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UNIQUE SUBLOCALES FROM IDEALS OF A LOCALE

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Abstract In a locale L, if I represents an ideal, which is closed under arbitrary join, we construct a complete lattice $M = \{K_a; a \in L\}$ of ideals of L with the property $I \subseteq K_a$ for all $a \in L$. M induces a frame congruence R_I on L and R_I determines a sublocale S of L.The topological properties such as subfit, fit, S_2', regularity, normal and compactness of the sublocale S of L thus constructed can be obtained using the class of core elements of L with respect to I. On the other hand, via a sublocale S attached to a locale L, an ideal I_S which is closed under arbitrary join can be obtained. We prove that the sublocale constucted using the congruence R_{I_S} as above is embeddable in the given sublocale S.

1 Introduction

Since 1914, it has been understood that a set with a lattice of open subsets is a topological space. Marshall Stone was one of the first person to investigate the connection that exists between lattice to topology. In his work “The point of pointless topology ”[5],Johnstone described pointless topology as “the complete lattice" that fulfils “infinite distributive law". The majority of topological concepts have now been researched against a localic context. Theory of frames is the opposite of the idea of theory of locales. Studies in localic settings are topological, while those utilising frame theory are more algebraic.

In the framework of point free topology, given an ideal I in a locale L, we construct a collection $\{K_a; a \in L\}$ of ideals of L with the property $I \subseteq K_a$ for all $a \in L$. It is demonstrated that for any $a \in L$, the ideals K_a are prime if the ideal I is so. We have shown that if the ideal I is closed under arbitrary join, then there arises a “complete join semilattice homomorphism" from the locale L to the complete lattice $M = (\{K_a; a \in L\}, \supseteq)$ and M induces a frame congruence R_I on L. This congruence determines a sublocale of L. The topological properties such as subfit, fit, S_2', regularity, normal and compactness of the sublocale S of L thus constructed can be obtained using the class of core elements of L with respect to I.

Conversely given a sublocale S associated with a locale L, an ideal I_S which is closed under arbitrary join is obtained. It is proved that the sublocale constucted using the congruence R_{I_S} is embeddable in the given sublocale S.

The concept of core element with respect to an ideal I is introduced. It is shown that the collection Ç of core elements is a congruence class with respect to R_I.

Sublocales of a locale are traditionally presented in terms of sublocale homomorphism, frame congruence and nucleus. In [6] Pultr and Picardo have shown that there exist a one-one correspondence between sublocales of a locale L and nuclei in L. The work in this paper discuss a method of construction of a sublocales using ideals of a locale L.

2 Preliminaries

“A frame (or a locale) is a complete lattice L satisfying the infinite distributivity law $a \wedge \bigvee B = \bigvee\{a \wedge b; b \in B\}$ for all a$\in L$ and B$\subseteq$ L [6]. Given the frames L, M, a frame homomorphism is a map $h : L \to M$ preserving all finite meets (including the top 1) and all joins (including the bottom 0). The category of frames is denoted by **Frm**. The opposite category of **Frm** is the category **Loc** of locales."

Example 2.1. i. "The lattice $\Omega(X)$ of open subsets of topological space $(X, \Omega(X))$."
ii. "The Boolean algebra B of all open subsets U of real line R such that $U = int(cl(U))$."

Definition 2.2. [4] "A lattice A is said to be a Heyting algebra if for each pair of elements (a,b) in A, there exist an element $a \to b$ such that $c \leq (a \to b)$ if and only if $c \wedge a \leq b$."

Example 2.3. Every Boolean algebra is a Heyting algebra, with $p \to q$ given by $\neg p \vee q$.

Definition 2.4. [4] "A subset I of a locale L is said to be an ideal if
i)I is a sub-join-semilattice of L; that is 0$\in I$ and $a \in I, b \in I$ implies $a \vee b \in I$;and
ii)I is a lower set;that is $a \in I$ and $b \leq a$ imply $b \in I$."

"If $a \in L$,the set $\downarrow(a) = \{x \in L; x \leq a\}$ is an ideal of L. $\downarrow(a)$ is the smallest ideal containing a and is called the principal ideal generated by a. A proper ideal I is prime if $x \wedge y \in I$ implies that either $x \in I$ or $y \in I$."

Definition 2.5. [6] "A subset F of locale L is said to be a filter if
i)F is a sub-meet-semilattice of L; that is 1$\in F$ and $a \in F, b \in F$ implies $a \wedge b \in F$.
ii)F is an upper set; that is $a \in F$ and $a \leq b$ imply $b \in F$."

Definition 2.6. [6] "A filter F is proper if $F \neq L$,that is if 0 $\notin F$."

Definition 2.7. [6] "A proper filter F in a locale L is prime if $a_1 \vee a_2 \in F$ implies that $a_1 \in F$ or $a_2 \in F$."

Definition 2.8. [6] "A proper filter F in a locale L is a completely prime filter if for any J and $a_i \in L, i \in J, \bigvee a_i \in F \Rightarrow \exists i \in J$ such that $a_i \in F$."

"Completely prime filters are denoted by c.p filters."

Example 2.9. "U(x)=$\{V \in \Omega(X); x \in V\}$ is a completely prime filter in the locale $\Omega(X)$."

"For $a \in L$, set $\Sigma_a = \{F \subseteq L; F \neq \phi, F \ \ is \ \ c.p \ \ filters \ \ ; a \in F\}$.Thus $\Sigma_0 = \phi$,$\Sigma_{\bigvee a_i}$=$\bigcup \Sigma_{a_i}$ $\Sigma_{a \wedge b} = \Sigma_a \cap \Sigma_b$ and $\Sigma_1 = \{all \ \ c.p \ \ filters\}$."

