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SUPPLY CHAIN INVENTORY MODEL FOR FIXED LIFE TIME PRODUCT WITH LINEAR AND FIXED BACK ORDER COST

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Abstract In this paper, we develop Supply chain vendor inventory model for fixed life time product to estimate coordination and non coordination stages with linear and fixed back order cost. The vendor analysis fixed and linear back order cost, which leads to improved efficiency of saving percentage, reduced costs, and increased customer satisfaction. We discuss the significant investment of time and resources to establish the effective partnerships. This model develops coordination stage more benefit for both buyer and vendor. The model includes a numerical example for illustration purposes.

1 Introduction

Buyer-vendor coordination and non-coordination are two contrasting approaches that are often employed in supply chain management to manage relationships between buyers and vendors. Buyer-vendor coordination refers to the collaborative efforts between buyers and vendors to improve supply chain efficiency and achieve mutual benefits. This approach involves information sharing, joint planning, and coordination of activities such as production, inventory management, and transportation. Coordination is often facilitated through the use of technology, such as electronic data interchange (EDI), which allows for the seamless sharing of information between buyers and vendors. In contrast, non-coordination refers to a more traditional approach where buyers and vendors operate independently without any significant collaboration or information sharing. In this approach, each party is primarily concerned with its own interests and aims to optimize its individual performance. When choices are made by each party independently of the other, without taking the supply chain as a whole into account, non-coordination can lead to inefficiencies including overproduction, stock-outs, and delays.

Both coordination and non-coordination approaches have their advantages and disadvantages.While coordination can result in higher productivity, lower expenses, and happier customers, building and maintaining successful partnerships takes a substantial time and resource commitment. Non-coordination may be simpler and more cost-effective in the short term, but it can lead to inefficiencies and missed opportunities in the long run. Furthermore, the choice between coordination and non-coordination may also depend on the specific industry and market conditions. For example, in industries with high levels of product customization, such as fashion or technology, coordination may be necessary to ensure timely delivery of unique products. In contrast, in industries with standardized products, such as consumer packaged goods, non-coordination may be more feasible as vendors can easily replace one another. It is important to note that while coordination may offer benefits, it requires a significant investment of time and resources to establish and maintain effective partnerships. The type of connection between buyers and vendors may also affect the amount of cooperation needed. For example, a closer coordination may be necessary for critical components, while less coordination may be required for less critical products. In conclusion, buyer-vendor coordination and non-coordination are two contrasting approaches to supply chain management that have their advantages and disadvantages.The decision between these strategies may be influenced by a number of variables, including as the product's nature, the market, the industry, and the degree of trust that exists between buyers and sellers. Effective coordination can improve supply chain efficiency and achieve mutual benefits, but it requires significant investment and ongoing maintenance. Non-coordination may be simpler and more cost-effective in the short term but can lead to inefficiencies and missed opportunities in the long run.

2 Literature Review

Inventory model is a numerical model that facilitates business in deciding the ideal level of inventories. In this trade world, deterioration is a characteristic procedure of all wares. In the event of product loss, degradation could be extremely noticeable.A supply chain with a single vendor and numerous retailers was the subject of a model of inventory analysis by Esmaeili, M., & Nasrabadi, M. [5]. This terminology offers a vendor-buyer coordination method in the face of inflation and trade credit deteriorating products. Deterioration of objects is a typical issue in daily life. Deterioration usually means that the products are getting worse. Many researchers on different average inventory models carry out research and hold debates on deteriorating products with variable demand patterns.

S. P. Aggarwal et al [20] considered the deteriorating products by introducing delay credit period.Time-dependent degrading rate was applied by P. Muniappan et al. [14] to analyse an EOQ model for deteriorating products. In their analysis they used different credit payment options which resulted in the benefit of both buyer and vendor. B. Sarkar et al. [1] established an enhanced inventory model when vendors where facing partial backlogging. For model formulation time varying deterioration and stock-dependent demand where considered. Price-sensitive demands for perishable goods were covered by Shib Sankar Sana [19]. The optimum production inventory model with degrading products was established by K.S. Swaminathan et al. [6]; the manufacturer benefited from this model, which produced inventory at the lowest possible cost.

A production model with decaying products was created by P. Muniappan et al. [15] employing a two-level trade credit period. Muniappan P et al. [16] developed mathematical model for computing optimal replenishment polices, it resulted in best accurate ordering period to run the business in positive manner. M. Ravithammal et al. [10] studied two ware house supply chain models. This model deals with the manufacturer who consumes first rental warehouse and then their own warehouses because of space adequacy. For analysis the deteriorating items were considered as ramp- type demand.

M. Ravithammal et al. [11] analyzed an integrated optimum production inventory system which develops vendor buyer coordination strategy. The buyer himself reworks or screens the damaged products. In this case the buyer and vendor both benefited compared with non coordination strategy. M. Ravithammal et al. [12] studied production inventory model by applying quantity discount provided by the manufacturer to their buyers. It deals buying of more products compared with regular lot size. Ravithammal. M et al. [13] developed inventory model by using price discount with various factors like shortages, back ordering and rework.

