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RADIUS OF STARLIKENESS FOR A SUBCLASS OF THE JANOWSKI STARLIKE FUNCTIONS

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Abstract Let φ be a normalized univalent function defined on the open unit disc $\mathbb{D}$ with positive real part whose image domain is starlike with respect to 1 and symmetric about the real axis. A normalized analytic function f is said to be Ma- Minda starlike if $zf'(z)/f(z)$ is subordinate to the function φ. For normalized Janowski starlike functions f_i defined on the unit disc $\mathbb{D}$ and $\alpha_i > 0$, we investigate the inclusion and the radii constants of Ma-Minda starlikeness of normalised analytic function g defined as $g(z) = z\prod_{i=1}^{k}(f_i/z)^{\alpha_i}$.

1 Introduction

Let $\mathbb{D}$ denote the open unit disc in the complex plane $\mathbb{C}$. The class $\mathcal{A}$ consists of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{S}$ denote a subclass of $\mathcal{A}$ consisting of univalent (one-to-one) functions. A function $f \in \mathcal{A}$ is said to be starlike if the image of the unit disc $\mathbb{D}$ is a starlike domain with respect to the origin. Similarly, $f \in \mathcal{A}$ is convex if $f(\mathbb{D})$ is a convex set. The subclasses of $\mathcal{A}$ consisting of starlike and convex functions are denoted as $\mathcal{ST}$ and $\mathcal{CV}$ respectively. An analytic function $p : \mathbb{D} \to \mathbb{C}$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ is a Carathéodory function if $p(0) = 1$ and $\Re(p(z)) > 0$ for every $z \in \mathbb{D}$ and is denoted by $\mathcal{P}$. Analytically, we say that a function is starlike if and only if $zf'(z)/f(z) \in \mathcal{P}$ and convex if and only if $1 + (zf''(z)/f'(z)) \in \mathcal{P}$. Alexander's theorem [3] gives a very useful correspondence between the classes $\mathcal{ST}$ and $\mathcal{CV}$, for $f \in \mathcal{A}$, $f \in \mathcal{CV}$ if and only if $zf' \in \mathcal{ST}$. Let $\mathcal{B}$ be the class of all analytic functions that maps unit disc $\mathbb{D}$ onto itself with $w(0) = 0$ and $|w(z)| < 1$. The function w is widely known as schwarz function. Let f and g be two analytic functions defined on the unit disc. The function f is said to be subordinate to g, represented as $f \prec g$, if there exists an analytic function $w \in \mathcal{B}$ such that $f(z) = g(w(z))$. When the superordinate function g is univalent (one-to-one), then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. Let $\varphi : \mathbb{D} \to \mathbb{C}$ be an univalent function with a positive real part normalized by the conditions $\varphi(0) = 1$ and $\varphi'(0) > 0$, such that the image domain $\varphi(\mathbb{D})$ is starlike with respect to 1 and is symmetric about the real axis. For such function φ, using subordination Ma and Minda [14] defined subclasses $ST(\varphi)$ and $CV(\varphi)$ by

$$\mathcal{ST}(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\} \tag{1.1}$$

and

$$\mathcal{CV}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}. \tag{1.2}$$

For different choices of φ these classes reduce to various subclasses of starlike and convex functions, respectively. For $-1 \leqslant B < A \leqslant 1$, when $\varphi(z) = (1 + Az)/(1 + Bz)$, the classes $ST(\varphi)$ and $CV(\varphi)$ are denoted as $\mathcal{ST}[A, B]$ and $\mathcal{CV}[A, B]$, as given in [8] and are called as class of Janowski starlike functions, and the class of Janowski convex functions, respectively.

For $-1 \leqslant B < A \leqslant 1$ and $p(z) = 1 + c_n z^n + \cdots$, $n \in \mathbb{N}$, we say that $p \in \mathcal{P}_n[A, B]$ if

$$p(z) \prec \frac{1 + Az}{1 + Bz}, \quad (z \in \mathbb{D}).$$

The functions $p \in \mathcal{P}_n[A, B]$ satisfy the following lemma:

Lemma 1.1. *[18] If $p \in \mathcal{P}_n[A, B]$, then*

$$\left| p(z) - \frac{1 - ABr^{2n}}{1 - B^2 r^{2n}} \right| \leqslant \frac{(A - B)r^n}{1 - B^2 r^{2n}} \quad (|z| \leqslant r < 1).$$

The class $\mathcal{ST}_n[A, B]$ consists of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z) \in \mathcal{P}_n[A, B]$. Our present study, deals with the class $\mathcal{ST}_n^{\alpha}[A, B]$ defined as follows:

$$\mathcal{ST}_n^{\alpha}[A, B] = \left\{ g \in \mathcal{A} : g(z) = z \prod_{i=1}^{k} \left(\frac{f_i(z)}{z} \right)^{\alpha_i}, \quad f_i \in \mathcal{ST}_n[A, B], \alpha_i > 0 \right\}.$$

Let $\mathcal{F}$ and $\mathcal{G}$ be two subclasses of $\mathcal{A}$, the largest number $\mathcal{R} \in (0, 1]$ such that for $0 < r < \mathcal{R}$, $f(rz)/r \in \mathcal{F}$ for every $f \in \mathcal{G}$ is known as the $\mathcal{F}$- radius of the class $\mathcal{G}$ and it is denoted as $\mathcal{R}_{\mathcal{F}}(\mathcal{G})$. Radius problems are being explored extensively in recent times [1, 9, 12, 13, 15, 19]. In order to obtain the radius, we find the largest positive number $\mathcal{R}$ less than 1 such that the image of the disc $\mathbb{D}_{\mathcal{R}} : \{z \in \mathbb{C} : |z| < \mathcal{R}\}$ under the mapping $zg'(z)/g(z)$ for g in the class defined, lies inside the image of the corresponding superordinate functions. We compute the $\mathcal{ST}(\varphi)$ radius for functions in the class $\mathcal{ST}_n^{\alpha}[A, B]$ for various subclasses of $\mathcal{A}$ such as starlike functions of order β, exponential function, cardiod, lune and so on. The radii obtained are sharp.

2 $\mathcal{ST}_n[C, D]$ RADIUS FOR FUNCTIONS IN THE CLASS $\mathcal{ST}_n^{\alpha}[A, B]$

A function $f \in \mathcal{A}$ satisfying $\Re(zf'(z)/f(z) > \beta)$ for $z \in \mathbb{D}$, where $0 \leqslant \beta < 1$ is said to be starlike of order β and the class of all such functions is denoted as $\mathcal{ST}(\beta)$. The following theorem gives the radius of Janowski starlikeness of functions in the class $\mathcal{ST}_n^{\alpha}[A, B]$ and, in particular, the $\mathcal{ST}(\beta)$ radius of the functions in the class $\mathcal{ST}_n^{\alpha}[A, B]$, which implies that the class $\mathcal{ST}_n^{\alpha}[A, B]$ is a subclass of starlike functions.

Theorem 2.1. *Let $\alpha := \sum_{i=1}^{k} \alpha_i$, $\alpha_i > 0$. Let $-1 \leqslant B \leqslant 0$ and $B < A \leqslant 1$. Let $-1 \leqslant D \leqslant 0$ and $D < C \leqslant 1$. For the class $\mathcal{ST}_n^{\alpha}[A, B]$, the inclusion $\mathcal{ST}_n^{\alpha}[A, B] \subset \mathcal{ST}_n[C, D]$ holds if*

$$|D\alpha(A - B) - B(C - D)| \leqslant (C - D) - \alpha(A - B).$$

When the inclusion fails, the $\mathcal{ST}_n[C, D]$ radius is given by

$$\mathcal{R}_{\mathcal{ST}_n[C,D]} = \left(\frac{C - D}{\alpha(A - B) + |B(C - D) - D\alpha(A - B)|} \right)^{1/n}.$$

Proof. Let $g \in \mathcal{ST}_n^{\alpha}[A, B]$. Then there will be functions $f_i \in \mathcal{ST}_n[A, B]$, satisfying

$$g(z) = z \prod_{i=1}^{k} \left(\frac{f_i(z)}{z} \right)^{\alpha_i}.$$

A computation shows that

$$\frac{zg'(z)}{g(z)} = 1 - \sum_{i=1}^{k} \alpha_i + \sum_{i=1}^{k} \alpha_i \left(\frac{zf_i'}{f_i} \right). \tag{2.1}$$

By Lemma 1.1 we have

$$\left| \frac{zf'(z)}{f(z)} - \frac{1 - ABr^{2n}}{1 - B^2 r^{2n}} \right| \leqslant \frac{(A - B)r^n}{1 - B^2 r^{2n}}.$$

Using the above inequality and (2.1), we get

$$\left|\frac{zg'(z)}{g(z)} - \frac{1 - \left[\sum_{i=1}^{k} \alpha_i B(A-B) + B^2\right] r^{2n}}{1 - B^2 r^{2n}}\right| \leqslant \frac{\sum_{i=1}^{k} \alpha_i (A-B) r^n}{1 - B^2 r^{2n}} \quad (|z| \leqslant r < 1).$$

Since $\alpha := \sum_{i=1}^{k} \alpha_i$, the above disc can be rewritten as

$$\left|\frac{zg'(z)}{g(z)} - \frac{1 - \left[\alpha B(A-B) + B^2\right] r^{2n}}{1 - B^2 r^{2n}}\right| \leqslant \frac{\alpha(A-B) r^n}{1 - B^2 r^{2n}} \quad (|z| \leqslant r < 1). \tag{2.2}$$

The centre and radius are given by

$$c(\alpha, r) := \frac{1 - \left[\alpha B(A-B) + B^2\right] r^{2n}}{1 - B^2 r^{2n}} \quad \text{and} \quad d(\alpha, r) := \frac{\alpha(A-B) r^n}{1 - B^2 r^{2n}}. \tag{2.3}$$

The diametric end points of the disc (2.2) are

$$c(\alpha, r) - d(\alpha, r) = \frac{1 - (B + \alpha(A-B)) r^n}{1 - B r^n} \tag{2.4}$$

and

$$c(\alpha, r) + d(\alpha, r) = \frac{1 + (B + \alpha(A-B)) r^n}{1 + B r^n}. \tag{2.5}$$

Let $-1 \leqslant D \leqslant 0$ and $D < C \leqslant 1$. The disc for the class $\mathcal{ST}_n[C, D]$ is given by

$$\left|\frac{zf'(z)}{f(z)} - \frac{1 - CD}{1 - D^2}\right| \leqslant \frac{(C-D)}{1 - D^2} \quad (|z| < 1). \tag{2.6}$$

The centre and radius for the disc (2.6) are given by

$$a = \frac{1 - CD}{1 - D^2} \quad \text{and} \quad b = \frac{(C-D)}{1 - D^2}, \tag{2.7}$$

and the diametric end points of the disc (2.6) are

$$a - b = \frac{1 - C}{1 - D} \quad \text{and} \quad a + b = \frac{1 + C}{1 + D}. \tag{2.8}$$

To establish $\mathcal{ST}_n^{\alpha}[A, B] \subset \mathcal{ST}_n[C, D]$, it is sufficient to show that

$$\{w : |w - c(\alpha, 1)| \leqslant d(\alpha, 1)\} \subseteq \{w : |w - a(1)| \leqslant b(1)\}$$

if and only if $|a - c(\alpha, 1)| \leqslant b - d(\alpha, 1)$ which is equivalent to the inequalities

$$c(\alpha, 1) + d(\alpha, 1) \leqslant a + b \tag{2.9}$$

and

$$a - b \leqslant c(\alpha, 1) - d(\alpha, 1). \tag{2.10}$$

To prove the inclusion $\mathcal{ST}_n^{\alpha}[A, B] \subset \mathcal{ST}_n[C, D]$ by considering the following three cases:

Case (i): Let $B = -1$ and $D = -1$. Using the diametric end points (2.4) at $r = 1$ and (2.8), it can be seen that the image of the function $g \in \mathcal{ST}_n^{\alpha}[A, B]$ lies in the half plane

$$\left\{w : \Re(w) > \frac{2 - \alpha(1 + A)}{2}\right\} \tag{2.11}$$

and also the image of the function $f \in \mathcal{ST}_n[C, D]$ lies in the half plane

$$\left\{w : \Re(w) > \frac{1 - C}{2}\right\}. \tag{2.12}$$

Since the condition $(1-C)/2 \leqslant (2-\alpha(A+1))/2$ holds is equivalent to the inequality (2.10), therefore the half plane given in (2.11) is contained in the half plane given by (2.12).