Definition 2.10. [6] "The spectrum of a locale is defined as follows.
Sp(L)=($\{all \ \ c.p \ \ filters\}, \{\Sigma_a : a \in L\}$). Then Sp(L)is a topological space with the topology $\Omega(Sp(L)) = \{\Sigma_a : a \in L\}$."

Definition 2.11. [6] "A nucleus in a locale L is a mapping $v : L \to L$ such that
1.$a \leq v(a)$,
2.$a \leq b \Rightarrow v(a) \leq v(b)$
3.$v(v(a)) = v(a)$ and
4.$v(a \wedge b) = v(a) \wedge v(b)$."

Definition 2.12. [6] "A subset $S \subseteq L$ is a sublocale of L if
1. S is closed under all meets
2. for every $s \in S$ and every $x \in L$,$x \to s \in S$."

Example 2.13. "Let L be a locale. For each $a \in L$, the closed sublocales are given by $c(a) = \{x \in L : a \leq x\}$ and open sublocales are given by $o(a) = \{a \to x : x \in L\}$."

Proposition 2.14. *[6] "Let L be a locale. A subset $S \subseteq L$ is a sublocale if and only if it is a locale in the induced order and the embedding map $j : S \subseteq L$ is a localic map."*

"Sublocales of a locale L have alternate representations.
1.Sublocales of a locale L are represented as onto frame homomorphism $g : L \to M$, a sublocale homomorphism. The translation between sublocale homomorphism to sublocales and vice versa is as follows.
$h \mapsto h_*[M]$ for an onto $h : L \to M$ and h_* is its right adjoint, and $S \mapsto j_S^* : L \to S$ for

$j_S : S \subseteq L$.
2.Sublocales of a locale can also be represented using frame congruence. A sublocale homomorphism $g : L \to M$ induces a frame congruence $E_g = \{(x, y) : g(x) = g(y)\}$ and a frame congruence gives rise to a sublocale homomorphism $x \mapsto Ex : L \to L/E$, where L/E denotes the quotient frame defined by the congruence E, and Ex denotes the E-class.
3. Sublocales of a locale can also be represented using nucleus. The translation between nuclei and frame congruence resp. sublocale homomorphism is straight forward:
$v \mapsto E_v = \{(x, y) : v(x) = v(y)\}$,
$E \mapsto v_E = (x \mapsto \bigvee Ex) : L \to L$;
$v \mapsto v_h = v$ restricted to $L \to v[L]$,
$h \mapsto v_h = (x \mapsto h_* h(x)) : L \to L$

We can relate sublocales and nuclei directly. For a sublocale $S \subseteq L$, set $v_S(a) = j_S^*(a) = \bigwedge\{s \in S : a \leq s\}$ and for a nucleus $v : L \to L$, set $S_v = v[L]$."

Proposition 2.15. *[6] "The formula $S \mapsto v_S$ and $v \mapsto S_v$ constitute a one-one correspondence between subloales of L and nuclei."*

Definition 2.16. [6] "An element $p \neq 1$ in a lattice L is said to be meet irreducible if for any $a, b \in L$,$a \wedge b \leq p$ implies that either $a \leq p$ or $b \leq p$."

Example 2.17. In a chain, all elements except the top one are meet-irreducible.

As in classical topology, the point free topology have seperation axioms. Subfit and fit correspond to T_1 axiom of classical topology.

Definition 2.18. [6] "A locale L is said to be subfit if for $a, b \in L, a \nleq b$, then $\exists c \in L$, such that $a \vee c = 1$ and $b \vee c \neq 1$."

Example 2.19. [6] Every T_1 space is subfit

Definition 2.20. [6] "A locale L is said to be fit if for $a, b \in L, a \nleq b$, then $\exists c \in L$, such that $a \vee c = 1$ and $c \to b \nleq b$."

Definition 2.21. [6] "In a locale L, for $a, b \in L$, we say that a is rather below b, denoted by $a \prec b$, if there exist $c \in L$ such that $a \wedge c = 0$ and $c \vee b = 1$."

Example 2.22. [6] In the locale $\Omega(X), V \prec U$, if there exist an open set W such that $V \cap W = \phi$ and $W \cup U = X$

Definition 2.23. [6] "A locale L is said to be regular if $a = \bigvee\{x : x \prec a\}$ for every $a \in L$."

Example 2.24. [6] If $(X, \Omega(X))$, is a regular topological space, then $\Omega(X)$ is a regular locale.

Definition 2.25. [6] "A locale L is said to have S_2' property if for any $a, b \in L$, if $a \vee b = 1$ with $a \neq 1$ and $b \neq 1$, then there exist $u, v \in L$ with $u \wedge v = 0$, $v \nleq a$, $u \nleq b$."

Definition 2.26. [6] "A locale L is called normal if it satisfies the condition:
If $a \vee b = 1$, then there exist $u, v \in L$ such that $a \vee v = 1, u \vee b = 1, u \wedge v = 0$."

Definition 2.27. [6] "A cover of a locale L is a subset $A \subseteq L$ such that $\bigvee A = 1$. A subcover of a cover A is a subset $B \subseteq A$ such that $\bigvee B = 1$. A locale is said to be compact if each cover has a finite subcover."

Example 2.28. [6] Every finite distributive lattice is a compact locale

Definition 2.29. [6] "Let **C** be a category and $A, B \in Obj(\mathbf{C})$. A morphism $f : A \to B$ is epimorphism if $f \circ g = f \circ h$ implies $g = h$ for all morphisms $g, h : B \to C$."