C.T. Chang et al. [4] discussed EOQ model with inspection errors of the product and using trade credit policy. S.C. Chen et al. [18] discussed retailer's minimum ordering policy by using deteriorating items with fixed lifetime. Sundara rajan. et al. [21] investigation reveals that the total profit gets maximized when the deficiency are absolutely backlogged. S. Amulu Priya et al. [17] investigation reveals that Supply chain management, which is focused on chronological progressin SCM and SC systems. Li. J et al.A supply chain with uncertain demand and dependable suppliers can benefit from the inventory management model put forward by [8]. While guaranteeing a high degree of customer service, the strategy seeks to reduce the overall cost of inventories. Wang, Y., et al.[22] developed a mathematical model for inventory control for a chain of supply with only one provider and several consumers. The model accounts for lead time variability and demand unpredictability in order to optimise the inventory strategy and maximise overall supply chain profit. A mathematical model for managing inventory in a supply network with several vendors and distributors was developed by Li. H. et al. [7]. The model's goal is to reduce total inventory costs while ensuring a high standard of customer care while accounting for demand fluctuation and lead time uncertainty.

3 Observations and Presumptions

The subsequent observations and presumptions were used in this paper
$\pi-$denote the fixed back order cost per unit.
π_1-denote the linear back order cost per unit.
$TC_b(Q,B)-$denote the buyer total cost.
$TC_v(m)-$denote the vendor total cost for non coordination.
$TC_v(n)-$denote the vendor total cost for coordination.
Assumption
The demand rate remains consistent and is predetermined for the entire planning period.

4 Fixed and Linear Back orders in an EOQ Model

Case(i): The EOQ approach is ineffective without co-ordination. In the event of no coordination, the consumer's overall expenditure is determined as follows:

$$TC_b(Q,B) = \frac{Dk_2}{Q} + \frac{(Q-B)^2h_2}{2Q} + \frac{\pi_1 B^2}{2Q} + \frac{\pi DB}{Q} \tag{4.1}$$

$$\frac{\partial TC_b(Q,B)}{\partial Q} = Dk_2(\frac{-1}{Q^2}) + \frac{(Q^2-B^2)h_2}{2Q^2} + \frac{\pi_1 B^2}{2}(\frac{-1}{Q^2}) + \pi DB(\frac{-1}{Q^2}) \text{ and}$$

$$\frac{\partial TC_b(Q,B)}{\partial B} = -\frac{(Q-B)h_2}{Q} + \frac{\pi_1 B}{Q} + \frac{\pi DB}{Q}$$

In order to be ideal $\frac{\partial TC_b(Q,B)}{\partial Q} = 0$ and $\frac{\partial TC_b(Q,B)}{\partial B} = 0$
Now $Q_0^* = \sqrt{\frac{2Dk_2(h_2+\pi_1)-\pi^2D^2}{h_2\pi_1}}$ and $B_0^* = \frac{h_2Q^*-\pi D}{h_2+\pi_1}$
Total Minimum Cost of Buyer is

$$TC_b^* = \frac{1}{h_2+\pi_1}(\sqrt{2Dk_2h_2\pi_1(h_2+\pi_1)-\pi^2D^2}) + \pi Dh_2)$$

The buyer's order quantity is determined when there is no coordination $Q_0 = \sqrt{\frac{2Dk_2(h_2+\pi_1)-\pi^2D^2}{h_2\pi_1}}$ with the annual cost

$$TC_b = \frac{1}{h_2+\pi_1}(\sqrt{2Dk_2h_2\pi_1(h_2+\pi_1)-\pi^2D^2}) + \pi Dh_2)$$

The order size of the vendor is mQ_0, due to a steady stream of demands at set intervals

$$t_0 = \sqrt{\frac{2k_2(h_2+\pi_1)-\pi^2D}{Dh_2\pi_1}}.$$

A vendor's typical inventory is

$$\frac{(m-1)Q_0 + (m-2)Q_0 + \cdots + Q_0 + 0Q_0}{m} = \frac{(m-1)Q_0}{2}.$$

Without cooperation, the vendor's total annual cost would be

$$TC_v(m) = \frac{Dk_1}{mQ_0} + \frac{(m-1)(Q-B)^2h_1}{2Q_0} + \frac{\pi_1(m-1)B^2}{2Q_0} + \frac{\pi DB}{Q_0}.$$

$$= \frac{1}{\sqrt{\frac{2Dk_2(h_2+\pi_1)-\pi^2D^2}{h_2\pi_1}}}\left[\frac{Dk_1}{m} + \frac{(m-1)(\frac{\pi_1 B+\pi D}{h_2})^2h_1}{2} + \frac{\pi_1(m-1)B^2}{2} + \pi DB\right].$$

In the absence of cooperation, the vendor issue is described as,

$$min \;\; TC_v(m) \;\; w.r.to \;\; \{mt_0 \leq L, m \geq 1, \tag{4.2}$$

where $mt_0 \leq L$ shows that goods are not past due before the consumer sells them.

Theorem 4.1. *If* $L^2 \geq \frac{2k_2(h_2+\pi_1)-\pi^2 D}{Dh_2\pi_1}$ *then,*

$$m^* = min\left\{\left\lceil\sqrt{\frac{2Dk_1}{(\frac{\pi_1 B+\pi D}{h_2})^2 h_1 + \pi_1 B^2} + \frac{1}{4}} - \frac{1}{2}\right\rceil, \left\lceil\frac{L}{\sqrt{\frac{2k_2(h_2+\pi_1)-\pi^2 D}{Dh_2\pi_1}}}\right\rceil\right\} \tag{4.3}$$

where $L^2 \geq \frac{2k_2(h_2+\pi_1)-\pi^2 D}{Dh_2\pi_1}$ *is to verify that* $m^* \geq 1$. *and* $\lceil x \rceil$ *is the smallest integer larger than or equivalent to* x.

