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MOBIUS CORDIAL LABELING OF SOME SNAKE GRAPHS

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Abstract The study of labelling vertices or edges of graphs with real numbers, subject to certain constraints, has numerous real-world applications across various fields. The concept of Mobius cordial labelling based on Mobius function, which was introduced by us, have been investigated to certain family of graphs. In Mobius cordial labelling, the integers -1, 0 and 1 are assigned to the edges of a graph $G(V, E)$ through an evaluating function defined on the vertices V and the mobius function defined on the edges E, ensuring the cordiality conditions are met. In this paper we are going to investigate this labelling technique for some snake related graphs and few other graphs.

1 Introduction

Graph labeling involves assigning integers to the vertices or edges of a graph based on specific conditions [6]. The fundamental concept of graph labeling is a key aspect of graph theory. The basic notion of graph labeling is found in [3]. For comprehensive information on graph labeling, references to the literature on the subject are available [1]. Harary's text book is recommended for understanding the notations, concepts and terminology in graph theory. The Mobius function, a significant multiplicative function in number theory and combinatorics [5], plays a crucial role in graph coloring problems and constructing flows in networks with specified properties [4]. Using Mobius function concept [7], we have introduced a new graph labeling known as Mobius cordial labeling of graphs and studied its existence for some standard classes of graphs and cycle related graphs [8, 9].

Definition 1.1. 'The Mobius function is a number theoretic function defined by

$$\mu(n) \equiv \begin{cases} 0 & \text{if } n \text{ has one or more repeated prime factors} \\ 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes} \end{cases}$$

so $\mu(n) \neq 0$ indicates that n is square free.'

The Mobius function, denoted by $\mu(n)$ was first introduced by August Ferdinand Mobius in 1832. However, Gauss had considered this function more than 3 decades ago, in 1801 writing about "The sum of all primitive roots (of a prime number p) is either $\equiv 0$ (when $p-1$ is divisible by a square), or $\equiv \pm 1 \pmod p$ (when $p-1$ is the product of unequal prime numbers; if the number of these is even the sign is positive but if the number is odd, the sign is negative". Mertens later introduced the modern notation $\mu(n)$ for the Mobius function in 1874.

The Mobius function is multiplicative, meaning that if a and b are coprime integers, then $\mu(ab) = \mu(a).\mu(b)$. This property is crucial for its application in number theory.

Here is a short table of values of $\mu(n)$:

n:	1	2	3	4	5	6	7	8	9	10
$\mu(n)$:	1	-1	-1	0	-1	1	-1	0	0	1

Definition 1.2. 'Let V be the vertex set and E be the edge set for the graph $G = (V, E)$. A 1-1 function $f : V \to N$ is said to be a Mobius cordial labeling of the graph G if the induced edge function $f^* : E \to \{-1, 0, 1\}$ defined by

$$f^*(uv) = |f(u) - f(v)|$$
$$= n = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes} \\ 0 & \text{if } n \text{ has one or more repeated prime factors} \end{cases}$$

where $uv \in E$, satisfies the following conditions:

(i) $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$

(ii) $|e_{f^*}(0) - e_{f^*}(-1)| \leq 1$

(iii) $|e_{f^*}(-1) - e_{f^*}(1)| \leq 1$.

'

Graph labeling techniques are widely used in various domains of computer science and this new labeling method could similarly be applied in those domains.

Definition 1.3. 'Mobius cordial graph G is a graph that admits the Mobius cordial labeling.'

Example 1.4. 'Mobius cordial labeling of a graph G is shown in Figure 1.

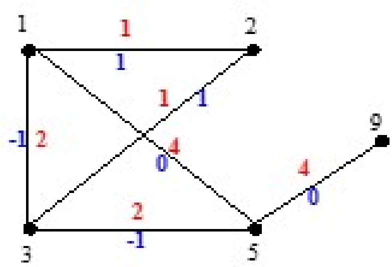

Figure 1. Mobius Cordial Labeling of G

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Definition 1.5. 'Snake graphs $C^m_{k,q}$ are the fusion of m cycle graphs, C_k, where q is the minimal number of edges between each point of fusion.'

Example 1.6. '$C^3_{5,2}$ with four vertices v_1, v_2, v_3, v_4 on the vertebrae.

