m < m + \frac{1}{2}\sum_{i=1}^{n}\left(d_i - \frac{2m}{n}\right)^2,$$

We analyze specific cases for d_i.

- **Case 1:** When $G = K_n$ and $d_i = \frac{2m}{n}$, for $n \geq 1$, this results in $2m < m$, which is a contradiction.
- **Case 2:** For a simple, connected graph where each d_i is at least one (assuming $n > 1$):

$$2m < m + \frac{1}{2}\sum_{i=1}^{n}\left(1 - \frac{2m}{n}\right)^2.$$

This inequality is examined further in subcases:

  - **Subcase 1:** If $m = n$, this suggests $2m < \frac{3m}{2}$, a contradiction.
  - **Subcase 2:** If $n > m$:

$$2m < -m + \frac{n}{2} + \frac{2m^2}{n}.$$

Substituting the maximum value of m,

$$2(n-1) < (n-1) + \frac{n}{2},$$

$$\frac{n}{2} - 1 < 0,$$

leads to a contradiction for $n \geq 2$.

  - **Subcase 3:** If $m > n$, considering

$$2m < -n + \frac{m}{2} + \frac{2m^2}{n}.$$

Substituting the maximum value for n, results in a contradiction, affirming $2m < M$.

Hence $M \leq LE(G)$ and the equality holds for K_1 since $M = 0$
To prove: $2\sqrt{M} \leq M$
Consider $(2\sqrt{M})^2 \leq M^2$ then we have $M(M-4) \geq 0$.
The critical points are $M = 0$ and $M = 4$. The areas that these points define must be tested.

(i) If $M < 0$, both M and $M-4$ are negative then $M(M-4)$ is positive. This condition is omitted since $M \geq 0$ and $n, m \geq 0$

(ii) If $M < 0 < 4$, M is positive and $M-4$ is negative then $M(M-4)$ is negative.

(iii) If $M > 4$, both M and $M-4$ are positive then $M(M-4)$ is positive.

The solution to the inequality $2\sqrt{M} \leq M$ is $M \geq 4$ and $M = 0$ □

Theorem 4.2. *If G be simple connected and finite (n, m) graph, then $LE(G) \leq 2m \leq 2M$*

Proof. We initiate our discussion by articulating $LE(G)$ as follows

$$\begin{aligned} LE(G) &= \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right| \\ &\leq \sum_{i=1}^{n}|\mu_i| \quad \text{(equality holds for } K_1) \\ &\leq 2m \end{aligned}$$

Next, to demonstrate that $2m \leq 2M$, assume the contrary, $2m > 2M$

$$2m > 2m + \sum_{i=1}^{n}\left(d_i - \frac{2m}{n}\right)^2$$

This assumption leads us to consider two specific cases:

- **Case 1:** If $d_i = \frac{2m}{n}$ for complete graphs, we find $2m > 2m$, a contradiction.
- **Case 2:** G is simple, connected and d_i is atleast 1 for $n > 1$, we have, $2m > 2m + \frac{n}{2}\left(1 - \frac{2m}{n}\right)^2$, a contradiction

Hence $2m \leq 2M$, and equality holds specifically for complete graphs. □

Theorem 4.3. $2\sqrt{M} < 3m < LE(G) < 4m < 2M$ *for a sunlet graph* $C_n \odot K_1$

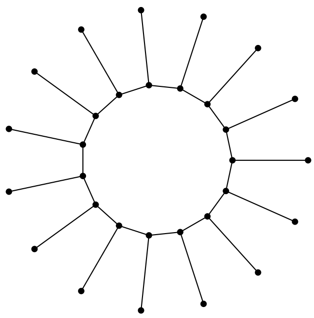

Figure 1. $C_{15} \odot K_1$

Proof. The sunlet graph can be described as the corona product $C_n \odot K_1$, where C_n is a cycle consisting of n vertices. Consequently, the sunlet graph comprises $2n$ vertices, and there are $m = 2n$ edges in the graph. First, we calculate M as follows,

$$M = m + \frac{1}{2}\sum_{i=1}^{2n}(d_i - 2)^2$$

Since half of the vertices have a degree of 3 (vertices from the cycle C_n each connected to an additional vertex K_1) and the other half have a degree of 1 (the vertices of K_1 each connected only to their corresponding vertex on the cycle), the computation simplifies to,

$$M = 2n + \frac{1}{2}\left[n \cdot 1^2 + n \cdot 1^2\right]$$

$$M = 2n + \frac{n}{2} = 3n = 3m$$

Thus, $2M = 6m$. Applying Theorems 4.1 and 4.2 for comparison and bounds, we observe the following inequalities, $2\sqrt{3m} < 3m < LE(G) < 4m < 6m$ □