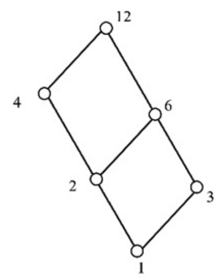
3 Sublocales from ideals of a locale

Definition 3.1. Let I represent any ideal at locale L. Let $K_a = \{x \in L : a \wedge x \in I\}$ be defined for each $a \in L$.

Proposition 3.2. *Consider the locale L and the ideal I within it. Then K_a represents an ideal in L for each $a \in L$.*

Proof. As $0 = 0 \wedge a \in I$, $0 \in K_a$. Hence K_a is non empty. Assume that K_a contains x, y. It follows that $a \wedge x, a \wedge y \in I$. As is closed under finite join, $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y) \in I$. Therefore $x \vee y \in K_a$. Consequently K_a is a subjoin semilattice of L. Assume that $z \in L$ has the property that $z \leq x$. Since $x \in K_a, a \wedge x \in I$. $z \leq x$ implies $z \wedge a \leq x \wedge a$. Since I is a lower set, $z \wedge a \in I$. Hence $z \in K_a$. Thus K_a is a lower set. □

Example 3.3. (i) Let us consider the following locale L and the ideal $I = \{1, 2\}$. $K_3 = K_6 =$



$\{1, 2, 4\}$, $K_4 = \{1, 2, 3, 6\}$, $K_2 = K_1 = L$ and $K_{12} = I$.

(ii) Consider a "frame homomorphism $f : L \to L$." For every $b \in L$, $(f)_b = \{x \in L : \Sigma_{f(x)} \subseteq \Sigma_b\}$ represent "ideals" in L. Consequently, $\langle a \rangle_f = \{x \in L : a \wedge x \in (f)_b\} = \{x \in L : \Sigma_{f(a \wedge x)} \subseteq \Sigma_b\}$ are ideals in L for all $a \in L$.

Definition 3.4. "An element $a \notin I$ in L is called partially prime to the ideal I if for any $x \in L$, $a \wedge x \in I$ implies $x \in I$."

Example 3.5. Consider a totally ordered set L and assume that $a \leq b$ in L. Then b is partially prime to the ideal $\downarrow a$.

Proposition 3.6. *When the ideal I is a prime in L, the ideals K_a are prime for all $a \in L$. If $a \in L$ is partially prime to I and K_a is prime, then I is prime.*

Proof. Assume that $x \wedge y \in K_a$ and that I represents a prime ideal . So $a \wedge (x \wedge y) \in I$. Because I is prime,either $a \wedge x \in I$, or $y \in I$. If $a \wedge x \in I$, then $x \in K_a$. If $y \in I$, then $a \wedge y \in I$ and hence $y \in K_a$. Conversely let a be partially prime to I and K_a be prime ideal in L. If $x \wedge y \in I$, then we have $a \wedge (x \wedge y) \in I$. Hence $x \wedge y \in K_a$. Since K_a is prime, x or y is in K_a . Fromwhich we can deduce that I is prime. □

Proposition 3.7. *Assume that L is a locale and I is an ideal within it.*

(i) For $a \leq b$ in L, $K_b \subseteq K_a$

(ii) $I \subseteq K_a$ for every $a \in L$

(iii) $K_a = L$ when and only when $a \in I$.

(iv) $K_1 = I$.

Proof. (i) For $a \leq b$, $x \in K_b$ implies $b \wedge x \in I$. As ideal possesses lower set property, $b \wedge x \in I$ implies that $a \wedge x \in I$. Hence $x \in K_b$ implies $x \in K_a$. Thus $K_b \subseteq K_a$.

(ii) Let $x \in I$. As ideal possesses lower set property, $a \wedge x \in I$ for all $a \in L$. Thus $x \in K_a$ for all $a \in L$. Hence $I \subseteq K_a$ for every $a \in L$.

(iii) If $K_a = L$, then the top element 1 is in $K_a = L$. Thus $a = a \wedge 1 \in I$. Hence $K_a = L$ implies $a \in I$. Conversely assume $a \in I$. Then for every $x \in L$, $a \wedge x \in I$. Hence $x \in K_a$ for all x in L. Thus $K_a = L$.

(iv) $K_1 = \{z \in L : z \wedge 1 \in I\} = \{z \in L : z \in I\} = I$.

□

Proposition 3.8. *Assume that L is a locale and I is an ideal within it. Suppose that a and b are any two elements of L, then*

(i) $K_a \cap K_b = K_{a \vee b}$

(ii) $K_a \cup K_b \subseteq K_{a \wedge b}$. *If* $a \wedge b$ *is partially prime to I,then* $K_a \cup K_b = K_{a \wedge b}$.

Proof. (i) $x \in K_a \cap K_b$ when and only when $a \wedge x \in I$ and $b \wedge x \in I$
when and only when $x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b) \in I$
when and only when $x \in K_{a \vee b}$.

Hence $K_a \cap K_b = K_{a \vee b}$.

(ii) $x \in K_a \cup K_b$ implies $a \wedge x \in I$ or $b \wedge x \in I$. Then $(a \wedge x) \wedge (b \wedge x) = (a \wedge b) \wedge x \in I$. Thus $x \in K_a \cup K_b$ implies $x \in K_{a \wedge b}$. Hence $K_a \cup K_b \subseteq K_{a \wedge b}$.
Let $a \wedge b$ be partially prime to I. $x \in K_{a \wedge b}$ implies $(a \wedge b) \wedge x \in I$. Since $a \wedge b$ is partially prime to I, $x \in I$. Hence $x \in K_a \cup K_b$.