Proof. Given the narrow convexity of $TC_v(m)$ in m, we have,

$$\frac{d^2 TC_v(m)}{dm^2} = \frac{2Dk_1}{m^3}\sqrt{\frac{h_2\pi_1}{2Dk_2(h_2+\pi_1) - \pi^2 D^2}} > 0.$$

Let us assume that m_1^* represents the ideal value of (4.2), as follows:
$m_1^* = max\{min\{m/TC_v(m) \leq TC_v(m+1)\}, 1\}$
$m_1^* = max\{min\{m/m(m+1) \geq \frac{2Dk_1}{(Q-B)^2 h_1 + \pi_1 B^2}\}, 1\}$
$= \left\lceil\sqrt{\frac{2Dk_1}{(\frac{\pi_1 B+\pi D}{h_2})^2 h_1 + \pi_1 B^2} + \frac{1}{4}} - \frac{1}{2}\right\rceil \geq 1.$
Use $t_0 = \sqrt{\frac{2k_2(h_2+\pi_1)-\pi^2 D}{Dh_2\pi_1}}$ into the above constraints then there exists the following inequality holds.
$m\sqrt{\frac{2k_2(h_2+\pi_1)-\pi^2 D}{Dh_2\pi_1}} \leq L$
Consider $m_2^* = \frac{L}{\sqrt{\frac{2k_2(h_2+\pi_1)-\pi^2 D}{Dh_2\pi_1}}} \geq 1$ because $L^2 \geq \frac{2k_2(h_2+\pi_1)-\pi^2 D}{Dh_2\pi_1}$
$m^* = m_1^*$ when $m_1^* \leq m_2^*$; else, $m^* = m_2^*$ In this case, $TC_v(m)$ is convex.
Let $L^2 \geq \frac{2k_2(h_2+\pi_1)-\pi^2 D}{Dh_2\pi_1}$, then

$$m^* = min\left\{\left\lceil\sqrt{\frac{2Dk_1}{(\frac{\pi_1 B+\pi D}{h_2})^2 h_1 + \pi_1 B^2} + \frac{1}{4}} - \frac{1}{2}\right\rceil, \left\lceil\frac{L}{\sqrt{\frac{2k_2(h_2+\pi_1)-\pi^2 D}{Dh_2\pi_1}}}\right\rceil\right\}$$

Hence we have without any coordination the vendor places $\frac{D}{m^*\left(\sqrt{\frac{2Dk_2(h_2+\pi_1)-\pi^2 D^2}{h_2\pi_1}}\right)}$ orders in every regular interval $\frac{m^*\left(\sqrt{\frac{2Dk_2(h_2+\pi_1)-\pi^2 D^2}{h_2\pi_1}}\right)}{D}$.
The order size for ventor is $m^*\left(\sqrt{\frac{2Dk_2(h_2+\pi_1)-\pi^2 D^2}{h_2\pi_1}}\right)$ and the optimum total cost is $TC_v(m^*)$. □

Case ii. EOQ model with coordination

$$TC_v(n) \;\; = \;\; \frac{Dk_1}{nKQ_0} + \frac{(n-1)K(Q-B)^2 h_1}{2Q_0} + \frac{\pi_1(n-1)KB^2}{2Q_0} + \frac{\pi DB}{Q_0} + Dd(K)p_1 \tag{4.4}$$

If there is insufficient coordination, the issue can be expressed as $minTC_v(n)$ with the appropriate

$$\begin{aligned} nKt_0 \;\; &\leq \;\; L, \\ \frac{Dk_2}{KQ_0} + \frac{K(Q-B)^2 h_2}{2Q_0} + \frac{\pi_1 KB^2}{2Q_0} + \frac{\pi DB}{Q_0} & \\ -\frac{1}{h_2+\pi_1}\left(\sqrt{2Dk_2 h_2\pi_1(h_2+\pi_1) - \pi^2 D^2} + \pi Dh_2\right) \;\; &\leq \;\; p_2 Dd(K), \\ n \;\; &\geq \;\; 1. \end{aligned} \tag{4.5}$$

$nKt_0 \leq$ signifies that goods are not distributed before being paid for by purchasers.

Theorem 4.2. *If m^* be the optimum value of the equation (4.2) and also n^* be the optimum value of (4.5), then*

$$TC_v(n^*) \quad \leq \quad TC_v(m^*). \tag{4.6}$$

Proof.

$$\begin{aligned} p_2 Dd(K) &= \frac{Dk_2}{KQ_0} + \frac{K(Q-B)^2 h_2}{2Q_0} + \frac{\pi_1 KB^2}{2Q_0} + \frac{\pi DB}{Q_0} \\ &\quad - \frac{1}{h_2+\pi_1}\left(\sqrt{2Dk_2h_2\pi_1(h_2+\pi_1) - \pi^2 D^2} + \pi D h_2\right), \\ d(k) &= \frac{1}{p_2 D}\left(\frac{Dk_2}{KQ_0} + \frac{K(Q-B)^2 h_2}{2Q_0} + \frac{\pi_1 KB^2}{2Q_0} + \frac{\pi DB}{Q_0} \right. \\ &\quad \left. - \frac{1}{h_2+\pi_1}\left(\sqrt{2Dk_2h_2\pi_1(h_2+\pi_1) - \pi^2 D^2} + \pi D h_2\right)\right) \end{aligned} \tag{4.7}$$

