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STRONG SUM AND PRODUCT CORDIAL LABELING OF SUBDIVISION OF TREE RELATED GRAPHS

A. Asha Rani, K. Thirusangu and B.J. Balamurugan

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Corresponding Author: A. Asha Rani

Abstract A graph $G(V, E)$ is said be strong sum and product cordial labeling if there exists an injective function f from the vertex set $V(G)$ to the set of positive real numbers so that the induced edge function f^* from the edge set $E(G)$ defined by $f^*(uv) = f(u) + f(v) = f(u) \times f(v)$ takes the value 1 if it satisfies or 0 otherwise. Also satisfies the cordiality condition that $|(e_{f^*}(0) - e_{f^*}(1)| \leq 1$. We are going to investigate this labeling for certain classes of subdivision of tree related graphs in this paper.

1 Introduction

According to the paper by Alex Rosa, the idea of graph labeling was originally conceived [11]. It is about the allocation of mathematical entities, including integers, prime numbers, modular integers, elements of a group, and so on. The evaluating function plays a crucial role in the utilization of the properties of a mathematical object. It assigns values to the edges and/or vertices of a graph $G = (V, E)$ based on specific criteria, Over the past three decades, a significant number of papers have been dedicated to various types of graph labeling [1–7, 10], which are updated by Gallian [8]. Labeled graphs have numerous applications in areas like coding theory, X-ray crystallography, the design of good radar type codes, astronomy, circuit design, communication network addressing, data base management. In graph theory, the concept of subdivision plays a crucial role in analyzing and understanding the properties of complex graphs by relating them to simpler graph structures. By subdividing edges of a graph, it is possible to derive insights about the original graph from the properties of the resulting subdivided graph.

2 Preliminaries

The graph being considered is a simple, connected and undirected graph with no loops or multiple edges between two vertices. In this paper we show that the subdivision of graphs such as binary tree,coconut tree, centipede tree, banana tree admits strong sum and product cordial labeling. For terminology and basic concepts we refer [9]. The following definition is needed for this paper.

Definition 2.1. ‘In a graph G, subdivision of an edge uv is the operation of replacing uv with a path u, w, v through a new vertex w.’

Subdivision of graphs is a fundamental operation in graph theory where edges of a given graph are replaced by paths. This operation results in a new graph that retains many properties of the original graph while potentially introducing additional vertices and edges. Subdivision plays a crucial role in various areas of graph theory, including graph coloring, planarity, and graph algorithms.

3 Main Results

In this paper, when discussing the subdivision of graph G, we use the notation $S(G)$ to represent the subdivided graph.

Definition 3.1. 'A graph $G(V, E)$ is said be strong sum and product cordial labeling if there exists an injective function $f : V(G) \to R^{+}\backslash\{0\}$ such that the induced edge function $f^* : E(G) \to \{0, 1\}$ satisfies the following conditions

(i) for every $uv \in E(G)$, $f^*(uv) = \begin{cases} 1 & \text{if } f(u) + f(v) = f(u) \times f(v) \\ 0 & \text{otherwise} \end{cases}$.

(ii) the condition for cordiality, $|(e_{f^*}(0) - e_{f^*}(1)| \leq 1$ where $e_{f^*}(0), e_{f^*}(1)$ denotes edges labeled 0, edges labeled 1 respectively.

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Definition 3.2. 'A graph G which admits the strong sum and product cordial labeling is called a strong sum and product cordial graph.'

Theorem 3.3. *The subdivision of perfect binary tree admits strong sum and product cordial labeling.*

Let $G = (V, E)$ be a binary tree of n levels with $2^n - 1$ vertices and $2^n - 2$ edges. For example $v_{2,1}$ denote the 1^{st} vertex (from the left) of the 2^{nd} level (from the top).
v_{ij} denotes a vertex at location (i, j). Note that for the vertex at location (i, j), its parent (if existent) is at location $(i - 1, \lceil \frac{j}{2} \rceil)$, and its left and right children (if existent) are at locations $(i + 1, 2j - 1)$ and $(i + 1, 2j)$, respectively.
Introduce a new vertex v'_{ij} by removing an edge joining a vertex at location (i, j) and its parent (if exist) which is at location $(i - 1, \lceil \frac{j}{2} \rceil)$,
The vertex set
$V(G) = \{v_{ij} : 1 \leq i \leq n \text{ and } 1 \leq j \leq 2^{i-1}\} \cup \{v'_{ij} : 2 \leq i \leq n \text{ and } 1 \leq j \leq 2^{i-1}\}$
The vertices are labeled by the injective function $f : V(G) \to R+\backslash\{0\}$ defined as

$$f(v_{ij}) = j + 2^{i-1}; 1 \leq i \leq n \text{ and } 1 \leq j \leq 2^{i-1}$$
$$f(v'_{ij}) = \frac{f(v_{ij})}{f(v_{ij}) - 1}; 2 \leq i \leq n \text{ and } 1 \leq j \leq 2^{i-1}$$

