

ON τ^* ii - OPEN SETS IN TOPOLOGICAL SPACES

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Abstract In this study, we introduce a novel category of open sets termed τ^* ii -open sets within the context of topological spaces. We elucidate the core properties of τ^* ii -open sets and explore their interrelations with various other types of open sets, including α -open sets. Moreover, we present fresh findings concerning the continuity of mappings, compactness, and connectedness pertaining to τ^* ii within topological spaces.

1 Introduction

The inception of i -open sets traces back to the seminal work of Mohammed and Askandar [1, 2, 3] in 2012. Their contributions extended to exploring separation axioms and i -continuous functions. Subsequently, Mohammed and Abdullah [4, 5] defined “ ii -open sets” as subsets of (X, τ) satisfying specific conditions. Dunham [6] introduced τ^* , a novel topology, along with the concept of the closure operator cl^* . This operator finds its definition in the \cap of all g -closed sets encompassing A . W -closed sets and αw -closed sets were further elucidated by Sundaram, Parimala [7, 8], and their colleagues in 2017. The exploration of preopen sets and precontinuous functions in topological spaces was undertaken by Velico, Levine, and Njastad [9, 10, 11] building upon their groundwork. This paper introduces τ^* ii -open sets and explores their core properties, including τ^* ii -continuous maps, τ^* ii -connectedness, and τ^* ii -compactness.

Throughout this article, the topological space (X, τ) is denoted as TS , with $O_s(X)$ representing its open sets, $C_s(X)$ its closed sets, $int(A)$ and $cl(A)$ representing the interior and closure of a set A , respectively. Additionally, τ^* ii -continuous maps are referred to as τ^* ii - C_s , τ^* ii -compactness is denoted as τ^* ii - Ct , τ^* ii -Connected as τ^* ii - Cd , Connected as Cd , cl^* as $\mathcal{C}l^*$ and mapping as Mpg .

2 On τ^* ii -Open Sets is Topological Spaces

Definition 2.1. A subset ξ of a TS is said to be τ^* ii -open set if \exists an τ^* -open set Γ satisfying (1) $\Gamma \neq \emptyset, X$ (2) $\xi \subseteq \mathcal{C}l^*(\xi \cap \Gamma)$ (3) $int(\xi) = \Gamma$.

Example 2.2. Let $X = \{l_1, l_2, l_3\}$ with $\tau = \{\emptyset, X, \{l_1\}\}$. Then $\{l_1\}$ is τ^* ii - $o_s(X)$.

Theorem 2.3. Every $o_s(X)$ is τ^* ii - $o_s(X)$.

Proof. Let ξ be a $o_s(\Gamma)$ satisfying $\xi \subseteq \mathcal{C}l^*(\xi \cap \Gamma)$ and $int(\xi) = \Gamma$. Since every $o_s(X)$ is τ^* - $o_s(X)$, ξ is a τ^* - $o_s(X)$. Choose $\Gamma = \xi$ as ξ is itself a τ^* -open set. Then $\mathcal{C}l^*(\xi \cap \Gamma) = \mathcal{C}l^*(\xi \cap \xi) = \mathcal{C}l^*(\xi)$. Since $\xi \subseteq \mathcal{C}l^*(\xi)$, $\xi \subseteq \mathcal{C}l^*(\xi \cap \Gamma)$. Also, since ξ is $o_s(\Gamma)$, $int(\xi) = \xi = \Gamma$. $\therefore \xi$ is τ^* ii - $o_s(X)$. \square

Remark 2.4. The subsequent outcome indicates that the inverse aspect of the aforementioned theorem is incorrect. Consider $X = \{l_1, l_2, l_3\}$ with $\tau = \{X, \emptyset, \{l_1\}, \{l_1, l_2\}\}$. Here $\{l_1, l_3\}$ is τ^* ii - $o_s(X)$ but not $o_s(X)$.