Case (ii): Let $B \neq -1$ and $D = -1$. The function $g \in \mathcal{ST}_n^\alpha[A,B]$ maps to a disc

$$\left|\frac{zg'(z)}{g(z)} - \frac{1-\left[\alpha B(A-B)+B^2\right]}{1-B^2}\right| \leqslant \frac{\alpha(A-B)}{1-B^2} \quad (|z|<1). \tag{2.13}$$

From (2.12) and (2.13), since $(1-C)/2 \leqslant (1-(B+\alpha(A-B)))/(1-B)$ holds and equivalent to the inequality (2.10), which proves the inclusion in this case.

Case(iii): When $B \neq -1$ and $D \neq -1$, we see that the inequality (2.9), becomes

$$\frac{1+C}{1+D} \geqslant \frac{1-(\alpha B(A-B)+B^2)+\alpha(A-B)}{1-B^2} = \frac{1+B+\alpha(A-B)}{1+B},$$

which reduces to

$$D\alpha(A-B) - B(C-D) \leqslant (C-D) - \alpha(A-B). \tag{2.14}$$

Similarly, the inequality (2.10) becomes

$$\frac{1-C}{1-D} \geqslant \frac{1-(\alpha B(A-B)+B^2)-\alpha(A-B)}{1-B^2} = \frac{1-B-\alpha(A-B)}{1-B},$$

that reduces to

$$-D\alpha(A-B) + B(C-D) \leqslant (C-D) - \alpha(A-B). \tag{2.15}$$

Therefore, by (2.14) and (2.15), the inclusion $\mathcal{ST}_n^\alpha[A,B] \subset \mathcal{ST}_n[C,D]$ holds if and only if

$$|D\alpha(A-B) - B(C-D)| \leqslant (C-D) - \alpha(A-B).$$

Case (iv): When $B = -1$ and $D \neq -1$, we see that for any function $g \in \mathcal{ST}_n^\alpha[A,B]$ has its the image lying in the half plane given in (2.11) and image of the function $f \in \mathcal{ST}_n[C,D]$ lies in the disc (2.6). Clearly, inclusion is not possible.

When either of the following conditions occur, we find the $\mathcal{ST}_n[C,D]$ radius for the class $\mathcal{ST}_n^\alpha[A,B]$:

(i) $B=-1$, $D=-1$ and $(1-C)/2 \geqslant (2-\alpha(A+1))/2$

(ii) $B \neq -1$, $D=-1$ and $(1-C)/2 \geqslant (1-(B+\alpha(A-B)))/(1-B)$

(iii) $B \neq -1$, $D \neq -1$ and $|D\alpha(A-B)-B(C-D)| \geqslant (C-D)-\alpha(A-B)$

(iv) $B=-1$ and $D \neq -1$.

Let $g \in \mathcal{ST}_n^\alpha[A,B]$. Then, by (2.2) we have $g(\mathbb{D}_r) \subset \{w : |w-c(\alpha,r)| \leqslant d(\alpha,r)\}$. For $r \leqslant \mathcal{R} = \mathcal{R}_{\mathcal{ST}_n[C,D]}$, we need to show that,

$$\{w : |w-c(\alpha,r)| \leqslant d(\alpha,r)\} \subseteq \{w : |w-a| \leqslant b\}$$

where a and b are given by (2.7) and $c(\alpha,r)$ and $d(\alpha,r)$ are given by (2.3). This containment holds if and only if $|a-c(\alpha,r)| \leqslant b-d(\alpha,r)$ or equivalently, if

$$c(\alpha,r)+d(\alpha,r) \leqslant a+b \tag{2.16}$$

and

$$a-b \leqslant c(\alpha,r)-d(\alpha,r). \tag{2.17}$$

The inequality (2.16), becomes

$$\frac{1+C}{1+D} \geqslant \frac{1-(\alpha B(A-B)+B^2)r^{2n}+\alpha(A-B)r^n}{1-B^2r^{2n}} = \frac{1+(B+\alpha(A-B))r^n}{1+Br^n}.$$

Solving the above inequality for r, we get

$$r \leqslant \left(\frac{C-D}{\alpha(A-B)(1+D)-B(C-D)}\right)^{1/n} := \rho_2.$$

Similarly, the inequality (2.17), becomes

$$\frac{1-C}{1-D} \leqslant \frac{1-(\alpha B(A-B)+B^2)r^{2n}-\alpha(A-B)r^n}{1-B^2r^{2n}} = \frac{1-(B+\alpha(A-B))r^n}{1-Br^n},$$

and solving for r gives

$$r \leqslant \left(\frac{C-D}{\alpha(A-B)(1-D)+B(C-D)}\right)^{1/n} := \rho_3.$$

The required radius is the $\min[\rho_2, \rho_3]$ which is given by

$$\mathcal{R} := \left(\frac{C-D}{\alpha(A-B)+|B(C-D)-D\alpha(A-B)|}\right)^{1/n}.$$

To prove the sharpness of $\mathcal{R}$, we first consider the function $\tilde{f}_i$ from the class $\mathcal{ST}_n[A,B]$ defined by $\tilde{f}_i(z) = z(1+Bz^n)^{\frac{A-B}{nB}}$. Then, we find the corresponding function $\tilde{g} \in \mathcal{ST}_n^\alpha[A,B]$ given by

$$\tilde{g}(z) = z(1+Bz^n)^{\frac{\alpha(A-B)}{nB}} \tag{2.18}$$

satisfying

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+(B+\alpha(A-B))z^n}{1+Bz^n}. \tag{2.19}$$

When $B(C-D)-D\alpha(A-B) < 0$, we have $\mathcal{R} = \rho_2$. Then for $z = \rho_2$, (2.19) gives

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+(B+\alpha(A-B))(\rho_2)^n}{1+B(\rho_2)^n} = \frac{1+C}{1+D},$$

which proves the sharpness for ρ_2.
When $B(C-D)-D\alpha(A-B) > 0$, we have $\mathcal{R} = \rho_3$. For $z = -\rho_3$, (2.19) gives

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+(B+\alpha(A-B))(-\rho_3)^n}{1+B(-\rho_3)^n} = \frac{1-C}{1-D},$$

which proves the sharpness for ρ_3. □

In particular, when $C = 1-2\beta$ and $D = -1$, Theorem 2.1 reduces to the following Corollary.

Corollary 2.2. *Let $\alpha > 0$ and $0 \leqslant \beta < 1$. Let $-1 \leqslant B \leqslant 0$ and $B < A \leqslant 1$. For the class $\mathcal{ST}_n^\alpha[A,B]$, the $\mathcal{ST}_n(\beta)$ radius is given by*

$$\mathcal{R}_{\mathcal{ST}_n(\beta)} = \min\left(1, \left(\frac{1-\beta}{\alpha(A-B)+B(1-\beta)}\right)^{1/n}\right).$$

When $n = 1$, Corollary 2.2 reduces to a Corollary of [7, Corollary 2.9, p.707].

3 $\mathcal{ST}_e$ RADIUS FOR FUNCTIONS IN THE CLASS $\mathcal{ST}_n^\alpha[A,B]$

Let the function g belong to the class $\mathcal{ST}_n^\alpha[A,B]$. By (2.2), we have $g(\mathbb{D}_r) \subset \{w : |w-c(\alpha,r)| \leqslant d(\alpha,r)\}$, where $c(\alpha,r)$ and $d(\alpha,r)$ are given in (2.3). For $0 \leqslant r \leqslant \mathcal{R} < 1$, we find the largest positive number $\mathcal{R}$, such that the disc $\{w : |w-c(\alpha,r)| \leqslant d(\alpha,r)\}$ is contained in $\varphi(\mathbb{D})$. To compute $\mathcal{R}$, we use the inclusion results obtained by various authors, wherein the image of the unit disc $\mathbb{D}$ under the function φ contains the largest disc with radius r_a centered at a. Since

$$c'(\alpha,r) = \frac{-2n\alpha B(A-B)r^{2n-1}}{(1-B^2r^{2n})^2},$$

it can be seen that $c(\alpha,r)$ is an increasing function of r when $B < 0$ and it is a decreasing function of r when $B > 0$. Also, for $B \leqslant 0$, we have $c(\alpha,r) \geqslant 1$ and when $B \geqslant 0$, $c(\alpha,r) \leqslant 1$. One immediate consequence is that for $B < 0$, $c(\alpha,r) \geqslant c(\alpha,0) = 1$. The following theorem gives the $\mathcal{ST}_e$ radius for functions in the class $\mathcal{ST}_n^\alpha[A,B]$.

Mendiratta et al. [16], introduced the class $\mathcal{ST}_e = \mathcal{ST}(\varphi_e) = e^z$, which consists of all functions $f \in \mathcal{A}$ such that $zf'(z)/f(z) \prec e^z$ or equivalently $|\log(zf'(z)/f(z))| < 1$.

Lemma 3.1. *[16] For $1/e < a < e$, let*

$$r_a = \begin{cases} a - \frac{1}{e} & \text{if } \frac{1}{e} < a \leqslant \frac{e+e^{-1}}{2} \\ e - a & \text{if } \frac{e+e^{-1}}{2} \leqslant a < e. \end{cases}$$

Then, $\{w : |w-a| < r_a\} \subset \Omega_e := \{w : |\log w| < 1\}$ where Ω_e is the image of the unit disc $\mathbb{D}$ under the exponential function.