□

Proposition 3.9. *Let the ideal I in a locale L be "closed under arbitrary join" and let* $M = \{K_a; a \in L\}$.*According to the partial order inclusion, M is a complete lattice.*

Proof. By 3.8, M is meet semilattice on the partial order inclusion. Suppose that arbitrary join of elements of I is in I and let $K_{a_\alpha} \in M$, $\alpha \in J$, for some index set J.
Then $x \in \cap K_{a_\alpha}$ when and only when $x \in K_{a_\alpha}$ for all $\alpha \in J$
when and only when $x \wedge a_\alpha \in I$ for all $\alpha \in J$
when and only when $\bigvee(x \wedge a_\alpha) = x \wedge \bigvee a_\alpha \in I$
when and only when $x \in K_{\vee a_\alpha}$.
Hence $\cap K_{a_\alpha} = K_{\vee a_\alpha} \in M$. Also $K_0 = L$ is the top element. Hence M is complete. □

Proposition 3.10. *If the ideal I in a locale L is "closed under arbitrary join", then there is a" complete join semilattice homomorphism" from the locale L to the complete lattice* $M = (\{K_a; a \in L\}, \supseteq)$.

Proof. Order $M = \{K_a; a \in L\}$ as $K_a \preceq K_b$ if and only if $K_a \supseteq K_b$. Then we have "$K_a \vee K_b = K_a \cap K_b$" and "$K_a \wedge K_b = K_a \cup K_b$". With respect to this ordering M is a complete lattice with bottom element $K_0 = L$ and top element $K_1 = I$. Define $f : L \to M$ by $f(a) = K_a$. $f(\bigvee a_\alpha) = K_{\bigvee a_\alpha} = \bigcap K_{a_\alpha} = \bigvee f(a_\alpha)$ and $f(0) = K_0 = L$. □

Lemma 3.11. *Assume that L is alocale and I is an ideal within it having the property that I is "closed under arbitrary join".Then for any* $a, b \in L$ *and* $S \subseteq L$, *we have*

(i) $K_a = K_b$ *implies* $K_{a \wedge c} = K_{b \wedge c}$

(ii) $K_a = K_b$ *implies* $K_{a \vee \bigvee S} = K_{b \vee \bigvee S}$.

Proof. (i) Let $K_a = K_b$. Then $x \in K_{a \wedge c}$ if and only if $a \wedge (c \wedge x) = (a \wedge c) \wedge x \in I$. That is if and only if $c \wedge x \in K_a = K_b$. Thus we have $b \wedge (c \wedge x) = x \wedge (b \wedge c) \in I$. Hence $x \in K_{a \wedge c}$ if and only if $x \in K_{b \wedge c}$.
Therefore $K_a = K_b$ implies $K_{a \wedge c} = K_{b \wedge c}$.

(ii) Let $K_a = K_b$ and $S \subseteq L$.
$x \in K_{a \vee \bigvee S}$ when and only when $x \wedge (a \vee \bigvee S) = x \wedge \bigvee(a \vee s) = \bigvee x \wedge (a \vee s) \in I$
when and only when $x \wedge (a \vee s) = (x \wedge a) \vee (x \wedge s) \in I$ for all $s \in S$
when and only when $x \wedge a \in I$ and $x \wedge s \in I$ for all $s \in S$
when and only when $x \in K_a = K_b$ and $x \wedge s \in I$ for all $s \in S$
when and only when $x \wedge b \in I$ and $x \wedge \bigvee s = \bigvee(x \wedge s) \in I$
when and only when $(x \wedge b) \vee (x \wedge \bigvee S) = x \wedge (b \vee \bigvee S) \in I$
when and only when $x \in K_{b \vee \bigvee S}$.

□

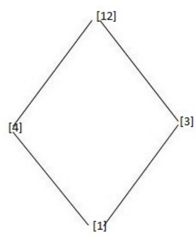
Definition 3.12. Assume that L is a locale and “I is an ideal" within it having the property that I is closed under arbitrary join. Form a relation R_I over L by $(a, b) \in R_I$ when and only when $K_a = K_b$.

The above lemma directly leads to the following proposition.

Proposition 3.13. *Assume that L is a locale and “I is an ideal" within it having the property that I is closed under arbitrary join. The binary relation R_I defined on L is a “congruence relation" on L.*

Proof. R_I is an “equivalence relation" on L. If $(a, b) \in R_I$, by above lemma $(a \wedge c, b \wedge c) \in R_I$ and $(a \vee \bigvee S, b \vee \bigvee S) \in R_I$. Hence R_I is a congruence relation on L. □

Since R_I is a congruence on L, by [2],L/R_I is a locale with respect to the partial order $[x] \leq [y]$ if and only if $x \leq y$ in L.
In example 3.3 (1), the congruence R_I gives $[1] = \{1, 2\}, [3] = \{3, 6\}, [4] = \{4\}$ and $[12] = \{12\}$ and the quotient locale L/R_I is given below.



