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SOME NEW SUBCLASSES OF BI-UNIVALENT FUNCTIONS CONNECTED WITH POLYLOGARITHM FUNCTION

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Abstract *In this work, the new subclass of bi-univalent functions related to differential operator of polylogarithm function has been examined. The convolution of two well-known differential operators define the new differential operator* $\mathcal{A}^{n}_{\lambda,\gamma,\mu}$*, using this operator a new bi-univalent subclass has been introduced and estimate the initial coefficients* $|a_2|$ *and* $|a_3|$*. Furthermore, by specializing the parameters in our primary findings, we show that various subclasses have been obtained and investigated the properties of the subclasses. The result of this study will fortify the field's theoretical foundations and open up fresh avenues for mathematical inquiry and application.*

1 Introduction

For the normalized analytic functions $\mathbb{A}$ *represent the class of*

$$f(\zeta) = \zeta + \sum_{k=2}^{\infty} a_k \zeta^k,$$

in the unit disc $\mathbb{U} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ *, which is analytical.*

Let $\mathcal{L}$ *represent the subclass of all functions in* $\mathbb{A}$*, it also satisfies* $f'(0) = 1$ *in* $\mathbb{U}$ *and the following constraints* $f(0) = 0$ *Let* $(f * g)(\zeta)$ *be the convolution between subsequence functions* $f(\zeta)$, $g(\zeta)$ *is written by*

$$(f * g)(\zeta) = \zeta + \sum_{k=2}^{\infty} a_k b_k \zeta^k,$$

where

$$g(\zeta) = \zeta + \sum_{k=2}^{\infty} b_k \zeta^k.$$

The polylogarithms function of $\gamma \in \mathbb{N}$, *with* $\gamma \geq 2$, *defined as*

$$Li_\gamma(\zeta) = \phi_\gamma(\zeta) = \sum_{k=1}^{\infty} \frac{\zeta^k}{(k)^\gamma}.$$

For $\lambda \in \mathbb{N}$, *Re*$(\lambda) > 1$ *and Re*$(c) > -1$, *the* λ^{th} *order polylogarithms function defined by*

$$\phi_\lambda(c;\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^k}{(k+c)^\lambda}.$$

For $f(\zeta) \in \mathcal{Q}$, Al-Shaqsi [2] introduced the following operator

$$\psi_\lambda(c;\zeta) = (1+c)^\lambda \phi_\lambda(c,\zeta) * f(\zeta) = \zeta + \sum_{k=2}^{\infty} \left(\frac{1+c}{k+c}\right)^\lambda a_k \zeta^k,$$

where $\lambda \in \mathbb{N}$, $Re(\lambda) > 1$ and $Re(c) > 0$.
For $f(\zeta) \in \mathcal{A}$, we define the differential operator

$$\mathcal{A}^n_{\lambda,\gamma,\mu} = \zeta + \sum_{k=2}^{\infty} [1 + (k\gamma\mu + \gamma - \mu)(k-1)]^n \left(\frac{1+c}{k+c}\right)^\lambda a_k \zeta^k.$$

Hence,

$$\begin{aligned}
\mathcal{A}^0_{\lambda,\gamma,\mu} &= \zeta + \sum_{k=2}^{\infty} \left(\frac{1+c}{k+c}\right)^\lambda a_k \zeta^k, \\
\mathcal{A}^1_{\lambda,\gamma,\mu} &= \zeta + \sum_{k=2}^{\infty} [1 + (k\gamma\mu + \gamma - \mu)(k-1)] \left(\frac{1+c}{k+c}\right)^\lambda a_k \zeta^k, \\
\mathcal{A}^2_{\lambda,\gamma,\mu} &= \mathcal{A}_{\lambda,\gamma,\mu}\left(\mathcal{A}_{\lambda,\gamma,\mu}\right).
\end{aligned}$$

Similarly,

$$\mathcal{A}^n_{\lambda,\gamma,\mu} = \mathcal{A}_{\lambda,\gamma,\mu}\left(\mathcal{A}^{n-1}_{\lambda,\gamma,\mu}\right) = \zeta + \sum_{k=2}^{\infty} [1 + (k\gamma\mu + \gamma - \mu)(k-1)]^n \left(\frac{1+c}{k+c}\right)^\lambda a_k \zeta^k, \quad (1.1)$$

which is convolution of the well known operators of Raducanu et al. [15] and Al-Shaqsi [2].