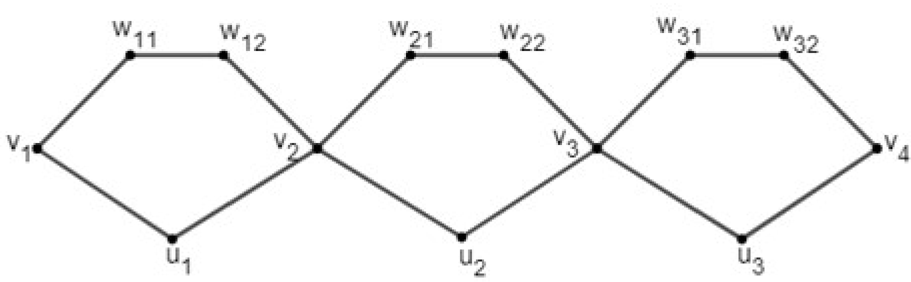

Figure 2. The graph $C^3_{5,2}$

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Definition 1.7. 'A Double Triangular Snake consists of two triangular snakes that have a common path. That is, a double triangular snake DT_n is a graph obtained from a path $v_1, v_2, \ldots v_n$ joining v_i and v_{i+1} to two new vertices u_i and w_i $(1 \le i \le n-1)$.'

Definition 1.8. 'An Alternate Double Triangular Snake graph $A(DT_n)$ is obtained from a path $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} alternatively $(i = 1, 3, 5, \ldots)$ to a new vertex u_i and w_i.'

Definition 1.9. 'An Irregular triangular Snake graph IT_n is obtained from a path $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+2} $(i = 1, 2, 3, \ldots, n-2)$ to a new vertex u_i.'

2 Main Result

Theorem 2.1. *The snake graph $C_{5,2}^m$ admits Mobius cordial labeling.*

Proof. Let the vertices on minimal number of edges be u_i $(1 \le i \le m)$ and w_{i1}, w_{i2} be vertices on maximal number of edges of each cycle C_5. Let v_i, $1 \le i \le m+1$ be $m+1$ vertices on vertebrae of the snake graph $C_{5,2}^m$.
It has $4m+1$ vertices and $5m$ edges.
The vertices are labelled as $f(v_{i+1}) = 1 + 3i$, $i = 0, 1, \ldots, m$
$f(w_{i1}) = f(v_i) + 4$, $i = 1, 2, \ldots, m$
$f(w_{i2}) = f(w_{i1}) - 2$, $i = 1, 2, \ldots, m$
$f(u_i) = f(v_{i+1}) + 7$, $i = m, m-3, \ldots, 1$
$f(u_i) = f(v_{i+1}) + 9$, $i = m-1, m-4, \ldots, 2$
$f(u_i) = f(v_{i+1}) + 30$, $i = m-2, m-5, \ldots, 3$

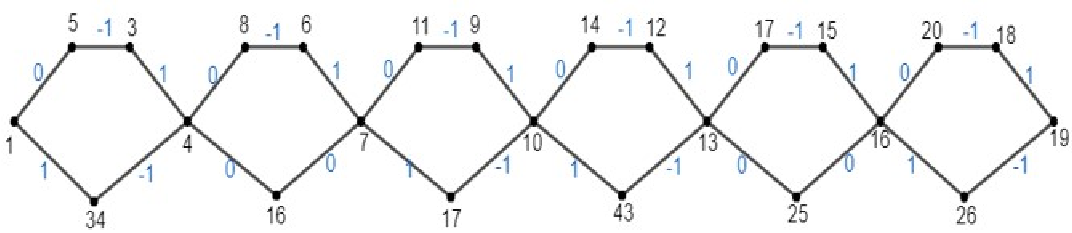

Figure 3. Mobius cordial labeling of $C_{5,2}^6$

By the above labeling pattern for vertices and by using the induced edge function f^* defined for edges as in Definition 1.2 we find the given snake graph $C_{5,2}^m$ admits Mobius cordial labeling. □

Theorem 2.2. *The Double triangular snake graph DT_n admits Mobius Cordial Labeling.*

Proof. Let the vertex set of DT_n be $V(DT_n) = \{u_i, w_i, v_i, (1 \le i \le n-1), v_n\}$
The total number of vertices of the double triangular snake graph DT_n is $|V(DT_n)| = 3n-2$
Let the edge set of DT_n be $E(DT_n) = \{u_iv_i, u_iv_{i+1}, w_iv_i, w_iv_{i+1}, v_iv_{i+1}, (1 \le i \le n-1)\}$
The total number of edges of the double triangular snake graph DT_n is $|E(DT_n)| = 5n-5$
The vertices are labelled as follows
Let $f(v_1) = 1$
$f(v_i) = f(v_{i-1}) + 4$ for $2 \le i \le n-1$
Let $f(u_1) = 4$
$f(u_i) = f(u_{i-1}) + 4$ for $2 \le i \le n-1$
$f(w_i) = f(v_{i-1}) + 1$, $i = 1, 4, 7, \ldots$
$f(w_i) = f(v_{i-1}) + 18$, $i = 2, 5, 8, \ldots$
$f(w_i) = f(v_{i-1}) + 9$, $i = 3, 6, 9, \ldots$
By the definition of induced edge function f^*, we find that the edges $u_iv_i, u_iv_{i+1}, w_iv_i, w_iv_{i+1}$
For all $1 \le i \le n-1$
$f^*(v_iv_{i+1}) = |f(v_i) - f(v_{i+1})| = 4$
$f^*(v_iu_i) = |f(v_i) - f(u_i)| = 3$
$f^*(v_{i+1}u_i) = |f(v_{i+1}) - f(u_i)| = 1$