Theorem 4.4. $2\sqrt{M} < 3n - \frac{4}{n} < LE(G) < 6n - 4 < 2M$ *for a ladder graph* L_n

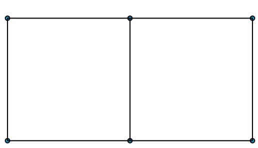

Figure 2. L_3

Proof. Consider a graph L_n composed of $2n$ vertices and $m = 3n - 2$ edges and we compute the proof as follows,

$$M = 3n - 2 + \frac{1}{2}\sum_{i=1}^{2n}\left(d_i - \frac{3n-2}{n}\right)^2$$

Given the vertex degree distribution d_i, we simplify the summation,

$$M = 3n - 2 + \frac{1}{2n^2}\left[4(-(n+2)^2 + (2n-4)(4)\right]$$

This reduces to,

$$M = 3n - 2 + \frac{1}{n^2}[2n^2 - 4n]$$

$$M = 3n - \frac{4}{n} \quad \text{and} \quad 2M = 6n - \frac{8}{n}$$

With these values, we apply the bounds established by Theorems 4.1 and 4.2,

$$2\sqrt{3n - \frac{4}{n}} < 3n - \frac{4}{n} < LE(G) < 6n - 4 < 6n - \frac{8}{n}$$

□

5 Comparison of Laplacian energy and Seidel laplacian energy

Theorem 5.1. *For a graph G with n vertices, let σ_i, for $i = 1, 2, \ldots, n$, represent the eigenvalues of the Seidel Laplacian matrix, and μ_i, for $i = 1, 2, \ldots, n$, denote the Laplacian eigenvalues. Then, the difference between the sum of the Seidel Laplacian eigenvalues and the sum of the Laplacian eigenvalues is given by $\sum_{i=1}^{n} \sigma_i - \sum_{i=1}^{n} \mu_i = n^2 - n - 6m$, where m is the number of edges in G.*

Proof.

$$\begin{aligned}\sum_{i=1}^{n} \sigma_i - \sum_{i=1}^{n} \mu_i &= 2m - (n(n-1) - 4m) \\ &= 2m - n^2 + n + 4m \\ &= n^2 - n - 6m\end{aligned}$$

□

Remark 5.2. If $SLE(G) = LE(G)$ then G is complete and conversely.

Proof. $SLE(G) - LE(G) = 0$ □

Theorem 5.3. $SLE(G) - LE(G) \leq 2n\sqrt{(2n+1)} - 2 - \sqrt{(4n-2)(3n-2)}$ *for a sunlet graph* $C_n \odot K_1$

Proof. Using Theorems 2.7 and 2.8, we have,

$$\begin{aligned}SLE(G) &\leq \sqrt{2n[2n(2n-1) + 4(10n) - 16n]} \\ &\leq \sqrt{4n^2(2n+11)} \\ LE(G) &\leq 2 + \sqrt{(2n-1)[6n-4]} \\ &\leq 2 + \sqrt{(4n-2)(3n-2)} \\ SLE(G) - LE(G) &\leq 2n\sqrt{(2n+1)} - 2 - \sqrt{(4n-2)(3n-2)}\end{aligned}$$

□

Theorem 5.4. *For a ladder graph L_n,*

$$SLE(G) - LE(G) \leq 2\sqrt{n\left[2n^2 - n + 8 - \frac{16}{n}\right]} - \frac{3n-2}{n} - \sqrt{\frac{2n-1}{(3n-2)^2}\left[38n^2 - 48n + 24 - \frac{32}{n}\right]}$$

Proof. Using Theorems 2.7 and 2.8, we obtain,

$$SLE(G) \leq 2\sqrt{n\left[2n^2 - n + 8 - \frac{16}{n}\right]}$$

$$LE(G) \leq \frac{3n-2}{n} + \sqrt{\frac{2n-1}{(3n-2)^2}\left[38n^2 - 48n + 24 - \frac{32}{n}\right]}$$