Remark 1.1. (i) For $\mu = 0$, and $c = 0$ in (1.1), we get the multiplier transformation $I^n f(\zeta)$ introduced by Flett [4].

(ii) For $\mu = 0, \gamma = 1, c = 0$ and $\lambda = -n, n \in \mathcal{N}_0$ in (1.1), we get the differential oprator $\mathcal{D}^n f(\zeta)$ introduced by Salagean [16].

(iii) For $\mu = 0$, $\gamma = 1$, $c = 1$ and $\lambda = n$ in (1.1), we get the differential oprator $\mathcal{I}^n f(\zeta)$ introduced by Uralegaddi et al. [18].

(iv) For $\mu = 0, \gamma = 1, c = 1, \lambda = \delta$ in (1.1), we get the mutiplier transformation oprator $\mathcal{I}^\delta f(\zeta)$ was introduced by Jung et al. [7].

(v) For $\mu = 0, \gamma = 1, c = a - 1$ $(a > 0)$, $\lambda = \delta$ in (1.1), we get the integral operator $\mathcal{I}^\delta_{a-1} f(\zeta)$ was introduced and studied by Komatu et al. [10].

In recent years, the researchers are fascinated by the differential and integral operators like Frasin et al. [5], Bansal et al. [3], Santosh Joshi et al. [8], Lashin [11] , Agnes et al. [13] Srivastava et al. [17], Kassim A. Jassim et al. [9], Al-Shbeil et al. [1], Hussen et al. [6], Murugusundaramoorthy et al. [12] and Yousef et al. [19] investigated a number of bi-univalent function subclasses and came up with bounds for the fundamental coefficients $|a_2|$ as well $|a_3|$.

By these inspiration, we present few bi-univalent functions new subclasses and compute the underlying coefficients $|a_2|$ and $|a_3|$.

In order to determine our principal results, we really want to review the accompanying lemma.

Lemma 1.2. *[14] If $p \in P$, then $|p_n| \leq 2$ for every n, where P is the symbol for family containing all functions. which is analytic in $\mathbb{U}$ for which*

$$Re\,(p(\zeta)) > 0, p(\zeta) = 1 + p_1\zeta + p_2\zeta^2 + \cdots$$

for $\zeta \in \mathbb{U}$.

2 Coefficient Estimation related to the operator $\mathcal{A}^n_{\lambda,\gamma,\mu}$

2.1 Coefficient Estimation for the class $\mathcal{L}^{\sigma}_{\lambda,\gamma,\mu}(\phi)$

It is expected that the sequel will

$$\varphi(\zeta) = 1 + B_1\zeta + B_2\zeta^2 + B_3\zeta^3 + \cdots, \quad B_1 > 0.$$

Let assume $u(\zeta)$ as well as $v(\zeta)$ are analytical in the unit disk $\mathbb{U}$ with $u_1(0) = v_1(0) = 0$, $|u_1(\zeta)| < 1$, $|v_1(\eta)| < 1$, and

$$\psi(\zeta) = 1 + p_1\zeta + p_2\zeta^2 + p_3\zeta^3 + \cdots, \quad (p_1 > 0),$$
$$\psi(\eta) = 1 + q_1\eta + q_2\eta^2 + q_3\eta^3 + \cdots, \quad (q_1 > 0).$$

Since $\psi(0) = 1, \psi'(0) > 0$

$$u_1(\zeta) = \frac{\psi(\zeta) - 1}{\psi(\zeta) + 1} = \frac{1}{2}p_1\zeta + \frac{1}{2}(p_2 - p_1^2)\zeta^2 + \cdots,$$

$$v_1(\eta) = \frac{\psi(\eta) - 1}{\psi(\eta) + 1} = \frac{1}{2}q_1\eta + \frac{1}{2}(q_2 - q_1^2)\eta^2 + \cdots.$$

It is widely acknowledged that

$$|p_1| \leq 1, \quad |p_2| \leq 1 - |p_1|^2, \quad |q_1| \leq 1, \quad |q_2| \leq 1 - |q_1|^2.$$