$f^*(v_iw_i) = |f(v_i) - f(w_i)| = 5$ for $i = 1, 4, 7, \ldots$
$f^*(v_{i+1}w_i) = |f(v_{i+1}) - f(w_i)| = 1$ for $i = 1, 4, 7, \ldots$
$f^*(v_iw_i) = |f(v_i) - f(w_i)| = 22$ for $i = 2, 5, 8, \ldots$
$f^*(v_{i+1}w_i) = |f(v_{i+1}) - f(w_i)| = 18$ for $i = 2, 5, 8, \ldots$
$f^*(v_iw_i) = |f(v_i) - f(w_i)| = 13$ for $i = 3, 6, 9, \ldots$
$f^*(v_{i+1}w_i) = |f(v_{i+1}) - f(w_i)| = 9$ for $i = 3, 6, 9, \ldots$

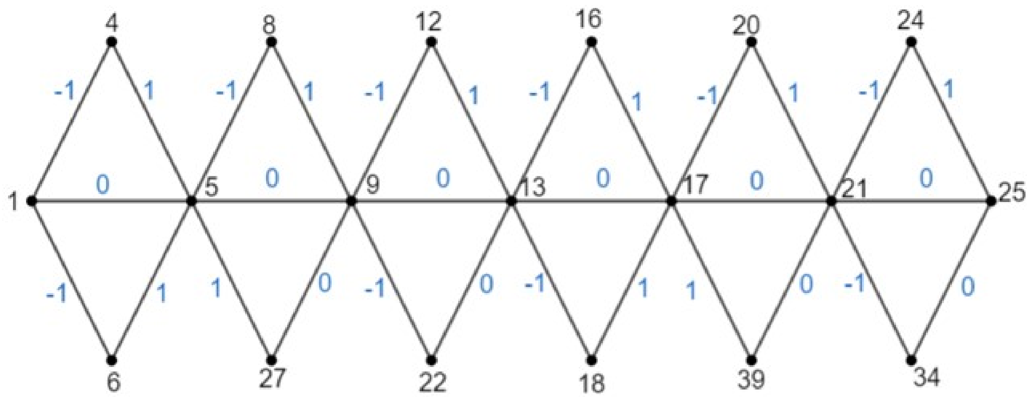

Figure 4. Mobius cordial labeling of DT_7

The edges satisfy the conditions,
$|e_{f^*}(0) - e_{f^*}(1)| \leq 1$
$|e_{f^*}(0) - e_{f^*}(-1)| \leq 1$
$|e_{f^*}(-1) - e_{f^*}(1)| \leq 1.$
Thus the graph, double triangular snake DT_n admits Möbius cordial labeling. □

Theorem 2.3. *An Alternate Double Triangular Snake graph $A(DT_n)$ admits Mobius Cordial Labelling.*

Proof. We obtain Double triangular snake graph from path $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} alternatively $(i = 1, 3, 5, \ldots)$ to a new vertex u_i and w_i, $1 \leq i \leq n-1$. Hence we get new vertices $u_1, w_1, u_3, w_3 \ldots, u_{n-1}, w_{n-1}$ (n is even) or $u_1, w_1, u_3, w_3 \ldots, u_{n-2}, w_{n-2}$ (n is odd)

$$|V(A(DT_n))| = \begin{cases} 2n & \text{when } n \text{ is even} \\ 2n-1 & \text{when } n \text{ is odd} \end{cases}$$

$$|E(A(DT_n))| = \begin{cases} 3n-1 & \text{when } n \text{ is even} \\ 3n-3 & \text{when } n \text{ is odd} \end{cases}$$