By the definition 3.1 of induced edge function f^*, the edge joining the vertices $v_{i-1,\lceil \frac{j}{2} \rceil}$ and v'_{ij} receive the label 0 and the edge joining the vertices v'_{ij} and v_{ij} receive the label 1 for $1 \leq i \leq n$ and $1 \leq j \leq 2^{i-1}$

Thus the condition for cordiality $|(e_{f^*}(0) - e_{f^*}(1)| \leq 1$ is attained. Hence the subdivision of perfect binary tree graph admits strong sum and product cordial labeling.

Theorem 3.4. *The subdivision of coconut tree $S(CT(n, n))$ admits strong sum and product cordial labeling.*

Proof. Let P_n be the path $v_1v_2 \ldots v_n$ and
let $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}$ are the vertex sets of $CT(n, n)$ and
edge set $E(CT(n, n)) = E(P_n) \cup E(K_{1,n})$ identifying the u with v_n.
The coconut tree $CT(n, n)$ has $2n - 1$ edges.
The vertex set and the edge set of the subdivision of coconut tree becomes
Let P_{2n-1} be the path $v_1v'_1v_2v'_2 \ldots v'_{n-1}v_n$ and
let $V(K_{2,n}) = \{u, u'_i, u_i : 1 \leq i \leq n\}$ are the new vertex sets of $S(CT(n, n))$ and
new edge set $E(S(CT(n, n))) = E(P_{2n-1}) \cup E(u, u'_i) \cup E(u'_i, u_i)$ identifying the u with v_n.
The subdivision of coconut tree $CT(n, n)$ has $4n - 2$ edges.
The vertices are labeled in the following way

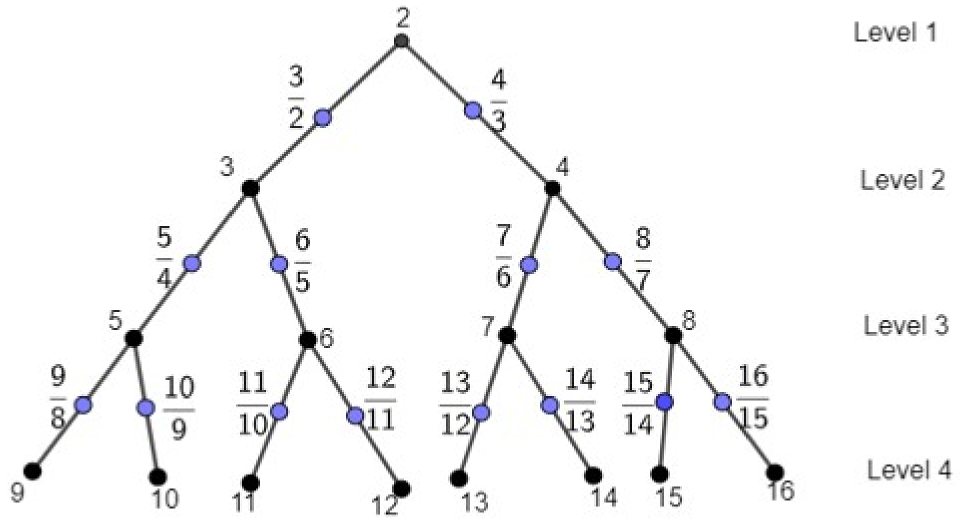

Figure 1. Strong sum and product cordial labeling of Subdivision of Binary Tree with 4 levels

Define a function $f : V(G) \to R^+ \setminus \{0\}$ such that

$$f(v_n) = f(u) = 2$$
$$f(v_i) = f(v_n) + (n - i) \text{ for } i = 1, 2, \ldots n - 1$$
$$f(u_i) = f(v_n) + ((n - 1) + i) \text{ for } i = 1, 2, \ldots n$$
$$f(v_i') = \frac{f(v_i)}{f(v_i) - 1} \text{ for } i = 1, 2, \ldots n - 1$$
$$f(u_i') = \frac{f(u_i)}{f(u_i) - 1} \text{ for } i = 1, 2, \ldots n$$