Theorem 3.2. *Let $\alpha > 0$, $-1 \leqslant B \leqslant 0$ and $B < A \leqslant 1$. For the class $\mathcal{ST}_n^\alpha[A,B]$, the inclusion $\mathcal{ST}_n^\alpha[A,B] \subset \mathcal{ST}_e$ holds if either*

$$\textbf{(i)}(-\alpha B(A-B))/(1-B^2) \leqslant (e+e^{-1}-2)/2 \quad \text{and} \quad (\alpha(A-B))/(1-B) \leqslant (e-1)/e$$
$$(or)$$
$$\textbf{(ii)}(-\alpha B(A-B))/(1-B^2) \geqslant (e+e^{-1}-2)/2 \quad \text{and} \quad (\alpha(A-B))/(1+B) \leqslant e-1.$$

If neither condition (i) nor condition (ii) holds, then the $\mathcal{ST}_e$ radius is given by

$$\mathcal{R}_{\mathcal{ST}_e} = \begin{cases} \left(\frac{e-1}{e\alpha(A-B)+(e-1)B}\right)^{1/n} & \text{if } \alpha(A-B) \geqslant 2|B| \\ \left(\frac{e-1}{\alpha(A-B)-(e-1)B}\right)^{1/n} & \text{if } \alpha(A-B) \leqslant 2|B|. \end{cases}$$

Proof. We first prove the inclusion $\mathcal{ST}_n^\alpha[A,B] \subset \mathcal{ST}_e$ by assuming that the condition (i) holds. The inequality $(-\alpha B(A-B))/(1-B^2) \leqslant (e+e^{-1}-2)/2$ is equivalent to $c(\alpha,1) \leqslant (e+e^{-1})/2$. The inequality $d(\alpha,1) \leqslant c(\alpha,1) - 1/e$ follows from $(-\alpha(A-B))/(1-B) \leqslant 1-(1/e)$. By (2.13), we get

$$\left|\frac{zg'(z)}{g(z)} - c(\alpha,1)\right| \leqslant d(\alpha,1) \leqslant c(\alpha,1) - \frac{1}{e}.$$

Therefore, using Lemma 3.1 we see that the disc in (2.13) is contained in Ω_e. Now assume that $(-\alpha B(A-B))/(1-B^2) \geqslant (e+e^{-1}-2)/2$ and $\alpha(A-B)/(1+B) \leqslant e-1$. The first inequality reduces to $c(\alpha,1) \geqslant (e+e^{-1})/2$. The condition $\alpha(A-B)/(1+B) \leqslant e-1$ is equivalent to the inequality $d(\alpha,1) \leqslant e - c(\alpha,1)$. Then, by (2.13), we get

$$\left|\frac{zg'(z)}{g(z)} - c(\alpha,1)\right| \leqslant d(\alpha,1) \leqslant e - c(\alpha,1).$$

Hence, using Lemma 3.1 we see that the disc in (2.13) is contained in Ω_e.

When the inclusion fails, we show that, for $0 \leqslant r \leqslant \mathcal{R} := \mathcal{R}_{\mathcal{ST}_e}$, the disc (2.2) is contained in Ω_e where $c(\alpha,r)$ and $d(\alpha,r)$ given by (2.3).

Case (i): Let $\alpha(A-B) \geqslant 2|B|$. The number

$$\rho_1 := \left(\frac{e+e^{-1}-2}{2\alpha|B|(A-B)+(e+e^{-1}-2)B^2}\right)^{1/2n}$$

be the unique root of the equation $c(\alpha,r) = (e+e^{-1})/2$ and let the number

$$\rho_2 := \left(\frac{e-1}{e\alpha(A-B)+(e-1)B}\right)^{1/n} < 1$$

be the positive root of the equation $d(\alpha,r) = c(\alpha,r) - 1/e$ or

$$\frac{1}{e} = \frac{1-\left(\alpha B(A-B)+B^2\right)r^{2n}}{1-B^2r^{2n}} - \frac{\alpha(A-B)r^n}{1-B^2r^{2n}} = \frac{1-(B+\alpha(A-B))r^n}{1-Br^n} := \zeta(r). \quad (3.1)$$

For $\alpha(A-B) \geqslant 2|B|$, we observe that $\rho_2 \leqslant \rho_1$. We shall now show that $\mathcal{R} = \mathcal{R}_{\mathcal{ST}_e} = \rho_2$. For $0 \leqslant r \leqslant \rho_2 < 1$, since $c(\alpha,r) \geqslant 1$, we have $c(\alpha,r) > 1/e$. Since $c(\alpha,r)$ is an increasing function, for $r \leqslant \rho_1$, we have $c(\alpha, r) \leqslant c(\alpha, \rho_1) = (e + e^{-1})/2$. Note that $c(\alpha, r) - d(\alpha, r)$ is a decreasing function of r, therefore, for $0 \leqslant r \leqslant \rho_2$, it follows that

$$c(\alpha, r) - d(\alpha, r) \geqslant c(\alpha, \rho_2) - d(\alpha, \rho_2) = 1/e$$

and hence

$$d(\alpha, r) \leqslant c(\alpha, r) - \frac{1}{e}. \tag{3.2}$$

For $0 \leqslant r \leqslant \mathcal{R} = \rho_2$, using (2.2) and (3.2) we have,

$$\left|\frac{zg'(z)}{g(z)} - c(\alpha, r)\right| \leqslant c(\alpha, r) - \frac{1}{e}. \tag{3.3}$$

Therefore, using the inclusion result in Lemma 3.1, for $1/(e) < a \leqslant (e + e^{-1})/2$, the disc in (3.3) lies inside the region Ω_e proving that $\mathcal{ST}_e$ radius for functions belonging to the class $\mathcal{ST}_n^{\alpha}[A, B]$ is at least ρ_2.

To prove the sharpness, for functions $\tilde{f}_i \in \mathcal{ST}_n[A, B]$, we find the corresponding function $\tilde{g} \in \mathcal{ST}_n^{\alpha}[A, B]$ defined (2.18). For $z = -\rho_2$, (2.19) gives

$$\left|\log\left(\frac{z\tilde{g}'(z)}{\tilde{g}(z)}\right)\right| = \left|\log\left(\frac{1 + (B + \alpha(A - B))(-\rho_2)^n}{1 + B(-\rho_2)^n}\right)\right| = \left|\log\left(\frac{1}{e}\right)\right| = 1,$$

which proves the sharpness for ρ_2.

Case (ii): Let $\alpha(A - B) \leqslant 2|B|$. The number

$$\rho_3 := \left(\frac{e - 1}{\alpha(A - B) - (e - 1)B}\right)^{1/n} < 1$$

be the positive root of the equation $d(\alpha, r) = e - c(\alpha, r)$ or

$$e = \frac{1 - \left[\alpha B(A - B) + B^2\right] r^{2n}}{1 - B^2 r^{2n}} + \frac{\alpha(A - B) r^n}{1 - B^2 r^{2n}} = \frac{1 + (B + \alpha(A - B)) r^n}{1 + B r^n} := \eta(r). \tag{3.4}$$

Observe that $\rho_3 \geqslant \rho_1$, for $\alpha(A - B) \leqslant 2|B|$. We now show that $\mathcal{R} = \mathcal{R}_{\mathcal{ST}_e} = \rho_3$. For $0 \leqslant r \leqslant \rho_3 < 1$, it follows that $c(\alpha, r) < e$. Since $c(\alpha, r)$ is an increasing function, for $r \leqslant \rho_1$, we have $c(\alpha, r) \leqslant c(\alpha, \rho_1) = (e + e^{-1})/2$. Clearly, $c(\alpha, r) + d(\alpha, r)$ is an increasing function of r, for $0 \leqslant r \leqslant \rho_3$, we have

$$c(\alpha, r) + d(\alpha, r) \leqslant c(\alpha, \rho_3) + d(\alpha, \rho_3) = e$$

and hence

$$d(\alpha, r) \leqslant e - c(\alpha, r). \tag{3.5}$$

For $0 \leqslant r \leqslant \mathcal{R} = \rho_3$, using (2.2) and (3.5) we have,

$$\left|\frac{zg'(z)}{g(z)} - c(\alpha, r)\right| \leqslant e - c(\alpha, r). \tag{3.6}$$

Therefore, using the inclusion result in Lemma 3.1, for $(e + e^{-1})/2 \leqslant a < e$, the disc in (3.6) lies inside the region Ω_e proving that $\mathcal{ST}_e$ radius for functions belonging to the class $\mathcal{ST}_n^{\alpha}[A, B]$ is at least ρ_3.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{ST}_n^{\alpha}[A, B]$ defined by $\tilde{g}(z) = z(1 + Bz^n)^{(\alpha(A-B)/nB)}$. For $z = \rho_3$ in (2.19), we have

$$\left|\log\left(\frac{z\tilde{g}'(z)}{\tilde{g}(z)}\right)\right| = \left|\log\left(\frac{1 + (B + \alpha(A - B))(\rho_3)^n}{1 + B(\rho_3)^n}\right)\right| = |\log e| = 1,$$

proving the sharpness for ρ_3.

For $0 < B < 1$, if neither condition (i) nor condition (ii) holds, then the $\mathcal{R}_{\mathcal{ST}_e}$ radius for the class $\mathcal{ST}_n^{\alpha}[A, B]$ is given by

$$\mathcal{R}_{\mathcal{ST}_e} = \left(\frac{e - 1}{e\alpha(A - B) + (e - 1)B}\right)^{1/n}. \qquad \square$$

4 $\mathcal{ST}_C$ RADIUS FOR FUNCTIONS IN THE CLASS $\mathcal{ST}_n^{\alpha}[A, B]$

Sharma et al. [20] studied the class $\mathcal{ST}_C = \mathcal{ST}(\varphi_C)$, where $\varphi_C(z) = 1 + (4/3)z + (2/3)z^2$, where the boundary of $\varphi_C(\mathbb{D})$ is a cardiod.

Lemma 4.1. *[20] For $1/3 < a < 3$, let*

$$r_a = \begin{cases} a - \frac{1}{3} & \text{if} \quad \frac{1}{3} < a \leqslant \frac{5}{3} \\ 3 - a & \text{if} \quad \frac{5}{3} \leqslant a < 3. \end{cases}$$

Then, $\{w : |w - a| < r_a\} \subset \varphi_C(\mathbb{D}) = \Omega_C$, where Ω_C is the region bounded by the cardiod $\{x + iy : (9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0\}$.

Theorem 4.2. *Let $\alpha > 0$, $-1 \leqslant B \leqslant 0$ and $B < A \leqslant 1$. For the class $\mathcal{ST}_n^{\alpha}[A, B]$, the inclusion $\mathcal{ST}_n^{\alpha}[A, B] \subset \mathcal{ST}_C$ holds if either*

$$\textbf{(i)}(-\alpha B(A - B))/(1 - B^2) \leqslant 2/3 \quad \text{and} \quad (\alpha(A - B))/(1 - B) \leqslant 2/3$$

$$(or)$$

$$\textbf{(ii)}(-\alpha B(A - B))/(1 - B^2) \geqslant 2/3 \quad \text{and} \quad (\alpha(A - B))/(1 + B) \leqslant 2.$$

If neither condition (i) nor condition (ii) holds, then the $\mathcal{R}_{\mathcal{ST}_C}$ radius is given by

$$\mathcal{R}_{\mathcal{ST}_C}(\mathcal{ST}_n^{\alpha}[A, B]) = \begin{cases} \left(\frac{2}{3\alpha(A-B)+2B}\right)^{1/n} & \text{if} \quad \alpha(A - B) \geqslant 2|B| \\ \left(\frac{2}{\alpha(A-B)-2B}\right)^{1/n} & \text{if} \quad \alpha(A - B) \leqslant 2|B|. \end{cases}$$

Proof. To prove the inclusion $\mathcal{ST}_n^{\alpha}[A, B] \subset \mathcal{ST}_C$, we first assume that condition (i) holds. The inequality $(-\alpha B(A - B))/(1 - B^2) \leqslant 2/3$ is equivalent to $c(\alpha, 1) \leqslant 5/3$. The inequality $d(\alpha, 1) \leqslant c(\alpha, 1) - 1/3$ follows from $(-\alpha(A - B))/(1 - B) \leqslant 2/3$. By (2.13), we get

$$\left|\frac{zg'(z)}{g(z)} - c(\alpha, 1)\right| \leqslant d(\alpha, 1) \leqslant c(\alpha, 1) - \frac{1}{3}.$$

Therefore, using Lemma 4.1 we see that the disc in (2.13) is contained in Ω_C. Next, we assume that $(-\alpha B(A - B))/(1 - B^2) \geqslant 2/3$ and $\alpha(A - B)/(1 + B) \leqslant 2$. The first inequality reduces to $c(\alpha, 1) \geqslant 5/3$. The condition $(\alpha(A - B))/(1 + B) \leqslant 2$ is equivalent to the inequality $d(\alpha, 1) \leqslant 3 - c(\alpha, 1)$. Then by (2.13), we get

$$\left|\frac{zg'(z)}{g(z)} - c(\alpha, 1)\right| \leqslant d(\alpha, 1) \leqslant 3 - c(\alpha, 1).$$

Using Lemma 4.1, it can seen that the disc in (2.13) is contained in Ω_C.