Theorem 2.5. Every $\alpha\text{-}o_s(X)$, is $\tau^* ii\text{-}o_s(X)$.

Proof. Since every $\alpha\text{-}o_s(X)$ is ii -open set and by the Theorem 2.3, the proof is straight forward. \square

Theorem 2.6. Every $o_s(X)$ is $\tau^* ii\text{-}o_s(X)$.

Proof. Since every semi- $o_s(X)$ is $ii\text{-}o_s(X)$ and by the Theorem 2.3, the proof is straightforward. \square

Theorem 2.7. Every $\delta\text{-}o_s(X)$ is $\tau^* ii\text{-}o_s(X)$.

Proof. Let Γ be a $\delta\text{-}o_s(X)$ then $X - \Gamma$ is δ -closed.

Let ξ be a $\delta\text{-}c_s(X)$ then $\xi = \mathfrak{Cl}_\delta(\xi)$.

Since $\xi \subseteq \mathfrak{Cl}_\delta(\xi)$.

Let $x \in (\xi \cap \text{int}(\text{cl}(\Gamma))) \neq \phi$.

$\Rightarrow x \in \xi$ and $x \in \text{int}(\text{cl}(\Gamma))$

$\Rightarrow x \in \xi$ and $x \in \text{int}(\text{cl}(\Gamma)) \subseteq \mathfrak{Cl}^*(\xi \cap \Gamma)$.

Since $\Gamma \neq \phi$, X

$\Rightarrow x \in \xi$ and $x \in \mathfrak{Cl}^*(\xi \cap \Gamma) \Rightarrow \xi \subseteq \mathfrak{Cl}^*(\xi \cap \Gamma)$.

To prove $\text{int}(\xi) = \Gamma$.

If $\text{int}(\xi) \neq \Gamma$ for all $\Gamma \in O(X)$ then $\text{cl}(\text{int}(\xi)) \neq \mathfrak{Cl}(\Gamma)$.

$\Rightarrow \xi \subseteq (\xi \cap \text{int}(\text{cl}(U))) = \phi$.

$\Rightarrow \xi \not\subseteq \mathfrak{Cl}_\delta(\xi)$.

This contradicts that ξ is a $\delta\text{-}o_s(X)$. Hence ξ is $\tau^* ii\text{-}o_s(X)$. \square

Theorem 2.8. Every $\tau^* ii\text{-}o_s(X)$ is $\Gamma g\text{-}c_s(X)$.

Proof. By the Theorem 2.3, the proof is straightforward. \square

Remark 2.9. $\tau^*\text{-}o_s(X)$ and $\tau^* ii\text{-}o_s(X)$ are independent to each other.

Consider $X = \{l_1, l_2, l_3\}$ with $\tau = \{X, \emptyset, \{l_1\}, \{l_1, l_2\}\}$. Here $\{l_1, l_3\}$ is $\tau^* ii\text{-}o_s(X)$ but not $\tau^*\text{-}o_s(X)$ and b is $\tau^*\text{-}o_s(X)$ but not $\tau^* ii\text{-}o_s(X)$.

This shows that $\tau^*\text{-}o_s(X)$ and $\tau^* ii\text{-}o_s(X)$ are independent to each other.

Theorem 2.10. Every $iw\text{-}c_s(X)$ is $\tau^* ii\text{-}o_s(X)$.

Proof. Consider an $iw\text{-}c_s(X)$ ξ in TS and U be any other $i\text{-}o_s(X)$. then $wcl(\xi) \subseteq U$ whenever $\xi \subseteq U$. Since $\Gamma \neq \phi$, X in (x, τ^*) . $\xi \subseteq wcl(\xi) \subseteq U \subseteq \mathfrak{Cl}^*(\xi \cap \Gamma) \Rightarrow \mathfrak{Cl}^*(\xi \cap \Gamma)$. Since $wcl(\xi) \subseteq U$, U is i -open then

$$\Gamma \subseteq \text{int}(\xi) \dots \dots \quad (2.1)$$

Also $\text{int}(\xi) \subseteq wcl(\xi) \subseteq \mathfrak{Cl}(\xi) \subseteq U$

$$\text{Int}(\xi) \subseteq \Gamma \dots \dots \quad (2.2)$$

From equations (2.1) & (2.2), $\text{int}(\xi) = \Gamma$.