A brief computation yields the following:

$$\phi(\zeta) = \varphi(u_1(\zeta)) = 1 + \frac{1}{2}B_1p_1\zeta + \left(\frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2p_1^2\right)\zeta^2 + \cdots, \tag{2.1}$$

and

$$\phi(\eta) = \varphi(v_1(\eta)) = 1 + \frac{1}{2}B_1q_1\eta + \left(\frac{1}{2}B_1(q_2 - \frac{q_1^2}{2}) + \frac{1}{4}B_2q_1^2\right)\eta^2 + \cdots. \tag{2.2}$$

Definition 2.1. If a function $f \in \mathcal{L}^{\Sigma}_{\lambda,\gamma,\mu}(\phi)$, then the subsequent subordinations are hold:

$$1 + \frac{1}{b}\left(\frac{\zeta(\mathcal{A}^n_{\lambda,\gamma,\mu}f(\zeta))'}{\mathcal{A}^n_{\lambda,\gamma,\mu}f(\zeta)} - 1\right) \prec \phi(\zeta) \tag{2.3}$$

and

$$1 + \frac{1}{b}\left(\frac{\eta(\mathcal{A}^n_{\lambda,\gamma,\mu}g(\eta))'}{\mathcal{A}^n_{\lambda,\gamma,\mu}g(\eta)} - 1\right) \prec \phi(\eta). \tag{2.4}$$

Theorem 2.2. *If $f(\zeta) \in \mathcal{L}^{\Sigma}_{\lambda,\gamma,\mu}(\phi)$, then*

$$|a_2| \leq \sqrt{\frac{2\,|b|\,B_1\,(p_2 + q_2) + |b|\,(B_2 - B_1)\,(p_1^2 + q_1^2)}{8\,[2X_1 - X_2]}}$$

and

$$|a_3| \leq \frac{2\,|b|\,B_1\,(p_2 + q_2) + |b|\,(B_2 - B_1)\,(p_1^2 + q_1^2)}{8\,[2X_1 - X_2]}$$
$$+ \frac{2\,|b|\,B_1\,(p_2 - q_2) + |b|\,(B_2 - B_1)\,(p_1^2 - q_1^2)}{16X_1},$$

where
$X_1 = (1 + 2\,(3\gamma\mu + \gamma - \mu))^n\,\left(\frac{1+c}{3+c}\right)^{\lambda}$,
$X_2 = (1 + 2\gamma\mu + \gamma - \mu)^{2n}\,\left(\frac{1+c}{2+c}\right)^{2\lambda}$,
$X_3 = (1 + 2\gamma\mu + \gamma - \mu)^{n}\,\left(\frac{1+c}{2+c}\right)^{2\lambda}$.

Proof. Let $f \in \mathcal{L}^{\Sigma}_{\lambda,\gamma,\mu}(\phi)$. There are also analytical functions $u, v : \mathbb{U} \to \mathbb{U}$ given by (2.3) and (2.4) such that

$$1 + \frac{1}{b}\left(\frac{\zeta(\mathcal{A}^n_{\lambda,\gamma,\mu} f(\zeta))'}{\mathcal{A}^n_{\lambda,\gamma,\mu} f(\zeta)} - 1\right) = \phi(\zeta)$$

and

$$1 + \frac{1}{b}\left(\frac{\eta(\mathcal{A}^n_{\lambda,\gamma,\mu} g(\eta))'}{\mathcal{A}^n_{\lambda,\gamma,\mu} g(\eta)} - 1\right) = \phi(\eta).$$

Since

$$1 + \frac{1}{b}\left(\frac{\zeta(\mathcal{A}^n_{\lambda,\gamma,\mu} f(\zeta))'}{\mathcal{A}^n_{\lambda,\gamma,\mu} f(\zeta)} - 1\right) = 1 + \frac{1}{b}\left[X_3 a_2\right]\zeta + \frac{1}{b}\left[X_1 a_3 - X_2 a_2^2\right]\zeta^2 + \cdots \tag{2.5}$$

and

$$1 + \frac{1}{b}\left(\frac{\eta(\mathcal{A}^n_{\lambda,\gamma,\mu} g(\eta))'}{\mathcal{A}^n_{\lambda,\gamma,\mu} g(\eta)} - 1\right) = 1 - \frac{1}{b}\left[X_3 a_2\right]\eta + \frac{1}{b}\left[-X_1 a_3\right]\eta^2 + \frac{1}{b}\left[(2X_1 - X_2)\, a_2^2\right]\eta^2 + \cdots \tag{2.6}$$