By this labeling pattern for vertices and by using the induced edge function f^* defined as in definition 3.1 for edges, we find that the $n - 1$ edges $\{v_iv_i', i = 1, 2, \ldots n - 1\}$ and the n edges $\{u_i'u_i : I = 1, 2, \ldots n\}$ are labeled 1 and the remaining $2n - 1$ edges are labeled 0. Thus we get $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$. Hence the given graph $CT(n, n)$ admits Strong Sum and Product cordial labeling. □

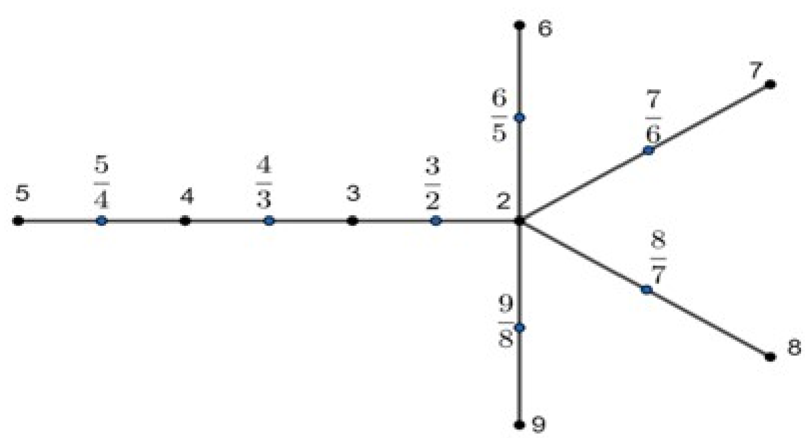

Figure 2. Strong sum and product cordial labeling of Subdivision of Coconut Tree $S(CT(4, 4))$

Theorem 3.5. *The subdivision of $(n, 2)$-centipede tree $S(C_{n,2})$ admits strong sum and product cordial labeling.*

Proof. 'The $(n, 2)$-centipede tree has n vertices on its spine, each vertex on the spine has two leaf nodes adjacent to it'. The vertex set of $C_{n,2} = \{u_i, v_i, w_i, 1 \leq i \leq n\}$ where u_i denotes vertices on its spine, v_i, w_i denotes vertices on its leaf nodes and
the edge set of $C_{n,2} = \{u_iu_{i+1}, 1 \leq i \leq n-1\} \cup \{u_iv_i, u_iw_i, 1 \leq i \leq n\}$. The $(n, 2)$-centipede tree $C_{n,2}$ has $3n$ vertices and $3n - 1$ edges.
The vertex set and the edge set of subdivision of $(n, 2)$-centipede tree $C_{n,2}$ is
$V(S(C_{n,2})) = \{u_i, v_i, w_i, 1 \leq i \leq n\} \cup \{u'_i, 1 \leq i \leq n-1\} \cup \{v'_i, w'_i, 1 \leq i \leq n\}$,
where u'_i denotes subdividing vertices on its spine, v'_i, w'_i denotes subdividing vertices on its leaf nodes.
$E(S(C_{n,2})) = \{u_iu'_i, u'_iu_{i+1}, 1 \leq i \leq n-1\} \cup \{u_iv'_i, v'_iv_i, u_iw'_i, w'_iw_i, 1 \leq i \leq n\}$
The subdivision of $(n, 2)$-centipede tree $S(C_{n,2})$ has $6n - 1$ vertices and $6n - 2$ edges whose vertices are labeled as follows.

$$f(v'_i) = i + 2 \text{ for } i = 1, 2, \ldots n$$
$$f(w'_i) = n + i + 2 \text{ for } i = 1, 2, \ldots n$$
$$f(u_i) = 2n + i + 2 \text{ for } i = 1, 2, \ldots n$$
$$f(u'_i) = \frac{f(u_i)}{f(u_i) - 1} \text{ for } i = 1, 2, \ldots n - 1$$
$$f(v_i) = \frac{f(v'_i)}{f(v'_i) - 1} \text{ for } i = 1, 2, \ldots n$$
$$f(w_i) = \frac{f(w'_i)}{f(w'_i) - 1} \text{ for } i = 1, 2, \ldots n$$