Lemma 3.14. *Assume that L is a locale and “I is an ideal" within it having the property that I is closed under arbitrary join. There exist a bijection between the locale L/R_I and the complete lattice $M = (\{K_a : a \in L\}, \supseteq)$.*

Proof. The function $f : L/R_I \to M$ defined by $f([a]) = K_a$ is a bijection. □

Lemma 3.15. *Assume that L is a locale and “I is a prime ideal" within it. Then $K_a = L$ for all $a \in I$ and $K_b = I$ for every $b \notin I$.*

Proof. If I is prime, by 3.7, $K_a = L$ for all $a \in I$. Let $b \notin I$. Then $K_b = \{x \in L : b \wedge x \in I\}$. If $x \in K_b$, then $b \wedge x \in I$. As “I is prime" $b \wedge x \in I$ implies $x \in I$. Therefore $K_b \subseteq I$. So $K_b = I$ for every $b \notin I$. □

Proposition 3.16. *Assume that L is a locale and “I is a prime ideal" within it. Then the locale L/R_I is isomorphic to the two element locale*

Proof. By above lemma, if I is prime $K_a = K_0$ for all $a \in I$ and $K_a = K_1$ for all $a \notin I$. Hence $L/R_I = \{[0], [1]\}$ which is isomorphic to the locale **2**. □

Corollary 3.17. *Let the locale L be a chain and “I be any ideal" of L. Then the locale L/R_I is isomorphic to the two element locale* **2**.

Proof. Every ideal of a chain is principal and prime. □

Remark 3.18. Given an ideal I that is closed under arbitrary join, we get a congruence and hence a sublocale of L. To the contrary given a sublocale S, we get the ideal I_S, which is closed under arbitrary join and the sublocale constructed using the congruence L/R_{I_S} is embeddable in the sublocale S of L.

In 3.3(1), the sublocale corresponding to the ideal $I = \{1, 2\}$ is the closed sublocale $c(2) = \uparrow 2$.

Lemma 3.19. *Assume that $c \in L$ is irreducible with respect to meet. Accordingly, $I = \downarrow(c)$ is a “prime ideal."*

Proof. Let $x \wedge y \in I$. That is $x \wedge y \leq c$. As c is meet irreducible, either $x \leq c$ or $y \leq c$. So either $x \in I$ or $y \in I$. Therefore "I is prime." □

Proposition 3.20. *Assume that $c \in L$ is irreducible with respect to meet and let $I =\downarrow (c)$. Let S be the sublocale corresponding to the ideal I. Then S is closed when and only when c is maximal element of the locale L.*

Proof. Since c is meet-irreducible element of L, by above lemma ideal I is prime.
By 2.15 lemma, $K_a = L, \forall a \in I$ and $K_a = I \forall a \notin I$. Then by construction, the corresponding sublocale $S = \{c, 1\}$.

Assume S is closed. Then $S =\uparrow (\bigwedge S) =\uparrow (c) = \{c, 1\}$. Thus there exist no element b such that $c \lneq b \lneq 1$. Hence c is maximal element of the locale L.

Conversely assume c is maximal element of the locale L.Then $\uparrow c = \{c, 1\} = S$. Hence the sublocale S is closed. □

Proposition 3.21. *Suppose that "S is a sublocale" of L and $j : S \to L$ be the inclusion. Accordingly $kerj_S^* = \{x \in L : j_S^*(x) = \bigwedge S\}$ represents an ideal of L and $kerj_S^*$ is closed under arbitrary join.*

Proof. $j_S^*(x) = \bigwedge\{s \in S : x \leq s\}$. So $j_S^*(0) = \bigwedge S$ and hence $0 \in kerj_S^*$. Thus $kerj_S^*$ is non empty.
Assume that $x \in kerj_S^*$ and $y \in L$ have the property that $y \leq x$. Then $j_S^*(y) = j_S^*(y \wedge x) = j_S^*(y) \wedge j_S^*(x) = \bigwedge S$. Thus $y \in kerj_S^*$. Hence $kerj_S^*$ is a lower set.
Let $x_i \in kerj_S^*$ for $i \in I$. Then we have $j_S^*(x_i) = \bigwedge S$ for all $i \in I$.
Also $j_S^*(\bigvee x_i) = \bigvee j_S^*(x_i) = \bigvee(\bigwedge S) = \bigwedge S$. Thus $\bigvee x_i \in kerj_S^*$. Hence $kerj_S^*$ is an ideal which is closed under arbitrary join. □

Denote the ideal $kerj_S^*$ by I_S. Let L/R_{I_S} be the corresponding quotient locale.

Proposition 3.22. *If $\bigwedge S$ is irreducible with respect to meet, then I_S is prime ideal.*

Proof. Let $x \wedge y \in I_S$.Then $j_S^*(x \wedge y) = \bigwedge S$. That is $j_S^*(x) \wedge j_S^*(y) = \bigwedge S$. Since $\bigwedge S$ is meet-irreducible element, either $j_S^*(x) = \bigwedge S$ or $j_S^*(y) = \bigwedge S$. Hence either $x \in I_S$ or $y \in I_S$. Thus the ideal I_S is prime. □

Proposition 3.23. *A locale L's "sublocale S is dense in L" when and only when I_S, the ideal, is trivial. T*

Proof. Let the sublocale S be dense in L. Then $0 \in S$ and hence $\bigwedge S = 0$. Then $I_S = \{x \in L : j_S^*(x) = \bigwedge S = 0\}$. Since j_S^* is a nucleus on L, we have $x \leq j_S^*(x)$ for all $x \in L$. $y \in I_S$ if and only if $y \leq j_S^*(y) = 0$. Hence $I_S = \{0\}$, the trivial ideal.
Conversely let the ideal I_S is trivial. By 3.7, $K_a = L$ when and only when $a \in I$. Since I_S is trivial ideal, $K_a = L$ if and only if $a = 0$. Thus $[0] = \{0\}$ and hence $0 \in S$. As a result, S is a dense sublocale in L. □

Proposition 3.24. *If S is closed sublocale of L, then the ideal I_S is principal.*

Proof. Consider the closed sublocale $S = C(a) =\uparrow (a)$ of L. Then the corresponding "nucleus" j_S^* is of the form $j_S^*(x) = a \vee x$ for each $x \in L$. $I_S = kerj_S^* = \{x \in L : j_S^*(x) = \bigwedge S = a\}$ $= \{x \in L; a \vee x = a\} = \{x \in L : x \leq a\} =\downarrow (a)$. Thus the ideal I_S is principal. □