□

6 Application

The Seidel Laplacian matrix, unlike the standard Laplacian matrix, accounts for both adjacency and non-adjacency interactions, making it ideal for in-depth investigation of fuzzy graphs [8]. Sunlet graphs (corona product) and ladder graphs (cartesian product) are frequently utilized in networks. Other graphs can also be investigated to examine their characteristics utilizing the energy idea, which helps comprehend complex networks and different chemical compounds, including pattern identification and other data safety measures [12]. The Python code for finding the Seidel laplacian energy of any graph is given below:

```
import numpy as np
import networkx as nx

def seidel_matrix(G):
    n = len(G.nodes)
    A = nx.adjacency_matrix(G).toarray()
    S = np.ones((n, n)) - 2 * A  # S = J - 2A, where J is the all-ones matrix
    np.fill_diagonal(S, 0)  # Diagonal entries are zero
    return S
def seidel_laplacian_matrix(G):
    n = len(G.nodes)
    degrees = np.array([d for _, d in G.degree()])
    D_S = np.diag(n - 1 - 2 * degrees)  # Compute D_S
    S = seidel_matrix(G)
    S_L = D_S - S  # Seidel Laplacian matrix
    return S_L
def seidel_laplacian_energy(G):
    n = len(G.nodes)
    m = len(G.edges)
    S_L = seidel_laplacian_matrix(G)
    eigenvalues = np.linalg.eigvalsh(S_L)  # Compute eigenvalues of S_L
    mean_value = (n * (n - 1) - 4 * m) / n
    SLE = np.sum(np.abs(eigenvalues - mean_value))  # Compute SLE
    return SLE
if __name__ == "__main__":
    G = nx.complete_graph(3)  # Example graph
    SLE = seidel_laplacian_energy(G)
    print("Seidel Laplacian Energy:", SLE)
```

```
# Example: Compute the Seidel Laplacian Energy for a complete graph K_3
if __name__ == "__main__":
    G = nx.complete_graph(3)  # Example graph
    SLE = seidel_laplacian_energy(G)
    print("Seidel Laplacian Energy of $K_3$:", SLE)
```

```
Seidel Laplacian Energy of $K_3$: 4.0
```

Figure 3. Example of the formulated code

The above code can be used to obtain the Seidel Laplacian energy of any graph.

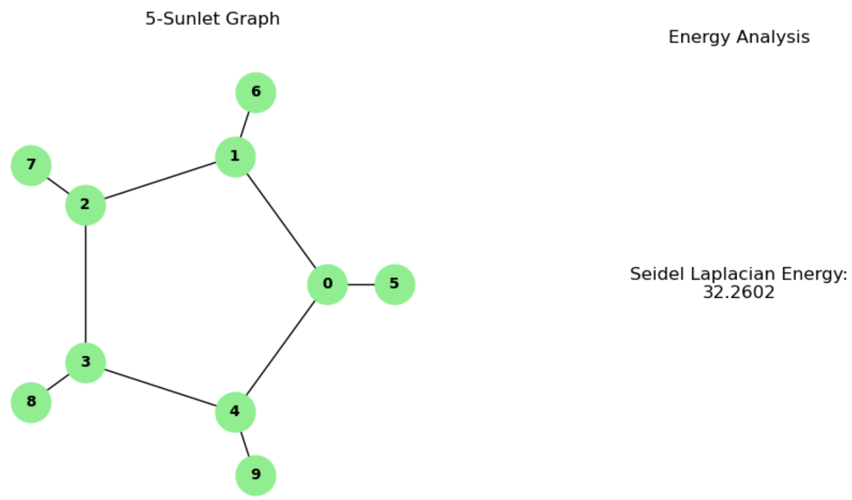

Figure 4. Seidel Laplacian energy of the Sunlet graph

```
Sunlet Graph on 12 vertices:
  Laplacian Energy: 18.155374460927145
  Seidel Laplacian Energy: 40.310748921854284
Ladder Graph on 12 vertices:
  Laplacian Energy: 16.92820323027551
  Seidel Laplacian Energy: 34.52307312721769
```

Figure 5. Output 1: $C_6 \odot K_1$, L_6

```
Sunlet Graph on 14 vertices:
  Laplacian Energy: 21.183587772959093
  Seidel Laplacian Energy: 48.80684060261717
Ladder Graph on 14 vertices:
  Laplacian Energy: 19.701710150654286
  Seidel Laplacian Energy: 42.391338716178474
```

Figure 6. Output 2: $C_7 \odot K_1$, L_7

From the output generated, we infer that $LE(G)$ values of the sunlet graph on 12 vertices and 14 vertices are 18.1553 and 21.18358 whereas the $SLE(G)$ values of the same is found to be 40.3107 and 48.8068 whose difference is large when compared to that of the laplacian energy obtained. The range difference between the Seidel laplacian energies of the graphs gives a better understanding than the laplacian energy values.

7 Conclusion remarks

This article obtained the Seidel Laplacian energy of complete, complete bipartite and star graphs. Although bounds for Laplacian energy existed in the literature, we aimed at attaining an upper bound less than $2M$ and a lower bound greater than $2\sqrt{M}$. The comparison of Laplacian energy to Seidel Laplacian energy for sunlet and ladder graphs was also examined. A comparative analysis can be done for other graphs also. Therefore, the significant results of this work make it fascinating and capable of further research.

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