Hence ξ is $\tau^* ii\text{-}o_s(X)$. \square

Theorem 2.11. Every $iiw\text{-}c_s(X)$ is $\tau^* ii\text{-}o_s(X)$.

Proof. Consider an $iiw\text{-}c_s(X)$ ξ in TS and U be $ii\text{-}o_s(X)$. Then $wcl(\xi) \subseteq U$ whenever $\xi \subseteq U$. Since $\Gamma \neq \phi$, x in (x, τ^*) , $\xi \subseteq wcl(\xi) \subseteq U$. \square

Theorem 2.12. If ξ is $ii\text{-}o_s(X)$ then ξ is a $\tau^* ii\text{-}o_s(X)$.

Proof. Let U be an $ii\text{-}o_s(X)$. Then \exists an $o_s(X)$, $\Gamma \in O(x)$ s.t

1) $\Gamma \neq \phi$ (2) $\xi \subseteq \mathfrak{Cl}(\xi \cap \Gamma) \subseteq \mathfrak{Cl}^*(\xi \cap \Gamma) \Rightarrow \mathfrak{Cl}^*(\xi \cap \Gamma) \Rightarrow \text{int}(\xi) = \Gamma$. Therefore ξ is τ^*

$ii-o_s(X)$.

$\Rightarrow \xi \subseteq \mathcal{C}l^*(\xi \cap \Gamma)$. Since $wcl(\xi) \subseteq U$ whenever U is ii -open set

$$\Rightarrow \Gamma \subseteq int(\xi) \dots \quad (2.3)$$

Also $int(\xi) \subseteq wcl(\xi) \subseteq \mathcal{C}l(\xi) \subseteq U$

$$\Rightarrow int(\xi) \subseteq \Gamma \quad (2.4)$$

From equations (2.3) and (2.4) $int(\xi) = \Gamma$

Hence ξ is $\tau^* ii-o_s(X)$. \square

Remark 2.13.

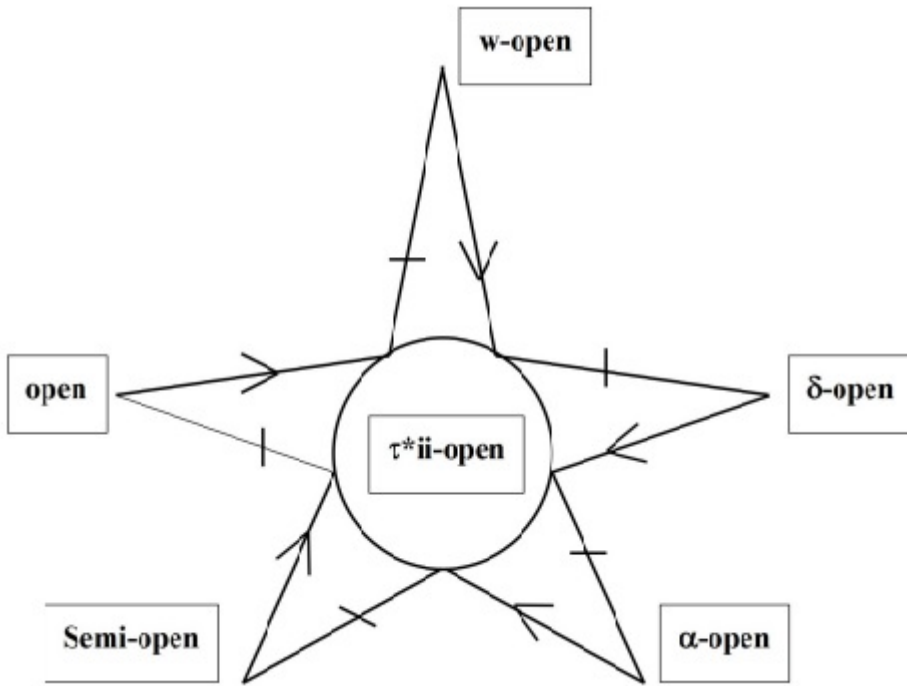


Figure 1.