By this labeling pattern for vertices and by using the induced edge function f^* defined as in definition 3.1 for edges, we find that the $3n - 1$ edges $\{v_iv'_i, w_iw'_i, i = 1, 2, \ldots n\} \cup \{u_iu'_i, i = 1, 2, \ldots n-1\}$ are labeled 1 and the remaining $3n-1$ edges are labeled 0. Thus we get $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$. Hence the subdivision of $(n, 2)$-centipede tree $C_{n,2}$ admits Strong Sum and Product cordial labeling. □

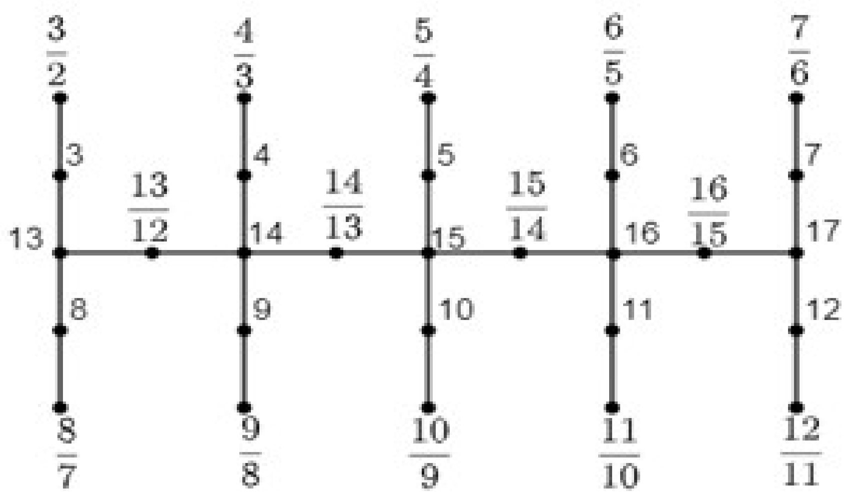

Figure 3. Strong sum and product cordial labeling of Subdivision of (5,2)-centipede tree $S(C_{5,2})$

Theorem 3.6. *The subdivision of Banana tree graph $S(B(m, n))$ admits strong sum and product cordial labeling.*

Proof. ‘An (m, n)-banana tree is a graph obtained by connecting one leaf of each of m copies of an n-star graph with a single root vertex that is distinct from all the stars’.
Let the vertex set of $B(m, n)$ banana tree graph be $V = \{u, w_1, w_2, \ldots, w_m, w_{ij}, i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n\}$ where $deg(u) = m$ which is the root vertex, $deg(w_i) = n$ for $i = 1, 2, \ldots, m$, $deg(w_{ij}) = 1$, $j = 2, \ldots, n$ and $deg(w_{i1}) = 2$, for all i.
Let the edge set of $B(m, n)$ banana tree graph be $E = \{uw_{i1}, w_i w_{ij} i = 1, 2, \ldots, m$ and $j = 2, \ldots, n\}$
An (m, n)-banana tree has $mn + m + 1$ vertices and $mn + m$ edges.
The vertex set and the edge set of subdivision of (m, n)-Banana tree $S(B(m, n))$ is
$V(S(B(m, n))) = \{u, u_i, w_1, w_2, \ldots, w_m, w_{ij}, w'_{ij}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n\}$
$E(S(B(m, n))) = \{uu_i, u_i w_{i1}, w_i w'_{ij}, w'_{ij} w_{ij} i = 1, 2, \ldots, m$ and $j = 2, \ldots, n\}$
A subdivision of (m, n)-Banana tree $S(B(m, n))$ has $2(mn + m) + 1$ vertices and $2(mn + m)$ edges.
The vertices are labeled as follows

$$f(u) = 2$$
$$f(w_i) = i + f(u) \text{ for } i = 1, 2, \ldots m$$
$$f(w_{ij}) = f(w_i) + (m - 1) + j \text{ for } i = 1 \text{ and } j = 1, 2, \ldots n$$
$$f(w_{ij}) = f(w_{i-1,n}) + j \text{ for } i = 2, 3, \ldots, m \text{ and } j = 1, 2, \ldots n$$
$$f(u_i) = \frac{f(w_{ij})}{f(w_{ij}) - 1} \text{ for } i = 1, 2, \ldots m \text{ and } j = 1$$
$$f(w'_{ij}) = \frac{f(w_i)}{f(w_i) - 1} \text{ for } i = 1, 2, \ldots m \text{ and } j = 1$$
$$f(w'_{ij}) = \frac{f(w_{ij})}{f(w_{ij}) - 1} \text{ for } i = 1, 2, \ldots m \text{ and } j = 2, 3, \ldots, n$$