On substituting K as 1 in (4.7), we obtain the value of d(1) as 0.

$$\begin{aligned} d(1) &= \frac{1}{p_2 D}\left(\frac{1}{h_2+\pi_1}\left(\sqrt{2Dk_2h_2\pi_1(h_2+\pi_1) - \pi^2 D^2} + \pi D h_2\right)\right) \\ &\quad - \frac{1}{p_2 D}\left(\frac{1}{h_2+\pi_1}\left(\sqrt{2Dk_2h_2\pi_1(h_2+\pi_1) - \pi^2 D^2} + \pi D h_2\right)\right) \\ &= 0. \end{aligned}$$

If we set $K = 1$ in (4.5), we get (4.2), which is the specific case of (4.5). Therefore, the inequality is true.

The following formula determines the ideal purchasing amount for both the seller and the consumer.

On substituting the value d(K) in (4.4), we obtain

$$\begin{aligned} TC_v(n) &= \frac{Dk_1}{nKQ_0} + \frac{(n-1)K(Q-B)^2 h_1}{2Q_0} + \frac{\pi_1(n-1)KB^2}{2Q_0} + \frac{\pi DB}{Q_0} \\ &\quad + p_2 D \times \frac{1}{p_2 D}\left(\frac{Dk_2}{KQ_0} + \frac{K(Q-B)^2 h_2}{2Q_0} + \frac{\pi_1 KB^2}{2Q_0} + \frac{\pi DB}{Q_0} \right. \\ &\quad \left. - \frac{1}{h_2+\pi_1}\left(\sqrt{2Dk_2h_2\pi_1(h_2+\pi_1) - \pi^2 D^2} + \pi D h_2\right)\right) \end{aligned} \tag{4.8}$$

In K, $d(K)$ is convex since (4.8) is a convex function. Let K^* be the smallest value of $TC_v(n)$. When $\frac{dTC_v(n)}{dK} = 0$ for flawlessness, we obtain

$$K^*(n) \quad = \quad \sqrt{\frac{2D\left(\frac{k_1}{n} + k_2)\right)}{(Q_0 - B)^2((n-1)h_1 + h_2) + \pi_1 n B^2}} \tag{4.9}$$

Now, put $K^*(n)$ and $t_0 = \sqrt{\frac{2k_2(h_2+\pi_1) - \pi^2 D}{Dh_2\pi_1}}$ into the first constraint of (4.5), we obtain

$$2n^2\left(\frac{k_1}{n} + k_2\right)(2k_2(h_2+\pi_1) - \pi^2 D) \quad \leq \quad L^2 h_2 \pi_1 \left(\left(\frac{\pi_1 B + \pi D}{h_2}\right)^2 ((n-1)h_1 + h_2) + \pi_1 n B^2\right)$$

Consider

$$f(n) = \left(-4k_2^2h_2(h_2+\pi_1) - 2k_2h_2\pi^2D\right)n^2 + \left(L^2\pi_1h_1(\pi_1B+\pi D)^2 + L^2\pi_1^2h_2^2B^2 - 4k_1k_2h_2(h_2+\pi_1) + 2k_1h_2\pi^2D\right)n + L^2\pi_1(h_2-h_1)(\pi_1B+\pi D)^2. \quad (4.10)$$

Thus, $nKt_0 \leq L$ is similar to $f(n) \geq 0$. Substitute $K^*(n)$ and $t_0 = \sqrt{\dfrac{2k_2(h_2+\pi_1)-\pi^2D}{Dh_2\pi_1}}$ in $TC_v(n)$, we get

$$TC_v(n) = \sqrt{\frac{h_2\pi_1}{2k_2(h_2+\pi_1)-\pi^2D}\left[2D\left(\frac{k_1}{n}+k_2\right)[(Q_0-B)^2[(n-1)h_1+h_2]+\pi_1nB^2]+4\pi^2D^2B^2\right]} - \left(\sqrt{\frac{2Dk_2h_2\pi_1(h_2+\pi_1)-\pi^2D^2+\pi^2D^2h_2^2}{(h_2+\pi_1)^2}}\right) \quad (4.11)$$

This is hence equal to $min\, TC_v(n)$ with respect to

$$\begin{cases} f(n) \geq 0, \\ n \geq 1. \end{cases} \quad (4.12)$$

For $x \geq 0$, the function $\sqrt{x}$ absolutely grows. (4.12) comparable to

$$\min T^{\sim}C_v(n) = \frac{h_2\pi_1}{2k_2(h_2+\pi_1)-\pi^2D^2}\left[2D\left(\frac{k_1}{n}+k_2\right)[(Q_0-B)^2[(n-1)h_1+h_2]+\pi_1nB^2] + 2\pi^2D^2B^2\right]$$

with respect to

$$\begin{cases} f(n) \geq 0, \\ n \geq 1. \end{cases} \quad (4.13)$$

To solve the preceding equation, consider $T^{\sim}C_v(n)$ and $f(n)$. Since $T^{\sim}C_v(n)$ is convex when $h_2 \geq h_1$, because

$$\min T^{\sim}C_v''(n) = \frac{h_2\pi_1}{2k_2(h_2+\pi_1)-\pi^2D^2}\left[\frac{2k_1(Q_0-B)^2(h_2-h_1)}{n^3}\right] > 0$$