Theorem 3.25. *Assume that S is a sublocale of locale L. Then the sublocale constructed using the congruence R_{I_S} is embeddable in S.*

Proof. Assume that S is a sublocale of locale L and let L/R_{I_S} be the quotient locale constructed using the congruence R_{I_S} in L. Let $\phi : L \to L/R_{I_S}$ be the corresponding extremal epimorphism in **Frm**. Then $\phi_*(L/R_{I_S})$ is the sublocale generated by the congruence R_{I_S}. We will show that the sublocale $\phi_*(L/R_{I_S})$ is embeddable in the sublocale S.
Let $y \in \phi_*(L/R_{I_S})$, then $y = \phi_*([x])$ for some $x \in L$. Thus y can be written as $y = \phi_*(\phi(x))$.
Define $h : \phi_*(L/R_{I_S}) \to S$ by $h(y) = j_S^*(x)$. Then the following triangle commutes.

The map $h : \phi_*(L/R_{I_S}) \to S$ is a one -one map. Hence the sublocale $\phi_*(L/R_{I_S})$ is embeddable in the sublocale S. □

$$\begin{tikzcd}
L \arrow[r, "\emptyset"] \arrow[d, "j_s^*"'] & L/R_{I_s} \arrow[r, "\emptyset_*"] & \emptyset_*(L/R_{I_s}) \arrow[dll, dotted, "h"] \\
S & &
\end{tikzcd}$$

4 Core element with respect to an ideal I

Assume that L is a locale and "I be an ideal" within it having the property that arbitrary join of elements of I is in I. The concept of core element with reference to the ideal I is introduced in this section.Throughout this section, I is used to denote ideal of a locale L, which is closed under arbitrary join.

Definition 4.1. With regard to the ideal I, an element $a \in L$ is referred to as a core element if $K_a = I$. Let Ç represent the collection of core elements of L.

By 3.7,1 $\in$Ç. Hence Ç is non empty.

Proposition 4.2. *The following is true for any "ideal" I of a locale L.*

(i) Concerning R_I, Ç is a congruence class.

(ii) "Ç is closed under finite meet and arbitrary join."

(iii) Ç is a filter of L.

(iv) If I is prime, Ç is a "completely prime filter."

Proof. Let Ç be the set of core elements of a locale L.

(i) By 3.7 (iv), 1 $\in$Ç. Now we will show that the equivalence class of 1 with respect to R_I is Ç. $[1]_{R_I} = \{t \in L : (1, t) \in R_I\} = \{t \in L : K_t = K_1\} = \{t \in L : K_t = I\} =$ Ç.

(ii) Let $x, y \in$ Ç. Then we have, by above part, $x, y \in [1]_{R_I}$ so that $(1, x) \in R_I$ and $(1, y) \in R_I$. Since R_I is a congruence, $(1, x) \in R_I$ implies $(1 \wedge y, x \wedge y) \in R_I$. That is $(y, x \wedge y) \in R_I$. Since R_I is an equivalence relation, $(1, y) \in R_I, (y, x \wedge y) \in R_I$ implies $(1, x \wedge y) \in R_I$. Hence $x \wedge y \in [1]_{R_I} =$ Ç. Thus Ç is closed under finite meet.
Now let $S = \{x_i; i \in J\} \subseteq$Ç. Then we have $(1, x_i) \in R_I$ for every $i \in J$. Since R_I is a congruence, we have $(1 \vee S, x_i \vee S) = (1, \vee S) \in R_I$. Hence $\vee S \in [1]_{R_I} =$ Ç. Thus Ç is closed under arbitrary join.

(iii) By 3.7 (iv), 1 $\in$Ç. By above part Ç is closed under finite meet. Now let $x \in$Ç and $y \in L$ with the property that $x \leq y$. Since $x \in$ Ç, we have $K_x = I$. By 3.7 (i), since $x \leq y$, $K_y \subseteq K_x = I$. Also by 3.7 (ii), $I \subseteq K_y$. Hence $K_y = I$. Thus $y \in$Ç. Hence Ç is a filter in L.

(iv) Let I be prime ideal. Then by 3.15, Ç $= \{x \in L : x \notin I\}$.
Let $\vee x_\alpha \in$ Ç. Then $\vee x_\alpha \notin I$. Since I is closed under arbitrary join, $x_\alpha \notin I$ for some α. Hence $x_\alpha \in$Ç and so Ç is completely prime filter of L.

□

Theorem 4.3. *Consider a locale L and an "ideal I" within it. Then the locale L/R_I is a Boolean algebra when and only when corresponding to each $x \in L$, there exist $y \in L$ with $x \wedge y \in I$ and $x \vee y \in$ Ç.*

Proof. Consider $x \in L$ and $[x] \in L/R_I$. Then L/R_I is a Boolean algebra if and only if there exist $[y] \in L/R_I$ such that $[x] \wedge [y] = [0]$, $[x] \vee [y] = [1]$. That is if and only if $[x \wedge y] = [0]$, $[x \vee y] = [1]$ or $K_{x \wedge y} = K_0 = L$ and $K_{x \vee y} = K_1 =$ Ç. So by 3.7, $x \wedge y \in I$ and $x \vee y \in$ Ç. □

Theorem 4.4. *Consider a locale L and an "ideal I" within it. If L/R_I represents a Boolean algebra, then R_I is the largest congruence relation with congruence class Ç.*

Proof. it is obvious that R_I is a congruence with Ç serving as the congruence class.With Ç being the congruence class, let θ represent any other congruence. Assume that $(x, y) \in \theta$. Consequently, $(x, y) \in \theta$ implies $(x \vee a, y \vee a) \in \theta$ for any $a \in L$. So $x \vee a \in$Ç if and only if $y \vee a \in$ Ç. That is $K_{x \vee a} = I$ if and only if $K_{y \vee a} = I$. Then by proposition 3.8, we have $K_x \cap K_a = I$ if and only if $K_y \cap K_a = I$.