When the inclusion fails, we shall show that, for $0 \leqslant r \leqslant \mathcal{R} := \mathcal{R}_{\mathcal{ST}_C}$, the disc $\mathbb{D}(c(\alpha, r); d(\alpha, r))$ given in (2.2) is contained in Ω_C, where $c(\alpha, r)$ and $d(\alpha, r)$ given by (2.3).

Case (i): Let $\alpha(A - B) \geqslant 2|B|$. The number

$$\rho_1 := \left(\frac{2}{3\alpha|B|(A - B) + 2B^2}\right)^{1/2n}$$

be the unique root of the equation $c(\alpha, r) = 5/3$ and let the number

$$\rho_2 := \left(\frac{2}{3\alpha(A - B) + 2B}\right)^{1/n} < 1$$

be the positive root of the equation $\zeta(r) = 1/3$ or $d(\alpha, r) = c(\alpha, r) - 1/3$.

For $\alpha(A - B) \geqslant 2|B|$, a computation shows that $\rho_2 \leqslant \rho_1$. We shall show that $\mathcal{R} = \mathcal{R}_{\mathcal{ST}_C} = \rho_2$. Since $c(\alpha, r) \geqslant 1$, for $0 \leqslant r \leqslant \rho_2 < 1$, we have $c(\alpha, r) > 1/3$. Also, since $c(\alpha, r)$

is an increasing function, for $r \leqslant \rho_1$, it follows that $c(\alpha, r) \leqslant c(\alpha, \rho_1) = 5/3$. Note that $c(\alpha, r) - d(\alpha, r)$ is a decreasing function of r, for $0 \leqslant r \leqslant \rho_2$, it follows that

$$c(\alpha, r) - d(\alpha, r) \geqslant c(\alpha, \rho_2) - d(\alpha, \rho_2) = 1/3$$

and hence

$$d(\alpha, r) \leqslant c(\alpha, r) - \frac{1}{3}. \tag{4.1}$$

Therefore, for $0 \leqslant r \leqslant \mathcal{R} = \rho_2$, using (2.2) and (4.1), we have

$$\left| \frac{zg'(z)}{g(z)} - c(\alpha, r) \right| \leqslant c(\alpha, r) - \frac{1}{3}. \tag{4.2}$$

Therefore, using the inclusion result in Lemma 4.1, for $1/3 < a \leqslant 5/3$, the disc in (4.2) lies inside the region Ω_C proving that $\mathcal{ST}_C$ radius for functions belonging to the class $\mathcal{ST}_n^\alpha[A, B]$ is at least ρ_2.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{ST}_n^\alpha[A, B]$ defined by (2.18). For $z = -\rho_2$ in (2.19), we have

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + (B + \alpha(A - B))(-\rho_2)^n}{1 + B(-\rho_2)^n} = \frac{1}{3} = \varphi_C(-1),$$

which proves the sharpness for ρ_2.

Case (ii): Let $\alpha(A - B) \leqslant 2|B|$. The number

$$\rho_3 := \left(\frac{2}{\alpha(A - B) - 2B} \right)^{1/n} < 1$$

be the positive root of the equation $\eta(r) = 3$ or $d(\alpha, r) = 3 - c(\alpha, r)$.

For $\alpha(A - B) \leqslant 2|B|$, observe that $\rho_3 \geqslant \rho_1$. We shall show that $\mathcal{R} = \mathcal{R}_{\mathcal{ST}_C} = \rho_3$. For $0 \leqslant r \leqslant \rho_3 < 1$, it follows that $c(\alpha, r) < 3$. Since $c(\alpha, r)$ is an increasing function, for $r \leqslant \rho_1$, we have $c(\alpha, r) \leqslant c(\alpha, \rho_1) = 5/3$. Note that $c(\alpha, r) + d(\alpha, r)$ is an increasing function of r, for $0 \leqslant r \leqslant \rho_3$, it follows that

$$c(\alpha, r) + d(\alpha, r) \leqslant c(\alpha, \rho_3) + d(\alpha, \rho_3) = 3$$

and hence

$$d(\alpha, r) \leqslant 3 - c(\alpha, r). \tag{4.3}$$

Therefore, for $0 \leqslant r \leqslant \mathcal{R} = \rho_3$, using (2.2) and (4.3), we have

$$\left| \frac{zg'(z)}{g(z)} - c(\alpha, r) \right| \leqslant 3 - c(\alpha, r). \tag{4.4}$$

Therefore, using the inclusion result in Lemma 4.1, for $5/3 \leqslant a < 3$, the disc in (4.4) lies inside the region Ω_C proving that $\mathcal{ST}_C$ radius for functions belonging to the class $\mathcal{ST}_n^\alpha[A, B]$ is at least ρ_3.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{ST}_n^\alpha[A, B]$ defined by (2.18). For $z = \rho_3$ in (2.19), we have

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + (B + \alpha(A - B))(\rho_3)^n}{1 + B(\rho_3)^n} = 3 = \varphi_C(1),$$

proving the sharpness for ρ_3.

For $0 < B < 1$, if neither condition (i) nor condition (ii) holds, then the $\mathcal{R}_{\mathcal{ST}_C}$ radius for the class $\mathcal{ST}_n^\alpha[A, B]$ is given by

$$\mathcal{R}_{\mathcal{ST}_C} = \left(\frac{2}{3\alpha(A - B) + 2B} \right)^{1/n}. \qquad \square$$

5 $\mathcal{ST}$ RADIUS FOR FUNCTIONS IN THE CLASS $\mathcal{ST}_n^\alpha[A,B]$

Raina and Sokól [17] studied the class $\mathcal{ST} = \mathcal{ST}(\varphi)$, where $\varphi(z) = z + \sqrt{1+z^2}$ and proved that $f \in \mathcal{ST}$ if and only if $zf'(z)/f(z) \in \Omega$, where Ω is the interior of a lune given by $\Omega := \{w \in \mathbb{C} : |w^2-1| < 2|w|\}$. Gandhi and Ravichandran [4] proved the following inclusion lemma:

Lemma 5.1. *[4] For $\sqrt{2}-1 < a \leqslant \sqrt{2}+1$, let*

$$r_a = 1 - |\sqrt{2} - a|.$$

Then, $\{w : |w-a| < r_a\} \subset \varphi(\mathbb{D}) = \Omega := \{w \in \mathbb{C} : |w^2-1| < 2|w|\}$.

Theorem 5.2. *Let $\alpha > 0$, $-1 \leqslant B \leqslant 0$ and $B < A \leqslant 1$. For the class $\mathcal{ST}_n^\alpha[A,B]$, the inclusion $\mathcal{ST}_n^\alpha[A,B] \subset \mathcal{ST}$ holds if either*

$$\textit{(i)}(-\alpha B(A-B))/(1-B^2) \leqslant \sqrt{2}-1 \quad \textit{and} \quad (\alpha(A-B))/(1-B) \leqslant 2-\sqrt{2}$$

$$(\textit{or})$$

$$\textit{(ii)}(-\alpha B(A-B))/(1-B^2) \geqslant \sqrt{2}-1 \quad \textit{and} \quad (\alpha(A-B))/(1+B) \leqslant \sqrt{2}.$$

If neither condition (i) nor condition (ii) holds, then the $\mathcal{ST}$ radius is given by

$$\mathcal{R}_{\mathcal{ST}} = \begin{cases} \left(\frac{2-\sqrt{2}}{\alpha(A-B)+(2-\sqrt{2})B}\right)^{1/n} & \textit{if} \quad \alpha(A-B) \geqslant 2|B| \\ \left(\frac{\sqrt{2}}{\alpha(A-B)-\sqrt{2}B}\right)^{1/n} & \textit{if} \quad \alpha(A-B) \leqslant 2|B|. \end{cases}$$

Proof. To prove the inclusion, assume that the condition (i) holds. The inequality $(-\alpha B(A-B))/(1-B^2) \leqslant \sqrt{2}-1$ is equivalent to $c(\alpha,1) \leqslant \sqrt{2}$. The condition $(-\alpha(A-B))/(1-B) \leqslant 2-\sqrt{2}$ is equivalent to the inequality $d(\alpha,1) \leqslant c(\alpha,1) - \sqrt{2} - 1$. By (2.13), we get

$$\left|\frac{zg'(z)}{g(z)} - c(\alpha,1)\right| \leqslant d(\alpha,1) \leqslant c(\alpha,1) - \sqrt{2} - 1.$$

Therefore, using Lemma 5.1, we see that the disc in (2.13) is contained in Ω. Assume that $(-\alpha B(A-B))/(1-B^2) \geqslant \sqrt{2}-1$ and $\alpha(A-B)/(1+B) \leqslant \sqrt{2}$. The first inequality reduces to $c(\alpha,1) \geqslant \sqrt{2}$. The condition follows $\alpha(A-B)/(1+B) \leqslant \sqrt{2}$ from the inequality $d(\alpha,1) \leqslant \sqrt{2}+1-c(\alpha,1)$. By (2.13), we get

$$\left|\frac{zg'(z)}{g(z)} - c(\alpha,1)\right| \leqslant d(\alpha,1) \leqslant \sqrt{2}+1-c(\alpha,1).$$

Using Lemma 5.1, we see that the disc in (2.13) is contained in Ω_C.

When the conditions (i) and (ii) fails, we show that, for $0 \leqslant r \leqslant \mathcal{R} := \mathcal{R}_{\mathcal{ST}}$, the disc $\mathbb{D}(c(\alpha,r); d(\alpha,r))$ given in (2.2) is contained in Ω.

Case (i): Let $\alpha(A-B) \geqslant 2|B|$. The number

$$\rho_1 := \left(\frac{\sqrt{2}-1}{\alpha|B|(A-B)+(\sqrt{2}-1)B^2}\right)^{1/n}$$

be the unique root of the equation $c(\alpha,r) = \sqrt{2}$ and let the number

$$\rho_2 := \left(\frac{2-\sqrt{2}}{3(\alpha(A-B)+(2-\sqrt{2})B}\right)^{1/n} < 1$$

be the positive root of the equation $\zeta(r) = \sqrt{2}-1$ or $d(\alpha,r) = c(\alpha,r) - (\sqrt{2}-1)$.