$f(n)$ is strictly concave since,

$$f''(n) = -2\left[4k_2^2h_2(h_2+\pi_1) - 2k_2h_2\pi^2D\right] < 0.$$

Case 1. For $n \geq 1$, we consider that n_1^* is the minimum of $T^{\sim}C_v(n)$, then

$$n_1^* = \begin{cases} \sqrt{\frac{k_1(h_2-h_1)(\pi_1B+\pi D)^2}{k_2(\pi_1B^2h_2^2+h_1(\pi_1B+\pi D)^2)}+\frac{1}{4}}-\frac{1}{2}, & \frac{k_1(h_2-h_1)(\pi_1B+\pi D)^2}{k_2(\pi_1B^2h_2^2+h_1(\pi_1B+\pi D)^2)} \geq 2, \\ 1 \text{ otherwise.} \end{cases} \quad (4.14)$$

Proof. $T^{\sim}C_v(n_1^*) \leq \min\{T^{\sim}C_v(n_1^*-1),\ T^{\sim}C_v(n_1^*+1)\}$ is true if the minimum of $T^{\sim}C_v(n)$ is n_1^*, $n \geq 1$.

$$T^{\sim}C_v(n_1^*) - T^{\sim}C_v(n_1^*-1) = \left(\frac{\pi_1B+\pi D}{h_2}\right)^2\left[k_2h_1 - \frac{k_1(h_2-h_1)}{n_1^*(n_1^*-1)}\right] + \pi_1k_2B^2 \leq 0.$$

We get,

$$\left(n_1^* - \frac{1}{2}\right)^2 \leq \frac{k_1(h_2-h_1)(\pi_1B+\pi D)^2}{k_2(\pi_1B^2h_2^2+h_1(\pi_1B+\pi D)^2)} + \frac{1}{4} \quad (4.15)$$

and also by

$$T^{\sim}C_v(n_1^*) - T^{\sim}C_v(n_1^*+1) \quad \leq \quad 0.$$

We get

$$\left(n_1^* - \frac{1}{2}\right)^2 \quad \geq \quad \frac{k_1(h_2-h_1)(\pi_1 B+\pi D)^2}{k_2(\pi_1 B^2 h_2^2 + h_1(\pi_1 B+\pi D)^2)} + \frac{1}{4} \tag{4.16}$$

Hence

$$n_1^* \quad = \quad \sqrt{\frac{k_1(h_2-h_1)(\pi_1 B+\pi D)^2}{k_2(\pi_1 B^2 h_2^2 + h_1(\pi_1 B+\pi D)^2)} + \frac{1}{4}} - \frac{1}{2}$$

when

$$\sqrt{\frac{k_1(h_2-h_1)(\pi_1 B+\pi D)^2}{k_2(\pi_1 B^2 h_2^2 + h_1(\pi_1 B+\pi D)^2)} + \frac{1}{4}} - \frac{1}{2} \leq n_1^* \leq \sqrt{\frac{k_1(h_2-h_1)(\pi_1 B+\pi D)^2}{k_2(\pi_1 B^2 h_2^2 + h_1(\pi_1 B+\pi D)^2)} + \frac{1}{4}} + \frac{1}{2}$$

Otherwise $n_1^* = 1$ when

$$\frac{k_1(h_2-h_1)(\pi_1 B+\pi D)^2}{k_2(\pi_1 B^2 h_2^2 + h_1(\pi_1 B+\pi D)^2)} + \frac{1}{4} < 0.$$

Also note that $n_1^* = 1$

$$0 < \frac{k_1(h_2-h_1)(\pi_1 B+\pi D)^2}{k_2(\pi_1 B^2 h_2^2 + h_1(\pi_1 B+\pi D)^2)} + \frac{1}{4} < 2.$$

Hence (4.14) is true.

Case 2. Let $n_{2(1)}^*$ and $n_{2(2)}^*$ be solutions of (4.10), then

if $Y^2 + 4XZ < 0$ or $Y^2 + 4XZ \geq 0$ and $n_{2(1)}^* < 1$, implies that $f(n) < 0$ for $n \geq 1$.

If $Y^2 + 4XZ \geq 0$ and $n_{2(1)}^* \geq 1$, then $n_{2(2)}^* \geq 1$, $f(n) \geq 0$.

For $\lceil n_{2(2)}^* \rceil \leq n \leq \lceil n_{2(1)}^* \rceil$; If $n_{2(2)}^* < 1$ and $n_{2(1)}^* \geq 1$, $f(n) \geq 0$ for $1 \leq n \leq \lceil n_{2(1)}^* \rceil$;

$$\begin{aligned} X &= -4k_2^2 h_2(h_2+\pi_1) - 2k_2 h_2 \pi^2 D, \\ Y &= L^2 \pi_1 h_1(\pi_1 B+\pi D)^2 + L^2 \pi_1^2 h_2^2 B^2 - 4k_1 k_2 h_2(h_2+\pi_1) + 2k_1 h_2 \pi^2 D, \\ Z &= L^2 \pi_1 (h_2-h_1)(\pi_1 B+\pi D)^2. \end{aligned}$$

Proof. Solving the quadratic equation $f(n) = 0$, we obtain

$$n_{2(1)}^* \quad = \quad \frac{Y + \sqrt{Y^2+4XY}}{2X},$$

and

$$n_{2(2)}^* \quad = \quad \frac{Y - \sqrt{Y^2+4XY}}{2X}.$$

Taking into account that $f(n)$ is a quadratic expression, the subsequent conclusions are possible.