Since L/R_I represents a Boolean algebra, by above theorem, x', a'$\in L$ with $x \wedge x', a \wedge a' \in I$ and $K_{x \vee x'} = I$, $K_{a \vee a'} = I$. Because $x \wedge x', a \wedge a' \in I$, we get $x' \in K_x$ and $a' \in K_a$. Thus $x' \wedge a' \in K_x \cap K_a = K_{x \vee a} = I$. $x' \wedge a' \in I$, implies $a' \in K_{x'}$. For appropriate $y' \in L$, we also obtain $a' \in K_{y'}$. Thus we have $a' \in K_{x'}$ if and only if $a' \in K_{y'}$. Thus $K_{x'} = K_{y'}$ or $(x', y') \in R_I$. Hence $x' \in$Ç if and only if $y' \in$ Ç. That is $K_{x'} = I$ if and only if $K_{y'} = I$. Hence $K_{x \vee x'} = K_x$ when and only when $K_{y \vee y'} = K_y$. Thus $K_x = I$ if and only if $K_y = I$. Hence $K_x = K_y$. Thus $(x, y) \in R_I$. □

Proposition 4.5. *The quotient locale L/R_I and hence the sublocale S constructed using R_I is subfit when and only when there is $c \in L$ witha $\vee$ $c \in$ Ç, $b \vee c \notin$ Ç, for any pair of numbers $a, b \in L$ where $a \nleq b$.*

Proof. Assume the quotient locale L/R_I is a subfit locale. If $a, b \in L$ with $a \nleq b$, then $[a] \nleq [b]$ in L/R_I. Since L/R_I is a subfit, there exist $[c] \in L/R_I$ such that $[a] \vee [c] = [1]$ and $[b] \vee [c] \neq [1]$. Hence by proposition 4.2, $a \vee c \in$ Ç, $b \vee c \notin$ Ç.

For converse, let $[a], [b] \in L/R_I$ with $[a] \nleq [b]$. Then $a, b \in L$ with $a \nleq b$. By assumption there exist $c \in L$ with $a \vee c \in$ Ç, $b \vee c \notin$ Ç. But $a \vee c \in$ Ç if and only if $[a \vee c] = [a] \vee [c] = [1]$. Hence the locale L/R_I is a subfit locale.

Since the sublocale S is isomorphic to the quotient locale L/R_I, the above result is true for the sublocale S. □

Proposition 4.6. *The quotient locale L/R_I and hence the sublocale S constructed using R_I is fit when and only when for every $a, b \in L$ with $a \nleq b$, there are $c, d \in L$such that $a \vee c \in$ Ç, $c \wedge d \leq b, d \nleq b$.*

Proof. Suppose the quotient locale L/R_I is fit.If $a, b \in L$ and $a \nleq b$, then $[a] \nleq [b]$ in L/R_I. Then $[a], [b] \in L/R_I$ with $[a] \nleq [b]$. Since L/R_I is fit, there exist $[c] \in L/R_I$ such that $[a] \vee [c] = [1]$ and $[c] \to [b] \nleq [b]$. But $[a] \vee [c] = [1]$ if and only if $a \vee c \in$ Ç. Also $[c] \to [b] \nleq [b]$ if and only if there exist $[d] \in L/R_I$such $[d] \wedge [c] \leq [b]$ and $[d] \nleq [b]$. That is when and only when there is a $d \in L$ with $c \wedge d \leq b, d \nleq b$.

Now suppose that $[a], [b] \in L/R_I$ with $[a] \nleq [b]$. Consequently, $a, b \in L$ with $a \nleq b$. By assumption there exist there exist $c, d \in L$such that $a \vee c \in$ Ç, $c \wedge d \leq b, d \nleq b$. Then $[c], [d] \in L/R_I$ with $[a] \vee [c] = [1]$ and $[c] \to [b] \nleq [b]$. Hence the quotient locale L/R_I is fit.

Since the sublocale S is isomorphic to the quotient locale L/R_I, the above result is true for the sublocale S. □

Proposition 4.7. *The quotient locale $L/R_I$and hence the sublocale S constructed using R_I is S_2' when and only when for every $a, b \in L$ with $a \vee b \in$ Ç, $a, b \notin$ Ç, there is $u, v \in L$ with $a \nleq u, b \nleq v$ and $u \wedge v \in I$.*

Proof. Suppose the locale L/R_I is S_2'. Let $a, b \in L$ and $a \vee b \in$ Ç, $a, b \notin$ Ç. Then $[a], [b] \in L/R_I$ with $[a] \vee [c] = [1], [a] \neq [1], [b] \neq [1]$. Since the locale L/R_I is S_2', there are $[u], [v] \in L/R_I$ with $[a] \nleq [u], [b] \nleq [v], [u] \wedge [v] = [0]$. But $[u] \wedge [v] = [0]$ if and only if $u \wedge v \in I$. In a similar manner we can prove the converse.