For $\alpha(A-B) \geqslant 2|B|$, a computation shows that $\rho_2 \leqslant \rho_1$. We now show that $\mathcal{R} = \mathcal{R}_{\mathcal{ST}} = \rho_2$. For $0 \leqslant r \leqslant \rho_2 < 1$, since $c(\alpha,r) \geqslant 1$, it can be seen that $c(\alpha,r) > \sqrt{2}-1$. Also since

$c(\alpha, r)$ is an increasing function, for $r \leqslant \rho_1$, we have $c(\alpha, r) \leqslant c(\alpha, \rho_1) = \sqrt{2}$. We note that $c(\alpha, r) - d(\alpha, r)$ is a decreasing function of r, hence for $0 \leqslant r \leqslant \rho_2$ it follows that,

$$c(\alpha, r) - d(\alpha, r) \geqslant c(\alpha, \rho_2) - d(\alpha, \rho_2) = \sqrt{2} - 1$$

and hence

$$d(\alpha, r) \leqslant c(\alpha, r) - (\sqrt{2} - 1). \tag{5.1}$$

Therefore, for $0 \leqslant r \leqslant \mathcal{R} = \rho_2$, using (2.2) and (5.1), we have

$$\left| \frac{zg'(z)}{g(z)} - c(\alpha, r) \right| \leqslant c(\alpha, r) - (\sqrt{2} - 1). \tag{5.2}$$

Thus, using the inclusion result in Lemma 5.1, for $2(\sqrt{2} - 1) < a \leqslant \sqrt{2}$, the disc in (5.2) lies inside the region Ω proving that $\mathcal{ST}$ radius for functions belonging to the class $\mathcal{ST}_n^{\alpha}[A, B]$ is at least ρ_2.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{ST}_n^{\alpha}[A, B]$ defined by (2.18). For $z = -\rho_2$ in (2.19), we have

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + (B + \alpha(A - B))(-\rho_2)^n}{1 + B(-\rho_2)^n} = \sqrt{2} - 1 = \varphi(-1),$$

which proves the sharpness for ρ_2.

Case (ii): Let $\alpha(A - B) \leqslant 2|B|$. The number

$$\rho_3 := \left(\frac{\sqrt{2}}{\alpha(A - B) - \sqrt{2}B} \right)^{1/n} < 1$$

be the positive root of the equation $\eta(r) = \sqrt{2} + 1$ or $d(\alpha, r) = \sqrt{2} + 1 - c(\alpha, r)$.

For $\alpha(A - B) \leqslant 2|B|$, we note that $\rho_3 \geqslant \rho_1$. We shall now show that $\mathcal{R} = \mathcal{R}_{\mathcal{ST}} = \rho_3$. For $0 \leqslant r \leqslant \rho_3 < 1$ it follows that $c(\alpha, r) \leqslant \sqrt{2} + 1$. Since $c(\alpha, r)$ is an increasing function, for $r \leqslant \rho_1$, we have $c(\alpha, r) \leqslant c(\alpha, \rho_1) = \sqrt{2}$. Also since $c(\alpha, r) + d(\alpha, r)$ is an increasing function of r, for $0 \leqslant r \leqslant \rho_3$, it follows that

$$c(\alpha, r) + d(\alpha, r) \leqslant c(\alpha, \rho_3) + d(\alpha, \rho_3) = \sqrt{2} + 1$$

and hence

$$d(\alpha, r) \leqslant \sqrt{2} + 1 - c(\alpha, r). \tag{5.3}$$

Therefore, for $0 \leqslant r \leqslant \mathcal{R} = \rho_3$, using (2.2) and (5.3), we have

$$\left| \frac{zg'(z)}{g(z)} - c(\alpha, r) \right| \leqslant \sqrt{2} + 1 - c(\alpha, r). \tag{5.4}$$

Therefore, using the inclusion result in Lemma 5.1, for $\sqrt{2} \leqslant a < \sqrt{2} + 1$, the disc in (5.4) lies inside the region Ω proving that $\mathcal{ST}$ radius for functions belonging to the class $\mathcal{ST}_n^{\alpha}[A, B]$ is at least ρ_3.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{ST}_n^{\alpha}[A, B]$ defined by (2.18). For $z = \rho_3$ in (2.19), we have

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + (B + \alpha(A - B))(\rho_3)^n}{1 + B(\rho_3)^n} = \sqrt{2} + 1 = \varphi_C(1),$$

proving the sharpness for ρ_3.

For $0 < B < 1$, if neither condition (i) nor condition (ii) holds, then the $\mathcal{R}_{\mathcal{ST}}$ radius for the class $\mathcal{ST}_n^{\alpha}[A, B]$ is given by

$$\mathcal{R}_{\mathcal{ST}} = \left(\frac{2 - \sqrt{2}}{\alpha(A - B) + (2 - \sqrt{2})B} \right)^{1/n}. \qquad \square$$

6 $\mathcal{ST}_{\wp}$ RADIUS FOR FUNCTIONS IN THE CLASS $\mathcal{ST}_n^{\alpha}[A,B]$

Kumar and Kamaljeet [11] introduced the class $\mathcal{ST}_{\wp} = \mathcal{ST}(\varphi_{\wp})$, where $\varphi_{\wp}(z) = 1 + ze^z$.

Lemma 6.1. *[11] For $1-(1/e) < a < 1+e$, let*

$$r_a = \begin{cases} (a-1)+\frac{1}{e} & \text{if} \quad 1-\frac{1}{e} < a \leqslant 1+\frac{e-e^{-1}}{2} \\ e-(a-1) & \text{if} \quad 1+\frac{e-e^{-1}}{2} \leqslant a < 1+e. \end{cases}$$

Then, $\{w : |w-a| < r_a\} \subset \varphi_{\wp}(\mathbb{D}) = \Omega_{\wp}$ where $\Omega_{\wp}$ is a cardiod.

Theorem 6.2. *Let $\alpha > 0$, $-1 \leqslant B \leqslant 0$ and $B < A \leqslant 1$. For the class $\mathcal{ST}_n^{\alpha}[A,B]$, the inclusion $\mathcal{ST}_n^{\alpha}[A,B] \subset \mathcal{ST}_{\wp}$ holds if either*

$$\textbf{(i)}(-\alpha B(A-B))/(1-B^2) \leqslant (e-e^{-1})/2 \quad \text{and} \quad (\alpha(A-B))/(1-B) \leqslant 1/e$$
$$(or)$$
$$\textbf{(ii)}(-\alpha B(A-B))/(1-B^2) \geqslant (e-e^{-1})/2 \quad \text{and} \quad (\alpha(A-B))/(1+B) \leqslant e.$$

If neither condition (i) nor condition (ii) holds, then the $\mathcal{R}_{\mathcal{ST}_{\wp}}$ radius is given by

$$\mathcal{R}_{\mathcal{ST}_{\wp}} = \begin{cases} \left(\frac{1}{e\alpha(A-B)+B}\right)^{1/n} & \text{if} \quad \alpha(A-B)(e-e^{-1}) \geqslant 2|B| \\ \left(\frac{e}{\alpha(A-B)-eB}\right)^{1/n} & \text{if} \quad \alpha(A-B)(e-e^{-1}) \leqslant 2|B|. \end{cases}$$

Proof. To prove the inclusion, we first assume that the condition (i) holds. The inequality $(-\alpha B(A-B))/(1-B^2) \leqslant (e-e^{-1})/2$ is equivalent to $c(\alpha,1) \leqslant 1+(e-e^{-1})/2$. The condition $(-\alpha(A-B))/(1-B) \leqslant 1/e$ is equivalent to the inequality $d(\alpha,1) \leqslant c(\alpha,1)-1+1/e$. By (2.13), we get

$$\left|\frac{zg'(z)}{g(z)} - c(\alpha,1)\right| \leqslant d(\alpha,1) \leqslant c(\alpha,1)-1+1/e.$$

Therefore, using Lemma 6.1, we see that the disc in (2.13) is contained in $\Omega_{\wp}$. Assume that $(-\alpha B(A-B))/(1-B^2) \geqslant (e-e^{-1})/2$ and $\alpha(A-B)/(1+B) \leqslant e$. The first inequality reduces to $c(\alpha,1) \geqslant 1+(e-e^{-1})/2$. If $d(\alpha,1) \leqslant e-(c(\alpha,1)-1)$ which directly follows from $(\alpha(A-B))/(1+B) \leqslant e$, then from (2.13), we get

$$\left|\frac{zg'(z)}{g(z)} - c(\alpha,1)\right| \leqslant d(\alpha,1) \leqslant e+1-c(\alpha,1).$$

Hence, using Lemma 6.1, we see that the disc in (2.13) is contained in $\Omega_{\wp}$.

When the inclusion fails, we shall show that, for $0 \leqslant r \leqslant \mathcal{R} := \mathcal{R}_{\mathcal{ST}_{\wp}}$, the disc $\mathbb{D}(c(\alpha,r);d(\alpha,r))$ given in (2.2) is contained in $\Omega_{\wp}$.

Case (i): Let $\alpha(A-B)(e-e^{-1}) \geqslant 2|B|$. The number

$$\rho_1 := \left(\frac{e-e^{-1}}{2\alpha|B|(A-B)+(e-e^{-1})B^2}\right)^{1/2n}$$

be the unique root of the equation $c(\alpha,r) = 1+(e-e^{-1})/2$ and let the number

$$\rho_2 := \left(\frac{1}{e\alpha(A-B)+B}\right)^{1/n} < 1$$

be the positive root of the equation $\zeta(r) = 1-(1/e)$ or $d(\alpha,r) = c(\alpha,r)-1+(1/e)$.

Observe that $\rho_2 \leqslant \rho_1$, for $\alpha(A-B)(e-e^{-1}) \geqslant 2|B|$. We shall show that $\mathcal{R} = \mathcal{R}_{\mathcal{ST}_{\wp}} = \rho_2$. For $0 \leqslant r \leqslant \rho_2 < 1$, since $c(\alpha,r) \geqslant 1$, we have $c(\alpha,r) > 1-(1/e)$. Since $c(\alpha,r)$ is an increasing function, for $r \leqslant \rho_1$, we have $c(\alpha, r) \leqslant c(\alpha, \rho_1) = 1 + (e - e^{-1})/2$. Note that $c(\alpha, r) - d(\alpha, r)$ is a decreasing function of r, for $0 \leqslant r \leqslant \rho_2$, it follows that

$$c(\alpha, r) - d(\alpha, r) \geqslant c(\alpha, \rho_2) - d(\alpha, \rho_2) = 1 - \frac{1}{e}$$

and hence

$$d(\alpha, r) \leqslant c(\alpha, r) - 1 + \frac{1}{e}. \tag{6.1}$$

Therefore, for $0 \leqslant r \leqslant \mathcal{R} = \rho_2$, using (2.2) and (6.1), we have

$$\left| \frac{zg'(z)}{g(z)} - c(\alpha, r) \right| \leqslant c(\alpha, r) - 1 + \frac{1}{e}. \tag{6.2}$$

Therefore, using the inclusion result in Lemma 6.1, for $1 - (1/e) < a \leqslant 1 + (e - e^{-1})/2$, the disc in (6.2) lies inside the region $\Omega_\wp$ proving that $\mathcal{ST}_\wp$ radius for functions belonging to the class $\mathcal{ST}_n^\alpha[A, B]$ is at least ρ_2.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{ST}_n^\alpha[A, B]$ defined by (2.18). For $z = -\rho_2$, (2.19) gives

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + (B + \alpha(A - B))(-\rho_2)^n}{1 + B(-\rho_2)^n} = 1 - \frac{1}{e} = \varphi_\wp(-1),$$

which proves the sharpness for ρ_2.