$f(n) < 0$ where $Y^2 + 4XY < 0$ for all n,

$n_{2(1)}^*$, $n_{2(2)}^*$ are all realvalues of $f(n)$ where $Y^2 + 4XY \geq 0$.

Given that $n \geq 1$,

$f(n) < 0$ where $n^*_{2(1)} < 1$, for $n \geq 1$;

$f(n) \geq 0$ where $n^*_{2(2)} \geq 1$, for $\lceil n^*_{2(2)} \rceil \leq n \leq \lceil n^*_{2(1)} \rceil$;

$f(n) \geq 0$ where $n^*_{2(2)} < 1$ and $n^*_{2(1)} \geq 1$, for $1 \leq n \leq \lceil n^*_{2(1)} \rceil$. □

Theorem 4.3.

$$\begin{aligned} n^* &= n^*_1 \text{ if } 1 \leq n^*_1 \leq \lceil n^*_{2(1)} \rceil, \\ n^* &= \lceil n^*_{2(1)} \rceil \text{ if } n^*_1 > \lceil n^*_{2(1)} \rceil. \text{ when } h_2 \geq h_1 n^*_{2(2)} \geq 1. \end{aligned}$$

If the minimum of $T^{\sim}C_v(n)$ is denoted by n^*_1, $(n \geq 1)$. $T^{\sim}C_v(n)$ is a convex function.

Hence if $n^* = n^*_1$, $1 \leq n^*_1 \leq \lceil n^*_{2(1)} \rceil$ else $n^* = \lceil n^*_{2(1)} \rceil$, $n^*_1 > \lceil n^*_{2(1)} \rceil$.

Thereby, $T^{\sim}C_v(n)$ is decreasing on the interval as $n^*_1 > \lceil n^*_{2(1)} \rceil$, so $n^* = \lceil n^*_{2(1)} \rceil$.

Theorem 4.4. *If $h_2 \geq h_1$, then $K^*(n^*) > 1$.*

Proof.

$$\begin{aligned} K^*(n) &= \sqrt{\frac{2D\left(\frac{k_1}{n} + k_2\right)}{(Q_0 - B)^2\left[(n-1)h_1 + h_2\right] + \pi_1 n B^2}} \\ &= \sqrt{\frac{2D\left(\frac{k_1}{n} + k_2\right)}{\left(\frac{\pi_1 B + \pi D}{h_2}\right)^2\left[(n-1)h_1 + h_2\right] + \pi_1 n B^2}} \end{aligned}$$

(i) If $\dfrac{k_1(h_2 - h_1)(\pi_2 B + \pi D)^2}{k_2(\pi_1 B^2 h_2^2 + h_1(\pi_1 B + \pi D)^2)} \geq 2$ then $n^* = n^*_1$.

(i.e), $n^* = n^*_1 = \left\lceil \sqrt{\dfrac{k_1(h_2 - h_1)(\pi_1 B + \pi D)^2}{k_2(\pi_1 B^2 h_2^2 + h_1(\pi_1 B + \pi D)^2)} + \dfrac{1}{4}} - \dfrac{1}{2} \right\rceil$

$K^*(n^*)$ is $\left\lceil \sqrt{x + \dfrac{1}{4}} - \dfrac{1}{2} \right\rceil \leq \sqrt{x} + 1$, then n is diminishing. This is true for $x \geq 0$.

To prove $K^*\left\lceil \sqrt{\dfrac{k_1(h_2 - h_1)(\pi_1 B + \pi D)^2}{k_2(\pi_1 B^2 h_2^2 + h_1(\pi_1 B + \pi D)^2)}} + 1 \right\rceil > 1$.

$$2Dk_1 + 2Dk_2\sqrt{\frac{k_1(h_2 - h_1)(\pi_1 B + \pi D)^2}{k_2(\pi_1 B^2 h_2^2 + h_1(\pi_1 B + \pi D)^2)}} + 2Dk_2 \quad >$$

$$\left(\frac{\pi_1 B + \pi D}{h_2}\right)^2\left(\frac{k_1(h_2 - h_1)(\pi_1 B + \pi D)^2}{k_2(\pi_1 B^2 h_2^2 + h_1(\pi_1 B + \pi D)^2)}\right)h_1$$
$$+\left(\frac{\pi_1 B + \pi D}{h_2}\right)^2\sqrt{\frac{k_1(h_2 - h_1)(\pi_1 B + \pi D)^2}{k_2(\pi_1 B^2 h_2^2 + h_1(\pi_1 B + \pi D)^2)}}h_1 + \pi_1 B^2\left(\frac{k_1(h_2 - h_1)(\pi_1 B + \pi D)^2}{k_2(\pi_1 B^2 h_2^2 + h_1(\pi_1 B + \pi D)^2)}\right)$$

Therefore the above equation holds if $k_1, k_2, h_1, h_2, B, D, \pi_1, \pi$ are all positive and $h_2 \geq h_1$.