It follows from (2.1), (2.2), (2.5) and (2.6), equating coefficient of ζ:

$$\frac{1}{b}\left[X_3\right] a_2 = \frac{1}{2} B_1 p_1. \tag{2.7}$$

Equating coefficient of ζ^2:

$$\frac{1}{b}\left[2X_1 a_3 - X_2 a_2^2\right] = \frac{1}{2} B_1 (p_2 - \frac{p_1^2}{2}) + \frac{1}{4} B_2 p_1^2. \tag{2.8}$$

Equating coefficient of η:

$$-\frac{1}{b}\left[X_3\right] a_2 = \frac{1}{2} B_1 q_1. \tag{2.9}$$

Equating coefficient of η^2:

$$\frac{1}{b}\left[-2X_1 a_3 + 4X_1 - (1+\delta)^{2n} a_2^2\right] = \frac{1}{2} B_1 (q_2 - \frac{q_1^2}{2}) + \frac{1}{4} B_2 q_1^2. \tag{2.10}$$

Adding (2.7) and (2.9), we get

$$p_1 = -q_1.$$

Squaring and adding the equations (2.7) and (2.9), we get

$$a_2^2 = \frac{b^2 B_1^2 \left(p_1^2 + q_1^2\right)}{8X_2}. \tag{2.11}$$

Substitute (2.8) in (2.10), we arrive

$$a_2^2 = \frac{2bB_1\left(p_2 + q_2\right) + b\left(B_2 - B_1\right)\left(p_1^2 + q_1^2\right)}{8\left[2X_1 - (1+\delta)^{2n}\right]}. \tag{2.12}$$

Subtracting (2.8) and (2.10), we get

$$a_3 = a_2^2 + \frac{2bB_1\left(p_2 - q_2\right) + b\left(B_2 - B_1\right)\left(p_1^2 - q_1^2\right)}{16X_1}. \tag{2.13}$$

Substitute (2.11) in (2.13), we obtain

$$a_3 = \frac{b^2 B_1^2 \left(p_1^2 + q_1^2\right)}{8X_2} + \frac{2bB_1\left(p_2 - q_2\right) + b\left(B_2 - B_1\right)\left(p_1^2 - q_1^2\right)}{16X_1}.$$

Substitute (2.12) in (2.13), we arrive

$$a_3 = \frac{2bB_1\left(p_2+q_2\right)+b\left(B_2-B_1\right)\left(p_1^2+q_1^2\right)}{8\left[2X_1-X_2\right]} + \frac{2bB_1\left(p_2-q_2\right)+b\left(B_2-B_1\right)\left(p_1^2-q_1^2\right)}{16X_1}.$$

Hence,

$$|a_2| \le \sqrt{\frac{2\left|b\right|B_1\left(p_2+q_2\right)+\left|b\right|\left(B_2-B_1\right)\left(p_1^2+q_1^2\right)}{8\left[2X_1-X_2\right]}}$$

and

$$|a_3| \le \frac{2\left|b\right|B_1\left(p_2+q_2\right)+\left|b\right|\left(B_2-B_1\right)\left(p_1^2+q_1^2\right)}{8\left[2X_1-X_2\right]} + \frac{2\left|b\right|B_1\left(p_2-q_2\right)+\left|b\right|\left(B_2-B_1\right)\left(p_1^2-q_1^2\right)}{16X_1}.$$

□

2.2 Coefficient Estimation for the class $\mathcal{L}^{\Sigma}_{\lambda,\gamma,\mu}(\varphi)$

If we set

$$\varphi(\zeta) = 1 + B_1\zeta + B_2\zeta^2 + B_3\zeta^3 + \cdots, \quad (B_1 > 0).$$