Case (ii): Let $\alpha(A - B)(e - e^{-1}) \leqslant 2|B|$. The number

$$\rho_3 := \left(\frac{e}{\alpha(A - B) - eB} \right)^{1/n} < 1$$

be the positive root of the equation or $\eta(r) = e + 1$ or $d(\alpha, r) = e + 1 - c(\alpha, r)$. Note that $\rho_3 \geqslant \rho_1$, for $\alpha(A - B)(e - e^{-1}) \leqslant 2|B|$. We shall show that $\mathcal{R} = \mathcal{R}_{\mathcal{ST}_\wp} = \rho_3$. For $0 \leqslant r \leqslant \rho_3 < 1$, we have $c(\alpha, r) < e + 1$. Since $c(\alpha, r)$ is an increasing function, for $r \leqslant \rho_1$, it follows that $c(\alpha, r) \leqslant c(\alpha, \rho_1) = 1 + (e - e^{-1})/2$. Clearly, $c(\alpha, r) + d(\alpha, r)$ is an increasing function of r, for $0 \leqslant r \leqslant \rho_3$, it follows that

$$c(\alpha, r) + d(\alpha, r) \leqslant c(\alpha, \rho_3) + d(\alpha, \rho_3) = e + 1$$

and hence

$$d(\alpha, r) \leqslant e + 1 - c(\alpha, r). \tag{6.3}$$

Therefore, for $0 \leqslant r \leqslant \mathcal{R} = \rho_3$, using (2.2) and (6.3), we have

$$\left| \frac{zg'(z)}{g(z)} - c(\alpha, r) \right| \leqslant e + 1 - c(\alpha, r). \tag{6.4}$$

Therefore, using the inclusion result in Lemma 6.1, for $1 + (e - e^{-1})/2 \leqslant a < 1 + e$, the disc in (6.4) lies inside the region $\Omega_\wp$ proving that $\mathcal{ST}_\wp$ radius for functions belonging to the class $\mathcal{ST}_n^\alpha[A, B]$ is at least ρ_3.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{ST}_n^\alpha[A, B]$ (2.18). For $z = \rho_3$, (2.19) gives

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + (B + \alpha(A - B))(\rho_3)^n}{1 + B(\rho_3)^n} = 1 + e = \varphi_\wp(1),$$

proving the sharpness for ρ_3.

For $0 < B < 1$, if neither condition (i) nor condition (ii) holds, then the $\mathcal{R}_{\mathcal{ST}_\wp}$ radius for the class $\mathcal{ST}_n^\alpha[A, B]$ is given by

$$\mathcal{R}_{\mathcal{ST}_\wp} = \left(\frac{1}{e\alpha(A - B) + B} \right)^{1/n}. \qquad \square$$

7 $\mathcal{ST}_{Ne}$ RADIUS FOR FUNCTIONS IN THE CLASS $\mathcal{ST}_n^{\alpha}[A,B]$

Wani and Swaminathan [23] studied the class $\mathcal{ST}_{Ne} = \mathcal{ST}(\varphi_{Ne})$ which consists of starlike functions associated with a nephroid domain, where $\varphi_{Ne}(z) = 1 + z - (z^3/3)$ that maps the unit circle onto a 2-cusped curve, $\left((u-1)^2 + v^2 - (4/9)\right)^3 - (4v^2/3) = 0$. The following lemma due to Wani and Swaminathan [22] provides the inclusion $\{w : |w-a| < r_a\} \subset \varphi_{Ne}(\mathbb{D}) = \Omega_{Ne}$.

Lemma 7.1. *[22] For $1/3 < a < 5/3$, let*

$$r_a = \begin{cases} a - \frac{1}{3} & \text{if} \quad \frac{1}{3} < a \leqslant 1 \\ \frac{5}{3} - a & \text{if} \quad 1 \leqslant a < \frac{5}{3}. \end{cases}$$

Then, $\{w : |w-a| < r_a\} \subset \varphi_{Ne}(\mathbb{D}) = \Omega_{Ne}$ where Ω_{Ne} is the region bounded by the nephroid φ_{Ne}.

Theorem 7.2. *Let $\alpha > 0$, $-1 \leqslant B \leqslant 0$ and $B < A \leqslant 1$. For the class $\mathcal{ST}_n^{\alpha}[A,B]$, the inclusion $\mathcal{ST}_n^{\alpha}[A,B] \subset \mathcal{ST}_{Ne}$ holds if either*

$$\textbf{\textit{(i)}} B \geqslant 0 \quad \textit{and} \quad (\alpha(A-B))/(1-B) \leqslant 2/3$$
$$(or)$$
$$\textbf{\textit{(ii)}} B \leqslant 0 \quad \textit{and} \quad (\alpha(A-B))/(1+B) \leqslant 2/3.$$

If neither condition (i) nor condition (ii) holds, then the $\mathcal{R}_{\mathcal{ST}_{Ne}}$ radius is given by

$$\mathcal{R}_{\mathcal{ST}_{Ne}} = \left(\frac{2}{3\alpha(A-B) + 2|B|}\right)^{1/n}.$$

Proof. We need to show that the inclusion $\mathcal{ST}_n^{\alpha}[A,B] \subset \mathcal{ST}_{Ne}$, therefore assume that $B \geqslant 0$ and $\alpha(A-B)/(1-B) \leqslant 2/3$. The inequality $B \geqslant 0$ is equivalent to $c(\alpha,1) \leqslant 1$. Since the inequality $d(\alpha,1) \leqslant c(\alpha,1) - 1/3$ follows from $\alpha(A-B)/(1-B) \leqslant 2/3$. By (2.13), we have

$$\left|\frac{zg'(z)}{g(z)} - c(\alpha,1)\right| \leqslant d(\alpha,1) \leqslant c(\alpha,1) - \frac{1}{3}.$$

Therefore, using Lemma 7.1 we see that the disc in (2.13) is contained in Ω_{Ne}.

Now assume that $B \leqslant 0$ and $\alpha(A-B)/(1+B) \leqslant 2/3$. The first inequality reduces to $c(\alpha,1) \geqslant 1$. The inequality $(\alpha(A-B))/(1+B) \leqslant 2/3$ is equivalent to $d(\alpha,1) \leqslant 5/3 - c(\alpha,1)$. Therefore, by (2.13), we have

$$\left|\frac{zg'(z)}{g(z)} - c(\alpha,1)\right| \leqslant d(\alpha,1) \leqslant \frac{5}{3} - c(\alpha,1).$$

Therefore, using Lemma 7.1 we see that the disc in (2.13) is contained in Ω_{Ne}.

When the inclusion fails, we show that, for $0 \leqslant r \leqslant \mathcal{R} := \mathcal{R}_{\mathcal{ST}_{Ne}}$, the disc $\mathbb{D}(c(\alpha,r); d(\alpha,r))$ given in (2.2) is contained in Ω_{Ne}. We prove the theorem by considering the cases $B \geqslant 0$ and $B \leqslant 0$.

Case (i): Let $B \geqslant 0$. Let the number

$$\rho_2 := \left(\frac{2}{3\alpha(A-B) + 2B}\right)^{1/n} < 1$$

be the positive root of the equation $\zeta(r) = 1/3$ or $d(\alpha,r) = c(\alpha,r) - (1/3)$.

We shall show that $\mathcal{R} = \mathcal{R}_{\mathcal{ST}_{Ne}} = \rho_2$. For $0 \leqslant r \leqslant \mathcal{R} < 1$, it follows that $1/3 < c(\alpha,r) \leqslant c(\alpha,\mathcal{R}) \leqslant 1$. Since $c(\alpha,r) - d(\alpha,r)$ is a decreasing function of r, for $0 \leqslant r \leqslant \rho_2$, it follows that

$$c(\alpha,r) - d(\alpha,r) \geqslant c(\alpha,\rho_2) - d(\alpha,\rho_2) = \frac{1}{3}$$

and hence

$$d(\alpha, r) \leqslant c(\alpha, r) - \frac{1}{3}. \tag{7.1}$$

Therefore, for $0 \leqslant r \leqslant \mathcal{R} = \rho_2$, using (2.2) and (7.1), we have

$$\left| \frac{zg'(z)}{g(z)} - c(\alpha, r) \right| \leqslant c(\alpha, r) - \frac{1}{3}. \tag{7.2}$$

Thus, using the inclusion result in Lemma 7.1, for $1/3 < a \leqslant 1$, the disc in (7.2) lies inside the region Ω_{Ne} proving that $\mathcal{ST}_{Ne}$ radius for functions belonging to the class $\mathcal{ST}_n^\alpha[A, B]$ is at least ρ_2.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{ST}_n^\alpha[A, B]$ defined by (2.18). For $z = -\rho_2$, (2.19) gives

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + (B + \alpha(A - B))(-\rho_2)^n}{1 + B(-\rho_2)^n} = \frac{1}{3} = \varphi_{Ne}(-1),$$

which proves the sharpness for ρ_2.

Case(ii): Let $B \leqslant 0$. Let the number

$$\rho_3 := \left(\frac{2}{3\alpha(A - B) - 2B} \right)^{1/n} < 1$$

be the positive root of the equation $\eta(r) = 5/3$ or $d(\alpha, r) = (5/3) - c(\alpha, r)$.

We shall show that $\mathcal{R} = \mathcal{R}_{\mathcal{ST}_{Ne}} = \rho_3$. For $0 \leqslant r \leqslant \mathcal{R} < 1$, it follows that $1 \leqslant c(\alpha, r) \leqslant c(\alpha, \mathcal{R}) < 5/3$. Since $c(\alpha, r) + d(\alpha, r)$ is an increasing function of r, for $0 \leqslant r \leqslant \rho_3$, it follows that

$$c(\alpha, r) + d(\alpha, r) \leqslant c(\alpha, \rho_3) + d(\alpha, \rho_3) = \frac{5}{3}$$

and hence

$$d(\alpha, r) \leqslant \frac{5}{3} - c(\alpha, r). \tag{7.3}$$

Therefore, for $0 \leqslant r \leqslant \mathcal{R} = \rho_3$, using (2.2) and (7.3), we have

$$\left| \frac{zg'(z)}{g(z)} - c(\alpha, r) \right| \leqslant \frac{5}{3} - c(\alpha, r). \tag{7.4}$$

Thus, using the inclusion result in Lemma 7.1, for $1 \leqslant a < 5/3$, the disc in (7.4) lies inside the region Ω_{Ne} proving that $\mathcal{ST}_{Ne}$ radius for functions belonging to the class $\mathcal{ST}_n^\alpha[A, B]$ is at least ρ_3.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{ST}_n^\alpha[A, B]$ defined by (2.18). We have, for $z = \rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(1 + B + \alpha(A - B))(\rho_3)^n}{1 + B(\rho_3)^n} = \frac{5}{3} = \varphi_{Ne}(1),$$

proving the sharpness for ρ_3. □

8 $\mathcal{ST}_{SG}$ RADIUS FOR FUNCTIONS IN THE CLASS $\mathcal{ST}_n^\alpha[A, B]$

Goel and Kumar [6] introduced the class $\mathcal{ST}_{SG} = \mathcal{ST}(\varphi_{SG})$, where $\varphi_{SG}(z) = 2/(1 + e^{-z})$.