By this labeling pattern for vertices and by using the induced edge function f^* defined as in definition 3.1 for edges, we find that the $mn + m$ edges $\{u_i w_{i1}, w_i w'_{i1}, w'_{ij} w_{ij} i = 1, 2, \ldots, m$ and $j = 2, \ldots, n\}$ are labeled 1 and the remaining $mn + m$ edges are labeled 0. Thus we get $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$. Hence the subdivision of Banana tree graph $S(B(m, n))$ admits Strong Sum and Product cordial labeling. □

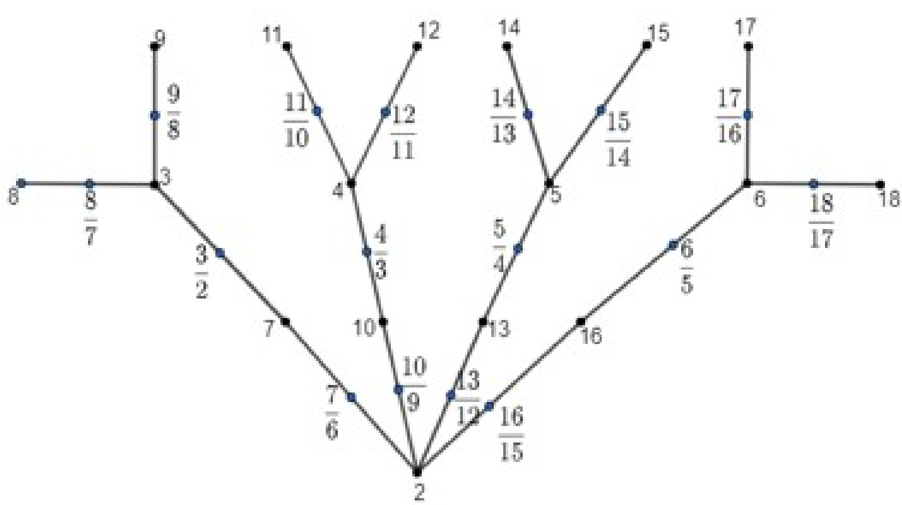

Figure 4. Strong sum and product cordial labeling of Subdivision of Banana tree $S(B(4, 3))$

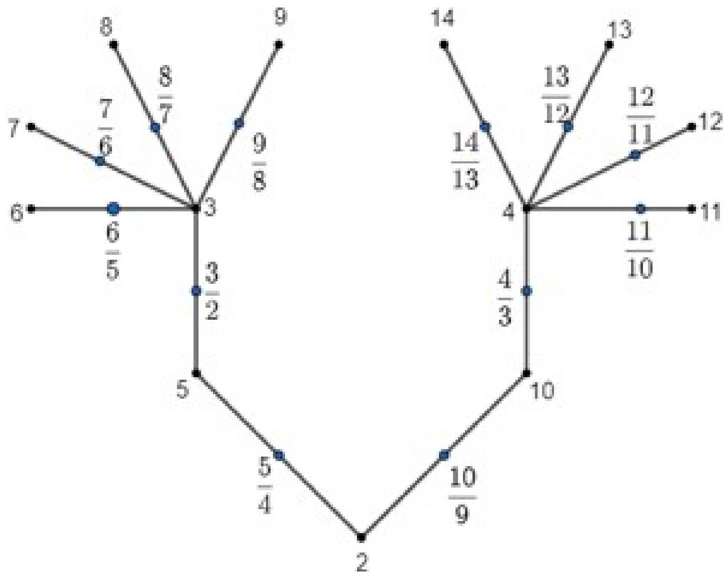

Figure 5. Strong sum and product cordial labeling of Subdivision of Banana tree $S(B(2,5))$

4 Conclusion

Strong sum and product labeling of subdivision of perfect binary tree graph, coconut tree graph, Centipede graph and Banana tree graph has been investigated. This concept of labeling can be investigated for other families of graphs.

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Author information

A. Asha Rani, Department of Mathematics with Computer Applications,
PSG College of Arts & Science, Coimbatore - 641014, Tamil Nadu, India.
E-mail: asharani.a@gmail.com

K. Thirusangu, Department of Mathematics,
SIVET College Gowrivakkam, Chennai - 60073, Tamil Nadu, India.
E-mail: kthirusangu@gmail.com

B.J. Balamurugan, School of Advanced Sciences,
VIT University, Chennai Campus, Chennai - 600127, Tamil Nadu, India.
E-mail: balamurugan.bj@vit.ac.in