Since $\varphi(0) = 1, \varphi'(0) > 0.$

Let assume $u(\zeta)$ *as well as* $v(\zeta)$ *are analytical about the unit disk* $\mathbb{U}$ *with* $u(0) = v(0) = 0$, $|u(\zeta)| < 1$, $|v(\eta)| < 1$, *and*

$$u(\zeta) = p_1\zeta + \sum_{k=2}^{\infty} p_k\zeta^k$$

and

$$v(\eta) = q_1\eta + \sum_{k=2}^{\infty} q_k\eta^k.$$

It is widely acknowledged that

$$|p_1| \le 1, \quad |p_2| \le 1 - |p_1|^2, \quad |q_1| \le 1, \quad |q_2| \le 1 - |q_1|^2.$$

A brief computation yields the following:

$$\varphi(u(\zeta)) = 1 + B_1p_1\zeta + (B_1p_2 + B_2p_1^2)\zeta^2 + \cdots, |\zeta| < 1$$

and

$$\varphi(v(\eta)) = 1 + B_1q_1\eta + (B_1q_2 + B_2q_1^2)\eta^2 + \cdots, |\eta| < 1.$$

Definition 2.3. If $f \in \mathcal{L}^{\Sigma}_{\lambda,\gamma,\mu}(\varphi)$, then the subsequent subordinations are hold:

$$1 + \frac{1}{b}\left(\frac{\zeta(\mathcal{A}^n_{\lambda,\gamma,\mu}f(\zeta))'}{\mathcal{A}^n_{\lambda,\gamma,\mu}f(\zeta)} - 1\right) \prec \varphi(\zeta)$$

and

$$1 + \frac{1}{b}\left(\frac{\eta(\mathcal{A}^n_{\lambda,\gamma,\mu}g(\eta))'}{\mathcal{A}^n_{\lambda,\gamma,\mu}g(\eta)} - 1\right) \prec \varphi(\eta).$$

Theorem 2.4. *If* $f(\zeta) \in \mathcal{L}^{\Sigma}_{\lambda,\gamma,\mu}(\varphi)$, *then*

$$|a_2| \le \sqrt{\frac{|b|\left[B_1\left(p_2+q_2\right)+B_2\left(p_1^2+q_1^2\right)\right]}{2\left[2X_1-X_2\right]}}$$

and

$$|a_3| \le \frac{|b|\,B_1\left(p_2+q_2\right)+|b|\,B_2\left(p_1^2+q_1^2\right)}{2\left[2X_1-X_2\right]} + \frac{|b|\,B_1\left(p_2-q_2\right)+|b|\,B_2\left(p_1^2-q_1^2\right)}{4X_1}.$$

2.3 Coefficient Estimation for the class $\mathcal{M}^{\Sigma}_{\lambda,\gamma,\mu}(\psi)$

If we set

$$\psi(\zeta) = 1 + p_1\zeta + p_2\zeta^2 + p_3\zeta^3 + \cdots, \quad (p_1 > 0)$$

and

$$\psi(\eta) = 1 + q_1\eta + q_2\eta^2 + q_3\eta^3 + \cdots, \quad (q_1 > 0).$$

Definition 2.5. If $f \in \mathcal{M}^{\Sigma}_{\lambda,\gamma,\mu}(\psi)$, then the subsequent subordinations are hold:

$$1 + \frac{1}{b}\left(\frac{\zeta(\mathcal{A}^n_{\lambda,\gamma,\mu}f(\zeta))'}{\mathcal{A}^n_{\lambda,\gamma,\mu}f(\zeta)} - 1\right) \prec \psi(\zeta)$$

and

$$1 + \frac{1}{b}\left(\frac{\eta(\mathcal{A}^n_{\lambda,\gamma,\mu}g(\eta))'}{\mathcal{A}^n_{\lambda,\gamma,\mu}g(\eta)} - 1\right) \prec \psi(\eta).$$

Theorem 2.6. *If $f(\zeta) \in \mathcal{M}^{\Sigma}_{\lambda,\gamma,\mu}(\psi)$, then*

$$|a_2| \le \sqrt{\frac{|b|\,(p_2 + q_2)}{2\,[2X_1 - X_2]}}$$

and

$$|a_3| \le \frac{|b|\,(p_2 + q_2)}{2\,[2X_1 - X_2]} + \frac{|b|\,(p_2 - q_2)}{4X_1}.$$