Lemma 8.1. *[6] For* $2/(1 + e) < a < 2e/(1 + e)$, *let*

$$r_a = \frac{e - 1}{e + 1} - |a - 1|.$$

Then, $\{w : |w - a| < r_a\} \subset \varphi_{SG}(\mathbb{D}) = \Omega_{SG} := \{w : |\log w/(2 - w)| < 1\}$ *where* Ω_{SG} *is a modified sigmoid.*

Theorem 8.2. *Let $\alpha > 0$, $-1 \leqslant B \leqslant 0$ and $B < A \leqslant 1$. For the class $\mathcal{ST}_n^{\alpha}[A,B]$, the inclusion $\mathcal{ST}_n^{\alpha}[A,B] \subset \mathcal{ST}_{SG}$ holds if either*

$$\textbf{\textit{(i)}} B \geqslant 0 \quad \textit{and} \quad (\alpha(A-B))/(1-B) \leqslant (e-1)/(e+1)$$
$$(or)$$
$$\textbf{\textit{(ii)}} B \leqslant 0 \quad \textit{and} \quad (\alpha(A-B))/(1+B) \leqslant (e-1)/(e+1).$$

If neither condition (i) nor condition (ii) holds, then the $\mathcal{R}_{\mathcal{ST}_{SG}}$ radius is given by

$$\mathcal{R}_{\mathcal{ST}_{SG}} = \left(\frac{e-1}{(e+1)\alpha(A-B)+(e-1)|B|}\right)^{1/n}.$$

Proof. To prove the inclusion, we assume that $B \geqslant 0$ and $\alpha(A-B)/(1-B) \leqslant (e-1)/(e+1)$. The inequality $B \geqslant 0$ is equivalent to $c(\alpha,1) \leqslant 1$. The condition $\alpha(A-B)/(1-B) \leqslant (e-1)/(e+1)$ is equivalent to the inequality $d(\alpha,1) \leqslant c(\alpha,1) + ((e-1)/(e+1)) - 1$. By (2.13), we get

$$\left|\frac{zg'(z)}{g(z)} - c(\alpha,1)\right| \leqslant d(\alpha,1) \leqslant c(\alpha,1) - \frac{e-1}{e+1} - 1.$$

Therefore, using Lemma 8.1 we see that the disc in (2.13) is contained in Ω_{SG}.

Now assume that $B \leqslant 0$ and $\alpha(A-B)/(1+B) \leqslant (e-1)/(e+1)$. The first inequality reduces to $c(\alpha,1) \geqslant 1$. The condition $\alpha(A-B)/(1+B) \leqslant (e-1)/(e+1)$ follows from the inequality $d(\alpha,1) \leqslant ((e-1)/(e+1)) + 1 - c(\alpha,1)$. using (2.13), we get

$$\left|\frac{zg'(z)}{g(z)} - c(\alpha,1)\right| \leqslant d(\alpha,1) \leqslant \frac{e-1}{e+1} + 1 - c(\alpha,1).$$

By Lemma 8.1, we see that the disc in (2.13) is contained in Ω_{SG}.

When the inclusion fails, we now show that, for $0 \leqslant r \leqslant \mathcal{R} := \mathcal{R}_{\mathcal{ST}_{SG}}$, the disc $\mathbb{D}(c(\alpha,r); d(\alpha,r))$ given in (2.2) is contained in Ω_{SG}.

Case (i): For $B \geqslant 0$, let the number

$$\rho_2 := \left(\frac{e-1}{(e+1)\alpha(A-B)+(e-1)B}\right)^{1/n} < 1$$

be the positive root of the equation $\zeta(r) = 1 - ((e-1)/(e+1))$ or $d(\alpha,r) = c(\alpha,r) + ((e-1)/(e+1)) - 1$.

We shall show that $\mathcal{R} = \mathcal{R}_{\mathcal{ST}_{SG}} = \rho_2$. For $0 \leqslant r \leqslant \mathcal{R} < 1$, it follows that $2/(1+e) < c(\alpha,r) \leqslant c(\alpha,\mathcal{R}) \leqslant 1$. Since $c(\alpha,r) - d(\alpha,r)$ is an decreasing function of r, it follows, for $0 \leqslant r \leqslant \rho_2$, that

$$c(\alpha,r) - d(\alpha,r) \geqslant c(\alpha,\rho_2) - d(\alpha,\rho_2) = 1 - \frac{e-1}{e+1}$$

and hence

$$d(\alpha,r) \leqslant c(\alpha,r) + \frac{e-1}{e+1} - 1. \tag{8.1}$$

Therefore, for $0 \leqslant r \leqslant \mathcal{R} = \rho_2$, using (2.2) and (8.1), we have

$$\left|\frac{zg'(z)}{g(z)} - c(\alpha,r)\right| \leqslant c(\alpha,r) + \frac{e-1}{e+1} - 1. \tag{8.2}$$

Therefore, using the inclusion result in Lemma 8.1, for $2/(1+e) < a \leqslant 1$, the disc in (8.2) lies inside the region Ω_{SG} proving that $\mathcal{ST}_{SG}$ radius for functions belonging to the class $\mathcal{ST}_n^{\alpha}[A,B]$ is at least ρ_2.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{ST}_n^{\alpha}[A,B]$ defined by (2.18). For $z = -\rho_2$, (2.19) gives

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+(B+\alpha(A-B))(-\rho_2)^n}{1+B(-\rho_2)^n} = \frac{2}{1+e} = \varphi_{SG}(-1),$$

which proves the sharpness for ρ_2.

Case (ii): When $B \leqslant 0$, let the number

$$\rho_3 := \left(\frac{e-1}{(e+1)\alpha(A-B)-(e-1)B}\right)^{1/n} < 1$$

be the positive root of the equation $\eta(r) = 1 + ((e-1)/(e+1))$ or $d(\alpha, r) = ((e-1)/(e+1)) + 1 - c(\alpha, r)$.

We shall show that $\mathcal{R} = \mathcal{R}_{\mathcal{ST}_{SG}} = \rho_3$. For $0 \leqslant r \leqslant \mathcal{R} < 1$, it follows that $1 \leqslant c(\alpha, r) \leqslant c(\alpha, \mathcal{R}) < 2e/(1+e)$. Since $c(\alpha, r) + d(\alpha, r)$ is an increasing function of r, for $0 \leqslant r \leqslant \rho_3$, it follows that

$$c(\alpha, r) + d(\alpha, r) \leqslant c(\alpha, \rho_3) + d(\alpha, \rho_3) = \frac{e-1}{e+1} + 1$$

and hence

$$d(\alpha, r) \leqslant \frac{e-1}{e+1} + 1 - c(\alpha, r). \tag{8.3}$$

Therefore, for $0 \leqslant r \leqslant \mathcal{R} = \rho_3$, using (2.2) and (8.3), we have

$$\left|\frac{zg'(z)}{g(z)} - c(\alpha, r)\right| \leqslant \frac{e-1}{e+1} + 1 - c(\alpha, r). \tag{8.4}$$

Thus, using the inclusion result in Lemma 8.1, for $1 \leqslant a < 2e/(1+e)$, the disc in (8.4) lies inside the region Ω_{SG} proving that $\mathcal{ST}_{SG}$ radius for functions belonging to the class $\mathcal{ST}_n^\alpha[A, B]$ is at least ρ_3.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{ST}_n^\alpha[A, B]$ defined by (2.18). For $z = \rho_3$, (2.19) gives

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + (B + \alpha(A-B))(\rho_3)^n}{1 + B(\rho_3)^n} = \frac{2e}{1+e} = \varphi_{SG}(1),$$

proving the sharpness for ρ_3. □

9 $\mathcal{ST}_{\sin}$ RADIUS FOR FUNCTIONS IN THE CLASS $\mathcal{ST}_n^\alpha[A, B]$

The class $\mathcal{ST}_{\sin} = \mathcal{ST}(\varphi_{\sin})$, where $\varphi_{\sin}(z) = 1 + \sin z$ was introduced by Cho et al. [2].

Lemma 9.1. *[2] For* $1 - \sin 1 < a < 1 + \sin 1$, *let*

$$r_a = \sin 1 - |a - 1|.$$

Then, $\{w : |w - a| < r_a\} \subset \varphi_{\sin}(\mathbb{D}) = \Omega_{\sin}$.

Theorem 9.2. *Let* $\alpha > 0$, $-1 \leqslant B \leqslant 0$ *and* $B < A \leqslant 1$. *For the class* $\mathcal{ST}_n^\alpha[A, B]$, *the inclusion* $\mathcal{ST}_n^\alpha[A, B] \subset \mathcal{ST}_{\sin}$ *holds if either*

$$\textbf{(i)} B \geqslant 0 \quad \text{and} \quad (\alpha(A-B))/(1-B) \leqslant \sin 1$$
$$(or)$$
$$\textbf{(ii)} B \leqslant 0 \quad \text{and} \quad (\alpha(A-B))/(1+B) \leqslant \sin 1.$$

If neither condition (i) nor condition (ii) holds, then the $\mathcal{R}_{\mathcal{ST}_{\sin}}$ *radius is given by*

$$\mathcal{R}_{\mathcal{ST}_{\sin}} = \left(\frac{\sin 1}{\alpha(A-B) + (\sin 1)|B|}\right)^{1/n}.$$

Proof. To prove the inclusion $\mathcal{ST}_n^\alpha[A, B] \subset \mathcal{ST}_{\sin}$, we assume that $B \geqslant 0$ and $\alpha(A-B)/(1-B) \leqslant \sin 1$. The inequality $B \geqslant 0$ is equivalent to $c(\alpha, 1) \leqslant 1$. The condition $\alpha(A-B)/(1-B) \leqslant \sin 1$ is equivalent to the inequality $d(\alpha, 1) \leqslant c(\alpha, 1) + (\sin 1) - 1$. By (2.13), we get

$$\left|\frac{zg'(z)}{g(z)} - c(\alpha, 1)\right| \leqslant d(\alpha, 1) \leqslant c(\alpha, 1) + (\sin 1) - 1.$$

Therefore, using Lemma 9.1 we see that the disc in (2.13) is contained in Ω_{sin}.

Now assume that $B \leqslant 0$ and $\alpha(A-B)/(1+B) \leqslant \sin 1$. The first inequality reduces to $c(\alpha,1) \geqslant 1$. The condition $\alpha(A-B)/(1+B) \leqslant \sin 1$ which directly follows from the inequality $d(\alpha,1) \leqslant (\sin 1)+1-c(\alpha,1)$. By (2.13), we get

$$\left|\frac{zg'(z)}{g(z)} - c(\alpha,1)\right| \leqslant d(\alpha,1) \leqslant (\sin 1)+1-c(\alpha,1).$$

Using Lemma 9.1, we see that the disc in (2.13) is contained in Ω_{sin}.

When the inclusion fails, we now show that, for $0 \leqslant r \leqslant \mathcal{R} := \mathcal{R}_{\mathcal{ST}_{\text{sin}}}$, the disc $\mathbb{D}(c(\alpha,r); d(\alpha,r))$ given in (2.2) is contained in Ω_{sin}.

Case (i): Let $B \geqslant 0$. The number

$$\rho_2 := \left(\frac{\sin 1}{\alpha(A-B)+(\sin 1)B}\right)^{1/n} < 1$$

be the positive root of the equation $\zeta(r) = 1-(\sin 1)$ or $d(\alpha,r) = c(\alpha,r)+(\sin 1)-1$.

We shall show that $\mathcal{R} = \mathcal{R}_{\mathcal{ST}_{\text{sin}}} = \rho_2$. For $0 \leqslant r \leqslant \mathcal{R} < 1$ it follows that $1-\sin 1 < c(\alpha,r) \leqslant c(\alpha,\mathcal{R}) \leqslant 1$. Since $c(\alpha,r)-d(\alpha,r)$ is an decreasing function of r, for $0 \leqslant r \leqslant \rho_2$, it follows that

$$c(\alpha,r)-d(\alpha,r) \geqslant c(\alpha,\rho_2)-d(\alpha,\rho_2) = 1-(\sin 1)$$

and hence

$$d(\alpha,r) \leqslant c(\alpha,r)+(\sin 1)-1. \tag{9.1}$$

Therefore, for $0 \leqslant r \leqslant \mathcal{R} = \rho_2$, using (2.2) and (9.1), we have

$$\left|\frac{zg'(z)}{g(z)} - c(\alpha,r)\right| \leqslant c(\alpha,r)+(\sin 1)-1. \tag{9.2}$$

Therefore, using the inclusion result in Lemma 9.1, for $1-(\sin 1) < a \leqslant 1$, the disc in (9.2) lies inside the region Ω_{sin} proving that $\mathcal{ST}_{\text{sin}}$ radius for functions belonging to the class $\mathcal{ST}_n^{\alpha}[A,B]$ is at least ρ_2.