Since the sublocale S is isomorphic to the quotient locale L/R_I,the above result is true for the sublocale S. □

Lemma 4.8. "$[a] \prec [b] \in L/R_I$ *if and only if there exist $c \in L$ such that $a \wedge c \in I$ and $b \vee c \in$ Ç."*

Proof. $[a] \prec [b] \in L/R_I$ if and only if there exist $[c] \in L/R_I$ such that $[a] \wedge [c] = [0]$ and $[b] \vee [c] = [1]$. But $[a] \wedge [c] = [0]$ if and only if $a \wedge c \in I$ and $[b] \vee [c] = [1]$ if and only if $b \vee c \in$ Ç. Hence the result. □

Proposition 4.9. *The quotient locale L/R_I and hence the sublocale S constructed using R_I is regular if and only if for every $a \in L$ there exist $x_i, b_i \in L$ for every $i \in J$, where J is an indexing set, such that $K_{\vee x_i} = K_a, x_i \wedge b_i \in I$ and $a \vee b_i \in$Ç.*

Proof. The quotient locale L/R_I is regular when and only when for every $[a] \in L/R_I$, there exist $[x_i] \in L/R_I$ such that $[a] = [\vee x_i]$ with $[x_i] \prec [a]$. But $[a] = [\vee x_i]$ if and only if $K_{\vee x_i} = K_a$. Also by above lemma, $[x_i] \prec [a]$ if and only if there exist $b_i \in L$ such that $x_i \wedge b_i \in I$ and $a \vee b_i \in$Ç.

Since the sublocale S is isomorphic to the quotient locale L/R_I, the above result is true for the sublocale S. □

Proposition 4.10. *The quotient locale L/R_I and hence the sublocale S constructed using R_I is normal when and only when for every $a, b \in L$ with $a \vee b \in$ Ç, there are $u, v \in L$ with $a \vee v \in$ Ç, $b \vee u \in$ Ç,$u \wedge v \in I$*

Proof. The quotient locale L/R_I is normal if and only if for every $[a], [b] \in L/R_I$ with $[a] \vee [b] = [1]$, there exist $[u], [v] \in L/R_I$ such that $[u] \wedge [v] = [0]$ and $[a] \vee [v] = [1] = [b] \vee [u]$. But $[u] \wedge [v] = [0]$ if and only $u \wedge v \in I$ and $[a] \vee [v] = [1] = [b] \vee [u]$ if and only if $a \vee v \in$ Ç, $b \vee u \in$ Ç.

Since the sublocale S is "isomorphic" to the quotient locale L/R_I, the above result is true for the sublocale S. □

Definition 4.11. "A filter F in a locale L is said to be weekly completely prime if $\bigvee a_\alpha \in F$, there exist $\alpha_1, \alpha_2, ...\alpha_n$ such that $a_{\alpha_1} \vee a_{\alpha_2} \vee a_{\alpha_3} \vee .... \vee a_{\alpha_n} \in F$."

Proposition 4.12. *The quotient locale L/R_I and hence the sublocale S constructed using R_I is compact when and only when the filter Ç is "weekly completely prime."*

Proof. Assume the quotient locale L/R_I is compact. Let $\bigvee a_\alpha \in$ Ç. Then $[\bigvee a_\alpha] = \bigvee [a_\alpha] = [1]$. Thus $\{[a_\alpha] : \alpha \in J\}$ is a cover for the locale L/R_I. Since the locale L/R_I is compact, there exist $\alpha_1, \alpha_2, ...\alpha_n \in J$ such that $[a_{\alpha_1}] \vee [a_{\alpha_2}] \vee .... \vee [a_{\alpha_n}] = [a_{\alpha_1} \vee a_{\alpha_2} \vee .... \vee a_{\alpha_n}] = [1]$. Thus $a_{\alpha_1} \vee a_{\alpha_2} \vee a_{\alpha_3} \vee .... \vee a_{\alpha_n} \in$ Ç. Hence the filter Ç is "weekly completely prime". In a similar manner we can prove the converse. □

5 Conclusion

This paper has explored the interplay between frame theory and point-free topology through the construction of ideals and the use of congruences. We have shown how the ideal I in a locale L leads to the formation of a collection of prime ideals $M = \{K_a; a \in L\}$ and how this structure induces a complete join semilattice homomorphism to a complete lattice M. Additionally, we have demonstrated that the congruence R_I determines a sublocale of L, where topological properties such as regularity, compactness, and normality can be analyzed using the core elements of L. The paper also establishes a correspondence between sublocales and ideals closed under arbitrary join, with the congruence R_I playing a central role in embedding sublocales and preserving topological properties. Furthermore, the core elements expressed as congruence classes within R_I and proved the result that when L/R_I forms a Boolean algebra, R_I is the largest congruence relation with the congruence class representing the core elements. This work contributes to the algebraic and topological understanding of locales and provides insights into the structure of sublocales via the framework of frame theory.

References

[1] C.H. Dowker and Dona Papert Strauss: Separation axioms for frames, pages 223-240 of Topics in Topology, ed A. Cs´asz´ar, North Holland (1972).

[2] C.H. Dowker and Dona Papert: Quotient frame and subspaces, Proc. London Math. Soc (3) 16 (1966) 275-96.

[3] H.Herrlich,G.E.Strecker,Category Theory:An Introduction,Allyn and Bacon, 1973.

[4] P.T.Johnstone,StoneSpaces,CambridgeUniversityPress,1982.

[5] P.T.Johnstone,The point of pointless topology,Bulletin of American Mathematical society,1983.

[6] J.Picado,Pultr, Frames and locales:Topology without points,Front.Math.,Springer,Basel,2012.

[7] S.Vickers, Topology via Logic, in: Cambridge Tracts in Theoretical Computer Science, Vol. 5, Cambridge University Press, Cambridge, 1985.

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