2.4 Coefficient Estimation for the class $\mathcal{M}^{\Sigma}_{\lambda,\gamma,\mu,\alpha}(\phi)$

If we set

$$\psi(\zeta) = 1 + p_1\zeta + p_2\zeta^2 + p_3\zeta^3 + \cdots, \quad (p_1 > 0)$$

and

$$\begin{aligned}
\psi(\eta) &= 1 + q_1\eta + q_2\eta^2 + q_3\eta^3 + \cdots, \quad (q_1 > 0)\\
\phi(\zeta) &= (\psi(\zeta))^{\alpha}\\
&= 1 + \alpha p_1\zeta + \left(\alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2\right)\zeta^2 + \cdots\\
\phi(\eta) &= (\psi(\eta))^{\alpha}\\
&= 1 + \alpha q_1\eta + \left(\alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2\right)w^2 + \cdots.
\end{aligned}$$

Definition 2.7. If $f \in \mathcal{M}^{\Sigma}_{\lambda,\gamma,\mu,\alpha}(\phi)$, then the subsequent subordinations are hold:

$$1 + \frac{1}{b}\left(\frac{\zeta(\mathcal{A}^n_{\lambda,\gamma,\mu}f(\zeta))'}{\mathcal{A}^n_{\lambda,\gamma,\mu}f(\zeta)} - 1\right) \prec \phi(\zeta)$$

and

$$1 + \frac{1}{b}\left(\frac{\eta(\mathcal{A}^n_{\lambda,\gamma,\mu}g(\eta))'}{\mathcal{A}^n_{\lambda,\gamma,\mu}g(\eta)} - 1\right) \prec \phi(\eta).$$

Theorem 2.8. *If $f(\zeta) \in \mathcal{M}^{\Sigma}_{\lambda,\gamma,\mu,\alpha}(\phi)$, then*

$$|a_2| \le \sqrt{\frac{2\,|b|\,\alpha\,(p_2 + q_2) + |b|\,\alpha\,(\alpha-1)\left(p_1^2 + q_1^2\right)}{4\,[2X_1 - X_2]}}$$

and

$$|a_3| \le \frac{2\,|b|\,\alpha\,(p_2 + q_2) + |b|\,\alpha\,(\alpha-1)\left(p_1^2 + q_1^2\right)}{4\,[2X_1 - X_2]} + \frac{2\,|b|\,\alpha\,(p_2 - q_2) + |b|\,\alpha\,(\alpha-1)\left(p_1^2 - q_1^2\right)}{8X_1}.$$

2.5 Coefficient Estimation for the class $\mathcal{L}^{\Sigma}_{\lambda,\gamma,\mu,\sigma}(\phi)$

If we set

$$\phi(\zeta) = \sigma + (1-\sigma)\psi(\zeta) = 1 + (1-\sigma)p_1\zeta + (1-\sigma)p_2\zeta^2 + \cdots$$

and

$$\phi(\eta) = \sigma + (1-\sigma)\psi(\eta) = 1 + (1-\sigma)q_1\eta + (1-\sigma)q_2\eta^2 + \cdots; \ \ 0 \le \sigma < 1.$$

Definition 2.9. If $f \in \mathcal{L}^{\Sigma}_{\lambda,\gamma,\mu,\sigma}(\phi)$, then the subsequent subordinations are hold:

$$1 + \frac{1}{b}\left(\frac{\zeta(\mathcal{A}^n_{\lambda,\gamma,\mu}f(\zeta))'}{\mathcal{A}^n_{\lambda,\gamma,\mu}f(\zeta)} - 1\right) \prec \phi(\zeta)$$

and

$$1 + \frac{1}{b}\left(\frac{\eta(\mathcal{A}^n_{\lambda,\gamma,\mu}g(\eta))'}{\mathcal{A}^n_{\lambda,\gamma,\mu}g(\eta)} - 1\right) \prec \phi(\eta).$$

Theorem 2.10. *If $f(\zeta) \in \mathcal{L}^{\Sigma}_{\lambda,\gamma,\mu,\sigma}(\phi)$, then*

$$|a_2| \le \sqrt{\frac{|b|\,(1-\sigma)\,(p_2+q_2)}{2\,[2X_1 - X_2]}}$$

and

$$|a_3| \le \frac{|b|\,(1-\sigma)\,(p_2+q_2)}{2\,[2X_1 - X_2]} + \frac{|b|\,(1-\sigma)\,(p_2-q_2)}{4X_1}.$$