(ii) If $n^* = n^*_1 = 1$ then

$$K^*(1) = \sqrt{\frac{2Dh_2(k_1 + k_2)}{(\pi_1 B + \pi D)^2 + \pi_1 B^2 h_2}}$$

Hence $K^*(1) > 1$ if $k_1, k_2, h_2, B, D, \pi_1, \pi$ are all positive.

(ii) If $n^* = \lceil n^*_{2(1)} \rceil$, $n^*_1 > \lceil n^*_{2(1)} \rceil$ then $K^*\left(\lceil n^*_{2(1)} \rceil\right) \geq K^*(n^*_1) > 1$.

From (i) to (iii), $K^*(n) > 1$ if $h_2 \geq h_1$. □

5 Numerical Analysis on various decision parameters

h_1	h_2	π	π_1	d(K)	Q	B	SP_b	SP_{v_1}	SP_{v_2}
5	10	0.03	1.2	0.0018	1363.5	1190.6	16.4538	11.5707	23.1415
5	10	0.04	1.3	0.0019	1313.8	1127.3	15.5082	11.0436	22.0872
5	10	0.05	1.4	0.0020	1269.1	1069.4	14.7079	10.5891	21.1783
5	10	0.06	1.5	0.0021	1228.5	1016.1	14.0240	10.1954	20.3908
5	9	0.05	1.4	0.0020	1277	1057.1	14.2213	10.2354	20.4709
5	10	0.06	1.5	0.0021	1229	1016.1	14.0240	10.1954	20.3908
5	10	0.01	1.0	0.0016	1482.9	1339.0	18.9643	12.9155	25.8309
6	11	0.02	1.1	0.0016	1413.0	1268.1	17.9752	12.4589	24.9178
7	12	0.03	1.2	0.0017	1351.7	1206.1	17.1784	12.0848	24.1696
8	13	0.04	1.3	0.0018	1297.2	1151.3	16.5226	11.7731	23.5462
9	14	0.05	1.4	0.0018	1248.5	1102.5	15.9732	11.5099	23.0197
10	15	0.06	1.5	0.0019	1204.4	1058.6	15.5064	11.2851	22.5702
5	10	0.01	1.0	0.0016	1482.9	1339.0	18.9643	12.9155	25.8309
6	10	0.02	1.1	0.0017	1419.3	1260.7	17.5855	12.1870	24.3740
7	10	0.03	1.2	0.0018	1363.5	1190.6	16.4538	11.5707	23.1415
8	10	0.04	1.3	0.0019	1313.8	1127.3	15.5082	11.0436	22.0872
9	10	0.05	1.4	0.0020	1269.1	1069.4	14.7079	10.5891	21.1783
10	10	0.06	1.5	0.0020	1228.5	1069	14.0240	10.1954	20.3908
5	10	0.01	1.3	0.0019	1318.2	1157.7	17.8470	12.1214	24.2428
5	10	0.01	1.4	0.0020	1275.9	1110.4	17.4659	11.8540	23.7081
5	10	0.01	1.5	0.0021	1238.0	1067.8	17.0824	11.5862	23.1724
5	10	0.01	1.6	0.0022	1203.9	1029.2	16.6971	11.3182	22.6364
5	10	0.02	1.0	0.0016	1481.9	1329.0	17.8805	12.4144	24.8288
5	10	0.03	1.0	0.0015	1480.2	1318.4	16.9452	11.9731	23.9462
5	10	0.04	1.0	0.0015	1477.8	1307.1	16.1348	11.5847	23.1694
5	10	0.05	1.0	0.0015	1474.8	1295.3	15.4306	11.2434	22.4869
5	10	0.02	1.1	0.0017	1419.3	1260.7	17.5855	12.1870	24.3740
5	10	0.03	1.2	0.0018	1363.5	1190.6	16.4538	11.5707	23.1415
5	10	0.04	1.3	0.0019	1313.8	1127.3	15.5082	11.0436	22.0872
5	10	0.05	1.4	0.0020	1269.1	1069.4	14.7079	10.5891	21.1783
5	10	0.06	1.5	0.0021	1228.5	1016.1	14.0240	10.1954	20.3908

Inference:

Holding costs (h_1 and h_2) play a significant role in any business organization and supply chain process. In our discussion, we explore scenarios where the buyer's holding cost is fixed while the vendor's holding cost increases. Our analysis shows that in these situations, both the buyer and the vendor can experience an increase in savings percentage, leading to a win-win outcome for both parties.

We have analyzed scenarios where both the buyer's and vendor's holding costs for the product are fixed. In this analysis, we determined the optimum order quantity, backorder costs, and the resulting savings percentage. Our findings indicate that this approach is more beneficial for both the buyer and the vendor.

Efficiency of the proposed model:

(i) Effect of Holding Costs: Increasing the values of the holding costs h_1 and h_2 increases the total saving percentage for both the buyer and the vendor. However, the vendor benefits more than the buyer in this scenario.

(ii) Impact of Linear Backorder Cost: The saving percentage decreases when the linear backorder cost increases, while the holding costs and fixed backorder costs remain constant.

(iii) Effect of Backorder and Holding Costs: The saving percentage decreases when either the backorder cost or the holding cost for the buyer or vendor increases.