Theorem 2.14. Let $\xi \subseteq X$. ξ is $ii-o_s(X)$ iff ξ is $\tau^* ii-o_s(X)$.

Proof. Let ξ be a $ii-o_s(X)$ in a TS . To prove ξ is $\tau^* ii-o_s(X)$, we need to find an $\tau^*-o_s(X)$, Γ satisfying $\xi \subseteq \mathcal{C}l^*(\xi \cap \Gamma)$, $int(\xi) = \Gamma$.

As ξ is $ii-o_s(X)$, and by the definition of $ii-o_s(X)$, \exists an $o_s(X)$, Γ s-t $\xi \subseteq \mathcal{C}l(\xi \cap \Gamma)$. Since every open set is $\tau^*-o_s(X)$, Γ is $\tau^*-o_s(X)$. Let $x \in \xi$. Since $\xi \subseteq \mathcal{C}l(\xi \cap \Gamma)$, $x \in \mathcal{C}l(\xi \cap \Gamma)$. This implies, $x \in$ the most closed set that includes $\xi \cap \Gamma$. Then, $x \in$ all closed sets containing intersection of ξ and Γ . Then $x \in \xi \cap \Gamma$ -closed sets containing $\xi \cap \Gamma$. $\Rightarrow x \in \mathcal{C}l^*(\xi \cap \Gamma)$. So $\xi \subseteq \mathcal{C}l^*(\xi \cap \Gamma)$ and also by definition $int(\xi) = \Gamma$. Hence ξ is $\tau^* ii-o_s(X)$.

Conversely, suppose ξ is $\tau^* ii-o_s(X)$. Then \exists an τ^* -open set Γ satisfying (1) $\Gamma \neq \emptyset, X$ (2) $\xi \subseteq \mathcal{C}l^*(\xi \cap \Gamma)$ (3) $int(\xi) = \Gamma$. By condition (3), it implies Γ is an open set. Since $\mathcal{C}l^*(\xi \cap \Gamma) \subseteq \mathcal{C}l(\xi \cap \Gamma)$ and by condition (2), $\xi \subseteq \mathcal{C}l(\xi \cap \Gamma)$. All 3 conditions of $ii-o_s(X)$ are satisfied. $\therefore \xi$ is $\tau^* ii-o_s(X)$. \square

Theorem 2.15. If ξ and γ are $\tau^* ii-o_s(X)$ then $\xi \cup \gamma$ is $\tau^* ii-o_s(X)$

Proof. Let $\xi, \gamma \subseteq (X, \tau)$. Let $\Gamma \neq \emptyset, X$ be a $\tau^*-o_s(X)$. Let ξ, γ are $\tau^* ii-o_s(X)$ then $\xi \subseteq \mathcal{C}l^*(\xi \cap \Gamma)$ and $\gamma \subseteq \mathcal{C}l^*(\gamma \cap \Gamma)$. Since $\mathcal{C}l^*(\xi \cap \Gamma) \cup \mathcal{C}l^*(\gamma \cap \Gamma) \subseteq \mathcal{C}l^*(\xi \cap \Gamma) \cup (\gamma \cap \Gamma) \subseteq \mathcal{C}l^*((\xi \cup \gamma) \cap \Gamma)$. $\xi \cup \gamma \subseteq \mathcal{C}l^*((\xi \cup \gamma) \cap \Gamma)$. $\Rightarrow int(\xi \cup \gamma) \subseteq \Gamma$ and $\Gamma \subseteq (\xi \cup \gamma)$. $\Rightarrow int(\xi \cup \gamma) = \Gamma$. \square

Theorem 2.16. $\xi \cap \gamma$ is τ^* ii - $o_s(X)$ if ξ and γ are τ^* ii - $o_s(X)$.