The vertices are labelled as follows
For some $n_1, n_2, n_3 \in N$
Let $f(v_1) = n_1$ (say 1)
$f(v_i) = f(v_{i-1}) + 4$ for $2 \leq i \leq n$
Let $f(u_1) = n_2$ (say 4)
$f(u_i) = f(u_{i-1}) + 8$ for $2 \leq i \leq n-1$
Let $f(w_1) = n_3$ (say 6)
$f(w_i) = f(w_{i-1}) + 8$ for $2 \leq i \leq n-1$
By the definition of induced edge function f^* defined as in Definition 1.2,
The edges V_iv_{i+1} for $1 \leq i \leq n-1$ receive the label 0, the edges v_iu_i and v_iw_i for $i = 1, 3, \ldots$ receive the label -1, the edges v_iu_{i-1} and v_iw_{i-1} for $i = 2, 4, \ldots$ receive the label 1. The edges satisfy the conditions,
$|e_{f^*}(0) - e_{f^*}(1)| \leq 1$
$|e_{f^*}(0) - e_{f^*}(-1)| \leq 1$
$|e_{f^*}(-1) - e_{f^*}(1)| \leq 1.$
Thus the graph, Alternate double triangular snake $A(DT_n)$ admits Möbius cordial labeling. □

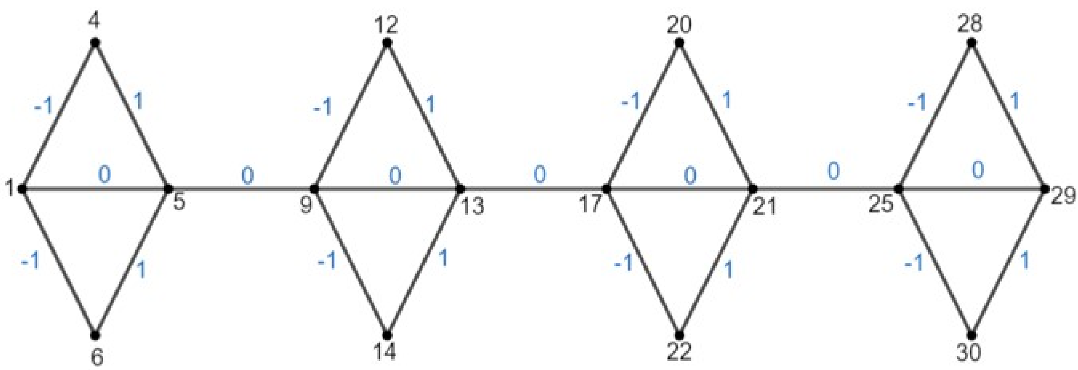

Figure 5. Mobius cordial labeling of $A(DT_8)$

Theorem 2.4. *An Irregular triangular Snake graph IT_n admits Mobius Cordial Labelling.*

Proof. Let the vertex set of Irregular triangular Snake graph IT_n be
$\{V_i, 1 \leq i \leq n, u_j, 1 \leq j \leq n-2\}$
And the edge set be $\{u_iu_{i+1}, 1 \leq i \leq n-1, u_jv_j, u_{j+2}v_j, 1 \leq j \leq n-2\}$.
$|V(IT_n)| = 2n-2$
$|E(IT_n)| = 3n-5$
The vertices are labelled by the definition as follows
Let $f(v_1) = n_0, n_0 \in N$
$f(v_i) = f(v_{i-1}) + 4, 2 \leq i \leq n$
$f(u_i) = f(v_{i+2}) - 1, 1 \leq i \leq n-2$
We find that the edges u_iu_{i+1}, $1 \leq i \leq n-1$ receive the label 0, the edges u_jv_j, $1 \leq j \leq n-2$ receive the label -1 and the edges $u_{j+2}v_j$, $1 \leq j \leq n-2$ receive the label 1. Thus the cordiality condition
$|e_{f^*}(0) - e_{f^*}(1)| \leq 1$
$|e_{f^*}(0) - e_{f^*}(-1)| \leq 1$
$|e_{f^*}(-1) - e_{f^*}(1)| \leq 1.$
got satisfied by edges. Hence an Irregular triangular Snake graph IT_n admits Mobius Cordial Labelling. □

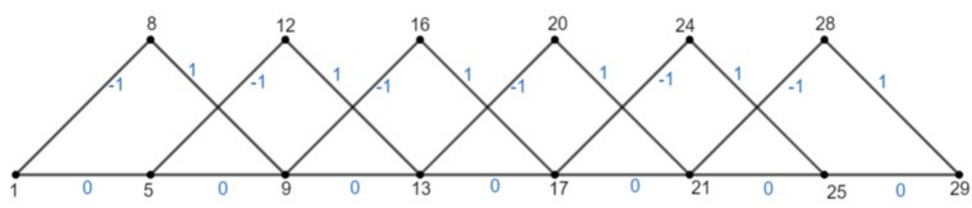

Figure 6. An Irregular triangular Snake graph IT_8 is a Mobius cordial graph

3 Conclusion

Mobius cordial labeling is studied for some snake related graphs. It remains open to find the Mobius cordial labeling of snake graphs $C^m_{k,q}$ for arbitrary values of m, k and q.

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