2.6 Coefficient Estimation for the class $\mathcal{M}^{\Sigma}_{\lambda,\gamma,\mu,\sigma}(\phi)$

If we set

$$\phi(\zeta) = \frac{1+(1-2\sigma)\zeta}{1-\zeta} = 1 + 2(1-\sigma)\zeta + 2(1-\sigma)^2\zeta^2 + \cdots$$

and

$$\phi(\eta) = \frac{1+(1-2\sigma)\eta}{1-\eta} = 1 + 2(1-\sigma)\eta + 2(1-\sigma)^2\eta^2 + \cdots; \ \ 0 \le \sigma < 1.$$

Definition 2.11. If a function $f \in \mathcal{M}^{\Sigma}_{\lambda,\gamma,\mu,\sigma}(\phi)$, then the subsequent subordinations are hold:

$$1 + \frac{1}{b}\left(\frac{\zeta(\mathcal{A}^n_{\lambda,\gamma,\mu}f(\zeta))'}{\mathcal{A}^n_{\lambda,\gamma,\mu}f(\zeta)} - 1\right) \prec \phi(\zeta)$$

and

$$1 + \frac{1}{b}\left(\frac{\eta(\mathcal{A}^n_{\lambda,\gamma,\mu}g(\eta))'}{\mathcal{A}^n_{\lambda,\gamma,\mu}g(\eta)} - 1\right) \prec \phi(\eta).$$

Theorem 2.12. *If $f(\zeta) \in \mathcal{M}^{\Sigma}_{\gamma,\lambda,\mu,\sigma}(\phi)$, then*

$$|a_2| \le \sqrt{\frac{2\,|b|\,(1-\sigma)^2}{2X_1 - X_2}}$$

and

$$|a_3| \le \frac{2\,|b|\,(1-\sigma)^2}{2X_1 - X_2}.$$

2.7 Coefficient Estimation for the class $\mathcal{L}^{\Sigma}_{\lambda,\gamma,\mu,\alpha}(\phi)$

If we set

$$\phi(\zeta) = \left(\frac{1+\zeta}{1-\zeta}\right)^{\alpha} = 1 + 2\alpha\zeta + 2\alpha^2\zeta^2 + \cdots ; \quad 0 < \alpha \leq 1.$$

and

$$\phi(\eta) = \left(\frac{1+\eta}{1-\eta}\right)^{\alpha} = 1 + 2\alpha\eta + 2\alpha^2\eta^2 + \cdots .$$

Definition 2.13. If $f \in \mathcal{L}^{\Sigma}_{\lambda,\gamma,\mu,\alpha}(\phi)$, then the subsequent subordination's are hold:

$$1 + \frac{1}{b}\left(\frac{\zeta(\mathcal{A}^n_{\lambda,\gamma,\mu}f(\zeta))'}{\mathcal{A}^n_{\lambda,\gamma,\mu}f(\zeta)} - 1\right) \prec \phi(\zeta)$$

and

$$1 + \frac{1}{b}\left(\frac{\eta(\mathcal{A}^n_{\lambda,\gamma,\mu}g(\eta))'}{\mathcal{A}^n_{\lambda,\gamma,\mu}g(\eta)} - 1\right) \prec \phi(\eta).$$

Theorem 2.14. *If $f(\zeta) \in \mathcal{L}^{\Sigma}_{\lambda,\gamma,\mu,\alpha}(\phi)$, then*

$$|a_2| \leq \sqrt{\frac{2\,|b|\,\alpha^2}{2X_1 - X_2}}$$

and

$$|a_3| \leq \frac{2\,|b|\,\alpha^2}{2X_1 - X_2}.$$

3 Conclusion remarks

In this current paper two initial coefficients $|a_2|$ and $|a_3|$ are estimated for a new sub-class. Furthermore, this work motivated the researchers to extend the results of this article into some new subclass of meromorphic functions, and q-calculus of bi-univalent functions. Also the researchers to extend into the famous inequalities like Fekete szego inequality and second and third Hankel determinants. Naturally it has a wide range of applications in science, engineering, and other related areas such as signal processing and control theory.

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