Proof. Let $\xi, \gamma \subseteq (X, \tau)$. Let $\Gamma \neq \emptyset$, X be a τ^* - $o_s(X)$. Let ξ, γ are τ^* ii - $o_s(X)$ then $\xi \subseteq \mathcal{Cl}^*(\xi \cap \Gamma)$ and $\gamma \subseteq \mathcal{Cl}^*(\xi \cap \Gamma)$. $\xi \cap \gamma \subseteq \mathcal{Cl}^*(\xi \cap \gamma \cap \Gamma) \subseteq \mathcal{Cl}^*(\xi \cap \gamma \cap \Gamma)$. $\xi \cap \gamma \subseteq \mathcal{Cl}^*(\xi \cap \gamma \cap \Gamma) \subseteq \mathcal{Cl}^*(\xi \cap \gamma \cap \Gamma)$. $\Rightarrow \xi \cap \gamma \subseteq \mathcal{Cl}^*(\xi \cap \gamma \cap \Gamma)$. \square

Corollary 2.17. If $\xi_1, \xi_2, \xi_3, \dots, \xi_n$ are τ^* ii - $o_s(X)$ then $\xi \subseteq \mathcal{Cl}^*(\xi \cap \Gamma)$,

- (i) $\xi_1 \cap \xi_2 \cap \xi_3 \cap \xi_4 \cap \dots \cap \xi_n$ is a τ^* ii - $o_s(X)$
- (ii) $\xi_1 \cap \xi_2 \cap \xi_3 \cap \xi_4 \cap \dots \cap \xi_n$ is a τ^* ii - $o_s(X)$.

3 On τ^* ii -compact

Definition 3.1. A subset ξ of a TS is considered τ^* ii - Ct if every open cover of ξ using τ^* ii - $o_s(X)$ contains a finite subcover.

Definition 3.2. A Mpg $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ is said to be τ^* ii - Cs iff the inverse image of every τ^* ii - $o_s(X)$, Γ in (Y, τ_2) is τ^* ii - $o_s(X)$ in (X, τ_1) .

Theorem 3.3. Let ξ be a τ^* ii -closed subset of τ^* ii - Ct TS X . Then ξ is τ^* ii - Ct .

Proof. Let $\{V_i\}$ be any collection of τ^* ii -open sets in X covering ξ . Now, τ^* ii -open set $X \setminus \xi$ together with $\{V_i\}$ forms an open cover of X by τ^* ii -open sets. This collection of open sets covering by X is τ^* ii -open sets possesses a finite sub cover $X \setminus \xi \cup \{V_i\}_{i=1}^n$, since X is Ct . Then ξ is covered by $\{V_i\}_{i=1}^n$. $\therefore \xi$ is τ^* ii - Ct . \square

Theorem 3.4. Finite union of τ^* ii - Ct sets is τ^* ii - Ct .

Proof. Let $\xi_1, \xi_2, \dots, \xi_n$ be τ^* ii - Ct sets in a TSX and $\{U_i\}$ be any cover by τ^* ii -open sets of $\xi_1 \cup \xi_2 \cup \dots \cup \xi_n$. Then since $\cup U_i \supseteq \xi_l$ and ξ_l is τ^* ii - Ct , for each $l = 1, 2, \dots, n$, there exists finite set of indices i_1, i_2, \dots, i_m such that $U_{i_k}_{k=1}^m \supseteq \xi_l$, for each $l = 1, 2, \dots, n$. Clearly, $U_{i_k}_{k=1}^m \supseteq \cup_{l=1}^n \xi_l$. Therefore, $\xi_1 \cup \xi_2 \cup \dots \cup \xi_n$ is τ^* ii - Ct . \square

Theorem 3.5. For any τ^* ii - Cs Mpg, if ξ is τ^* ii - Ct then $f(\xi)$ is τ^* ii - Ct .