6 Conclusion:

Effective inventory management of vendor-manufactured, buyer-supplied commodities with short shelf lives requires careful control of backorder costs. In this paper, we have developed fixed and linear backorders inventory model for vendor with production and without production. Analytically optimized decisions are arrived for the model. The vendor and the buyer's respective saving percentages are always raised by the cooperative strategy. The model handles a variety of change scenarios, including production increases, holding costs, and both linear and fixed backorder costs. We proved that system optimization is possible with the decentralized quantity discount strategy. As a result, over time, both the seller and the customer benefit. The model is demonstrated with a numerical example.

References

[1] B. Sarkar and S. Sarkar *An improved inventory model with partial backlogging, time varying deterioration and stock-dependent demand*, Economic Modelling, 30(2013), pp.924-932.

[2] Chen, L., Gao, J., Huang, X., and Zhang, X. *Coordination of inventory and price decisions in a supply chain under demand uncertainty.*, Applied Mathematical Modelling,(2018) 57, pp.105-119

[3] Chen, Q., Wang, C., Liu, C., and Wang, M.*A newsvendor model with uncertain demand and customer returns for a dual-channel supply chain*, Journal of Industrial and Management Optimization(2018), 14(3), 1253-1268.

[4] C.T. Chang, M.C. Cheng, and P.Y. Soong, *Impacts of inspection errors and trade credits on the economic order quantity model for items with imperfect quality*, International Journal of Systems Science: Operations and Logistics, 3(2016), pp.34-48.

[5] Esmaeili, M., and Nasrabadi, M. *An inventory model for single vendor multi retailer supply chain under inflationary conditions and trade credit,* Journal of Industrial and Production Engineering,(2021), 38(2), 75-88.

[6] K. S. Swaminathan and P. Muniappan,*Mathematical model for optimum production inventory with deteriorating items,* Applied Mathematical Sciences, 9(18), 2015, 895-900.

[7] Li, H., Li, Y., Li, X., & Li, Q. *Pricing and inventory control for supply chains with demand uncertainty and reference effects,* International Journal of Production Research,(2018). 56(16), 5344-5358.

[8] Li, J., Li, Z., Huang, Y., & Li, Y. *Inventory management in a supply chain with demand uncertainty and supplier reliability*, Journal of Industrial and Management Optimization,(2017). 13(3), 1215-1232.

[9] Liu, M., Liu, J., & Shen, B. *Supply chain coordination with multiple retailers under lead-time uncertainty*, (2020). Journal of the Operational Research Society, 71(1), 105-123.

[10] M. Ravithammal, R. Uthayakumar, S. Ganesh, P. Muniappan, *Two ware house supply chain model for deteriorating items with ramp- type,*International Journal of Pure and Applied Mathematics, 115, 5 (2017), 1001-1011.

[11] M. Ravithammal, R. Uthayakumar and S. Ganesh, *An Integrated Production Inventory System for Perishable Items with Fixed and Linear Backorders*, International Journal of Mathematical Analysis, 8 (2014), 1549 – 1559.

[12] M. Ravithammal, R. Uthayakumar and S. Ganesh, *A deterministic production inventory model for buyer-manufacturer with quantity discount and completely backlogged shortages for fixed life time product*, Global Journal of Pure and Applied Mathematics, 11 (2015), 3583-3600.

[13] M. Ravithammal, R. Uthayakumar and S. Ganesh, *Obtaining inventory model for price discount with shortage, back ordering and rework*,Asia Life Science, 14 (2017), 25-32.

[14] P. Muniappan, R. Uthayakumar & S. Ganesh,*An EOQ model for deteriorating items with inflation and time value of money considering time-dependent deteriorating rate and delay payments*, Systems Science & Control Engineering, 3(2015), 427–434.

[15] P. Muniappan, R. Uthayakumar and S. Ganesh, *An economic lot sizing production model for deteriorating items under two level trade credit*, Applied Mathematical Sciences, 8(2014), 4737 – 4747.

[16] P. Muniappan and R. Uthayakumar, *Mathematical analyze technique for computing optimal replenishment polices,* International Journal of Mathematical Analysis, 8(2014), 2979 – 2985.

[17] S. Amulu Priya,, Maheswari, V., & Balaji, V. *Operations Research for Supply Chain management–An Overview of Issue and Contributions.* In Journal of Physics: Conference Series(2021, July) (Vol. 1964, No. 2, p. 022010). IOP Publishing.

[18] S. C. Chen, and J. T. Teng, *Retailer's optimal ordering policy for deteriorating items with maximum lifetime under supplier's trade credit financing,* Applied Mathematical Modelling, 38 (2014), 4049-4061.

[19] Shib Sankar Sana, *Price-sensitive demand for perishable items – an EOQ model,* Applied Mathematics and Computations, 217(2011), 6248-6259.

[20] S. P. Aggarwal and C.K. Jaggi, *Ordering policies of deteriorating items under permissible delay in payments,* Journal of the Operational Research Society, 46 (1995), 658–662.

[21] Sundara Rajan, R. & Uthayakumar, R. *Optimal pricing and replenishment policies for instantaneous deteriorating items with backlogging and trade credit under inflation* J Ind Eng Int (2017) 13: 427.

[22] Wang, Y., Wang, X., & Hu, X. *Inventory control in a supply chain with a single supplier and multiple retailers under demand uncertainty and lead time variability,* Journal of Industrial and Management Optimization,(2017).13(4), 1823-1838

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