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w-a| < r_a\} \subseteq \Omega_{SG}$ holds which proves that $\mathcal{ST}_{SG}$ radius for functions in the class $\mathcal{ST}_n^{\alpha}[A,B]$ is at least $\mathcal{R} = \rho_2$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{ST}_n^{\alpha}[A,B]$ defined by (2.18). For $z = -\rho_2$, (2.19) gives

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+(B+\alpha(A-B))(-\rho_2)^n}{1+B(-\rho_2)^n} = 1-(\sin 1) = \varphi_{\text{sin}}(-1),$$

which proves the sharpness for ρ_2.

Case (ii): Let $B \leqslant 0$. The number

$$\rho_3 := \left(\frac{\sin 1}{\alpha(A-B)-(\sin 1)B}\right)^{1/n} < 1$$

be the positive root of the equation $\eta(r) = 1+(\sin 1)$ or $d(\alpha,r) = (\sin 1)+1-c(\alpha,r)$.

We shall show that $\mathcal{R} = \mathcal{R}_{\mathcal{ST}_{\text{sin}}} = \rho_3$. For $0 \leqslant r \leqslant \mathcal{R} < 1$, it follows that $1 \leqslant c(\alpha,r) \leqslant c(\alpha,\mathcal{R}) < 1+(\sin 1)$. Since $c(\alpha,r)+d(\alpha,r)$ is an increasing function of r, for $0 \leqslant r \leqslant \rho_3$, it follows that

$$c(\alpha,r)+d(\alpha,r) \leqslant c(\alpha,\rho_3)+d(\alpha,\rho_3) = (\sin 1)+1$$

and hence

$$d(\alpha,r) \leqslant (\sin 1)+1-c(\alpha,r). \tag{9.3}$$

Therefore, for $0 \leqslant r \leqslant \mathcal{R} = \rho_3$, using (2.2) and (9.3), we have

$$\left|\frac{zg'(z)}{g(z)} - c(\alpha,r)\right| \leqslant (\sin 1)+1-c(\alpha,r). \tag{9.4}$$

Hence, using the inclusion result in Lemma 9.1, for $1 \leqslant a < 1 + (\sin 1)$, the disc in (9.4) lies inside the region $\Omega_{\sin}$ proving that $\mathcal{ST}_{\sin}$ radius for functions belonging to the class $\mathcal{ST}_n^{\alpha}[A, B]$ is at least ρ_3.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{ST}_n^{\alpha}[A, B]$ defined by (2.18). For $z = \rho_3$, (2.19) gives

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + (B + \alpha(A - B))(\rho_3)^n}{1 + B(\rho_3)^n} = 1 + \sin 1 = \varphi_{\sin}(1),$$

proving the sharpness for ρ_3. □

10 $\mathcal{ST}_h$ RADIUS FOR FUNCTIONS IN THE CLASS $\mathcal{ST}_n^{\alpha}[A, B]$

Kumar and Arora [10] introduced the class $\mathcal{ST}_h = \mathcal{ST}(\varphi_h)$, where $\varphi_h(z) = 1 + \sinh^{-1}(z)$.

Lemma 10.1. *[10] For $1 - \sinh^{-1}(1) < a < 1 + \sinh^{-1}(1)$, let*

$$r_a = \begin{cases} a - (1 - \sinh^{-1}(1)) & \text{if} \quad 1 - \sinh^{-1}(1) < a \leqslant 1 \\ 1 + \sinh^{-1}(1) - a & \text{if} \quad 1 \leqslant a < 1 + \sinh^{-1}(1). \end{cases}$$

Then $\{w : |w - a| < r_a\} \subset \varphi_h(\mathbb{D}) = \Omega_h := \{w \in \mathbb{C} : |\sinh(w - 1)| < 1\}$. The boundary of $\varphi_h(\mathbb{D})$ is petal shaped.

Theorem 10.2. *Let $\alpha > 0$, $-1 \leqslant B \leqslant 0$ and $B < A \leqslant 1$. For the class $\mathcal{ST}_n^{\alpha}[A, B]$, the inclusion $\mathcal{ST}_n^{\alpha}[A, B] \subset \mathcal{ST}_h$ holds if either*

$$\textbf{\textit{(i)}} B \geqslant 0 \quad \textit{and} \quad \alpha(A - B)/(1 - B) \leqslant \sinh^{-1}(1)$$
$$(\textit{or})$$
$$\textbf{\textit{(ii)}} B \leqslant 0 \quad \textit{and} \quad \alpha(A - B)/(1 + B) \leqslant \sinh^{-1}(1).$$

If neither condition (i) nor condition (ii) holds, then the $\mathcal{R}_{\mathcal{ST}_h}$ radius is given by

$$\mathcal{R}_{\mathcal{ST}_h} = \left(\frac{\sinh^{-1}(1)}{\alpha(A - B) + (\sinh^{-1}(1))|B|} \right)^{1/n}.$$

Proof. To prove the inclusion, we assume that $B \geqslant 0$ and $\alpha(A - B)/(1 - B) \leqslant \sinh^{-1}(1)$. The inequality $B \geqslant 0$ is equivalent to $c(\alpha, 1) \leqslant 1$. The condition $\alpha(A - B)/(1 - B) \leqslant \sinh^{-1}(1)$ is equivalent to the inequality $d(\alpha, 1) \leqslant c(\alpha, 1) + \sinh^{-1}(1) - 1$. By (2.13), we get

$$\left| \frac{zg'(z)}{g(z)} - c(\alpha, 1) \right| \leqslant d(\alpha, 1) \leqslant c(\alpha, 1) + \sinh^{-1}(1) - 1.$$

Therefore, using Lemma 10.1 we see that the disc in (2.13) is contained in Ω_h. Now assume that $B \leqslant 0$ and $\alpha(A - B)/(1 + B) \leqslant \sinh^{-1}(1)$. The first inequality reduces to $c(\alpha, 1) \geqslant 1$. The condition $\alpha(A - B)/(1 + B) \leqslant \sinh^{-1}(1)$ directly follows from the inequality $d(\alpha, 1) \leqslant 1 + \sinh^{-1}(1) - c(\alpha, 1)$. By (2.13), we get

$$\left| \frac{zg'(z)}{g(z)} - c(\alpha, 1) \right| \leqslant d(\alpha, 1) \leqslant 1 + \sinh^{-1}(1) - c(\alpha, 1).$$

Using Lemma 10.1, we see that the disc in (2.13) is contained in Ω_h.

When the inclusion fails, we now show that, for $0 \leqslant r \leqslant \mathcal{R} := \mathcal{R}_{\mathcal{ST}_h}$, the disc $\mathbb{D}(c(\alpha, r); d(\alpha, r))$ given in (2.2) is contained in Ω_h.

Case (i): Let $B \geqslant 0$. Let the number

$$\rho_2 := \left(\frac{\sinh^{-1}(1)}{\alpha(A - B) + (\sinh^{-1}(1))B} \right)^{1/n} < 1$$

be the positive root of the equation $\zeta(r) = 1 - \sinh^{-1}(1)$ or $d(\alpha, r) = c(\alpha, r) + \sinh^{-1}(1) - 1$.

We shall show that $\mathcal{R} = \mathcal{R}_{\mathcal{ST}_h} = \rho_2$. For $0 \leqslant r \leqslant \mathcal{R} < 1$, it follows that $1 - \sinh^{-1}(1) < c(\alpha, r) \leqslant c(\alpha, \mathcal{R}) \leqslant 1$. Since $c(\alpha, r) - d(\alpha, r)$ is a decreasing function of r, for $0 \leqslant r \leqslant \rho_2$, it follows that

$$c(\alpha, r) - d(\alpha, r) \geqslant c(\alpha, \rho_2) - d(\alpha, \rho_2) = 1 - \sinh^{-1}(1)$$

and hence

$$d(\alpha, r) \leqslant c(\alpha, r) + \sinh^{-1}(1) - 1. \tag{10.1}$$

Therefore, for $0 \leqslant r \leqslant \mathcal{R} = \rho_2$, using (2.2) and (10.1), we have

$$\left| \frac{zg'(z)}{g(z)} - c(\alpha, r) \right| \leqslant c(\alpha, r) + \sinh^{-1}(1) - 1. \tag{10.2}$$

Hence, using the inclusion result in Lemma 10.1, for $1 - \sinh^{-1}(1) < a \leqslant 1$, the disc in (10.2) lies inside the region Ω_h proving that $\mathcal{ST}_h$ radius for functions belonging to the class $\mathcal{ST}_n^\alpha[A, B]$ is at least ρ_2.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{ST}_n^\alpha[A, B]$ defined by (2.18) that satisfies (2.19). For $z = -\rho_2$, (2.19) gives

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + (B + \alpha(A - B))(-\rho_2)^n}{1 + B(-\rho_2)^n} = 1 - \sinh^{-1}(1) = \varphi_h(-1),$$

which proves the sharpness for ρ_2.

Case (ii): Let $B \leqslant 0$. Let the number

$$\rho_3 := \left(\frac{\sinh^{-1}(1)}{\alpha(A - B) - (\sinh^{-1}(1))B} \right)^{1/n} < 1$$

be the positive root of the equation $\eta(r) = 1 + \sinh^{-1}(1)$ or $d(\alpha, r) = 1 + \sinh^{-1}(1) - c(\alpha, r)$.

We shall show that $\mathcal{R} = \mathcal{R}_{\mathcal{ST}_h} = \rho_3$. For $0 \leqslant r \leqslant \mathcal{R} < 1$, it follows that $1 \leqslant c(\alpha, r) \leqslant c(\alpha, \mathcal{R}) < 1 + \sinh^{-1}(1)$. Since $c(\alpha, r) + d(\alpha, r)$ is an increasing function of r, for $0 \leqslant r \leqslant \rho_3$, it follows that

$$c(\alpha, r) + d(\alpha, r) \leqslant c(\alpha, \rho_3) + d(\alpha, \rho_3) = 1 + \sinh^{-1}(1)$$

and hence

$$d(\alpha, r) \leqslant 1 + \sinh^{-1}(1) - c(\alpha, r). \tag{10.3}$$

Therefore, for $0 \leqslant r \leqslant \mathcal{R} = \rho_3$, using (2.2) and (10.3), we have

$$\left| \frac{zg'(z)}{g(z)} - c(\alpha, r) \right| \leqslant 1 + \sinh^{-1}(1) - c(\alpha, r). \tag{10.4}$$

Hence, using the inclusion result in Lemma 10.1, for $1 \leqslant a < 1 + \sinh^{-1}(1)$, the disc in (10.4) lies inside the region Ω_h proving that $\mathcal{ST}_h$ radius for functions belonging to the class $\mathcal{ST}_n^\alpha[A, B]$ is at least ρ_3.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{ST}_n^\alpha[A, B]$ defined by (2.18). For $z = \rho_3$ in (2.19), we have

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + (B + \alpha(A - B))(\rho_3)^n}{1 + B(\rho_3)^n} = 1 + \sinh^{-1}(1) = \varphi_h(1),$$

proving the sharpness for ρ_3. □

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