Proof. A Mpg $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ is τ^* ii -continuous. Let Γ_i be any collection of τ^* ii -open sets which cover $f(\xi)$. Then $f^{-1}(\Gamma_i)$ is the collection of τ^* ii -open sets which cover ξ , since f is τ^* ii -continuous. Since ξ is τ^* ii -compact, $f^{-1}(\Gamma_i)_{i=1}^n$ covers ξ (i.e., $\cup_{i=1}^n f^{-1}(\Gamma_i) \supseteq \xi$). Then we have, $f^{-1}(\cup_{i=1}^n \Gamma_i) \supseteq \xi$ and this implies $\cup_{i=1}^n \Gamma_i \supseteq f(\xi)$, a finite subcover of $f(\xi)$. Hence $f(\xi)$ is τ^* ii -compact. \square

4 On τ^* ii -connected Sets

Definition 4.1. A subset ξ of TS X is called τ^* ii -connected (τ^* ii - Cd) if there are no τ^* ii - $o_s(X)$ U and V ($\neq \emptyset$) s-t $\xi = U \cup V, U \cap V \neq \emptyset$.

Theorem 4.2. A TS X is τ^* ii - Cd iff \emptyset and X are the only τ^* ii -clopen sets.

Proof. Suppose ξ is τ^* ii -clopen set in X , but ξ is neither empty nor equal to X . Then ξ and $X - \xi$ are nonempty, τ^* ii -open and disjoint with $\xi \cup (X - \xi) = X$, so X is not τ^* ii - Cd . Conversely, suppose ξ, γ are nonempty, τ^* ii -open and disjoint with $\xi \cup \gamma = X$. This means that ξ is neither empty nor equal to X . On the other hand, $\gamma = X - \xi$ is τ^* ii - $o_s(X)$, so ξ is τ^* ii -clopen in X . \square

Theorem 4.3. Every τ^* ii - Cd is Cd subset of X .

Proof. Assume that ξ is not a Cd subset of X . Then $\exists U, V \neq \emptyset$ disjoint open subset of X s-t $\xi = U \cup V$. By Theorem 4.2., U and V are τ^* ii -open sets, whose union is X . Then ξ is not a τ^* ii - Cd set. \square

Theorem 4.4. A τ^* ii - Cs image of any τ^* ii - Cd subset of X is τ^* ii - Cd .

Proof. Assume that $\delta : (X, \tau_1) \rightarrow (Y, \tau_2)$ is τ^* ii - Cs mapping and ξ is τ^* ii - Cd subset of X . Let us assume that $\delta(\xi)$ is not τ^* ii - Cd . Then \exists disjoint τ^* ii -open sets U, V ($\neq \emptyset$) $\ni \xi = U \cup V$. By the definition of τ^* ii - Cs , $\delta^{-1}(U)$ and $\delta^{-1}(V)$ are non-empty disjoint τ^* ii -open sets of $\xi \subseteq \delta^{-1}(U \cup V) = \delta^{-1}(U) \cup \delta^{-1}(V)$. Then ξ is not τ^* ii - Cd , a contradiction. \square

5 Conclusion Remarks

By utilizing the *ii*-open set, a new open set called τ^* *ii*-open sets is introduced through this article. Study was done on the relationship with τ^* *ii*-open sets and other sets. It facilitates the expansion of our study into more topological spaces, such as soft topology, fuzzy topological spaces, Nano topological spaces, and bi-topological spaces. As a result, the findings can be used in any kind of future research.

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