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Parameter Uniform Convergence for a Weakly Coupled System of Two Partially Singularly Perturbed Delay Differential Equations of Convection-Diffusion Type

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Abstract This article presents a numerical method framed with a classical finite difference scheme and a piecewise uniform Shishkin mesh, which is used to resolve boundary and interior layers for a weakly coupled system of two partially singularly perturbed delay differential equations of convection-diffusion type on the interval [0, 2]. Here the first equation is considered as a singularly perturbed delay differential equation, and the second equation is an ordinary delay differential equation. The suggested numerical method is of first order uniform convergence in the maximum norm which is independent of the perturbation parameter. Numerical illustration is provided in support of the theoretical result.

1 Introduction

Delay differential equations (DDEs) of convection-diffusion of partially singularly perturbed weakly coupled systems explain systems in which the evolution of a quantity is affected by both convection and diffusion processes, with delays in the system's response. The term "weakly coupled" denotes that the interactions between convection and diffusion are not dominant, whereas the term "partially singularly perturbed" refers to conditions where one or more parameters or terms in the equations approach critical values, resulting in regions where convection or diffusion dominates.

Diffusion terms in these equations describe the expansion or Efficiency resulting from random motion, whereas convection terms often represent the advection of the quantity by a fluid flow. The system's behavior becomes dependent on both its previous and present states due to the memory effects introduced by the delays. Convection, diffusion, and delays together create complicated dynamics that make analysis and numerical modeling difficult.

Asymptotic analysis, perturbation methods, and numerical integration approaches for DDEs are among the specialized mathematical techniques frequently used in the solution of partially singularly perturbed weakly coupled delay differential equations of convection-diffusion. In many domains, including fluid dynamics, heat transfer, chemical processes, and population dynamics, where the interaction of convection, diffusion, and delays greatly influences the behavior and stability of the system, it is essential to analyze these equations. In engineering applications such as reaction-diffusion systems, transport phenomena, and biological systems modeling, an understanding of the dynamics of such systems is crucial for process prediction and optimization. However, because of the combined effects of diffusion, convection, and delays, their analysis is difficult, which makes them an interesting subject for applied mathematics and engineering study.

The article discusses the numerical treatment of a weakly coupled system of two partially singularly perturbed delay differential equations (PSPDDEs) on the interval [0,2]. In the first equation, the singular perturbation parameter is positive and multiplied by leading terms; in the second equation, however, there is no unique perturbation. There is an overlapping layer in the first component, and less severe overlapping layers in the subsequent components. Non-smooth behavior is caused by the existence of a perturbation parameter and delay term, especially close to boundary points and the delay point. To address this, the article proposes a classical finite difference scheme using a Shishkin mesh, which positions the delay term on nodal points after spatial discretization. The proposed method is rigorously analyzed and proven to be first-order convergent in the maximum norm, independent of the perturbation parameter. This ensures the method remains stable and accurate even as the perturbation parameter varies. The article presents a reliable tool for analyzing weakly coupled systems computationally.

The system of two partially singularly perturbed delay differential equations under consideration is

$$\vec{L}\vec{y}(s) = E\vec{y}''(s) + A(s)\vec{y}'(s) - B(s)\vec{y}(s) - D(s)\vec{y}(s-1) = \vec{f}(s)\ on(0,2) \tag{1.1}$$

$$with\ \vec{y} = \vec{\phi}\ on\ [-1,0]\ and\ \vec{y}(2) = \vec{r} \tag{1.2}$$

here for all $s \in [0,2], \vec{y} = ((y_1(s), y_2(s))^T, \vec{f} = ((f_1(s), f_2(s))^T. E,\ A(s),\ B(s)$ and $D(s)$ are 2×2 matrices. $E = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}$, $A(s) = \begin{pmatrix} a_1(s) & 0 \\ 0 & a_2(s) \end{pmatrix}$, $B(s) = \begin{pmatrix} b_{11}(s) & b_{12}(s) \\ b_{21}(s) & b_{22}(s) \end{pmatrix}$ and $D(s) = \begin{pmatrix} d_1(s) & 0 \\ 0 & d_2(s) \end{pmatrix}$.

Assumptions: Here we make the following the assumptions.
- The function $\vec{\phi}$ is sufficiently smooth on $[-1,0]$
- The singular perturbation parameter satisfies $0 < \varepsilon << 1$.
- For all $s \in [0,2]$, the entries $a_i(s)$ of $A(s)$, $b_{ij}(s)$ of $B(s)$ and the components $d_i(s)$ of $D(s)$ satisfy

$$a_i(s) \geq \alpha > 0, b_{ii}(s) + b_{ij}(s) \geq \beta > 0 \tag{1.3}$$

$$b_{ij}(s),\ d_i(s) \leq 0, \text{for } 1 \leq i \neq j \leq 2 \text{ and } b_{ii}(s) > \sum_{i\neq j} |b_{ij}(s) + d_i(s)|. \tag{1.4}$$

$$\text{and } 0 < \gamma < \min_{s\in[0,2]} \left(\sum_{j=1}^{2} b_{ij}(s) + d_i(s) \right) \text{ for } i = 1,2 \tag{1.5}$$

- The functions $f_i(s), a_i(s), b_{ij}(s)$ and $d_i(s)$, $1 \leq i,j \leq 2$ are in $C^2([0,2])$

From the above assumptions that the problem (1.1)-(1.2) has a unique solution $\vec{y}$ and $\vec{y} \in C^2$, where $C = C^0[0,2] \cap C^1(0,2) \cap C^2((0,1) \cup (1,2))$.
The problem (1.1)-(1.2) can be written as

$$\vec{L}_1\vec{y}(s) = E\vec{y}''(s) + A(s)\vec{y}'(s) - B(s)\vec{y}(s) = \vec{f}(s) + D(s)\vec{\phi}(s-1) = \vec{g}(s) \text{ on } (0,1) \tag{1.6}$$

$$\vec{L}_2\vec{y}(s) = E\vec{y}''(s) + A(s)\vec{y}'(s) - B(s)\vec{y}(s) - D(s)\vec{y}(s-1) = \vec{f}(s) \text{ on } (1,2) \tag{1.7}$$

$$\vec{y}(0) = \vec{\phi}(0),\ \vec{y}(2) = \vec{r},\ \vec{y}(1-) = \vec{y}(1+), \vec{y}'(1-) = \vec{y}'(1+) \tag{1.8}$$

and when perturbation parameter to zero, the corresponding reduced problem is given by

$$E_0\vec{y_0}''(s) + A(s)\vec{y_0}'(s) - B(s)\vec{y_0}(s) = \vec{g}(s) \text{ on } (0,1)$$

$$E_0\vec{y_0}''(s) + A(s)\vec{y_0}'(s) - B(s)\vec{y_0}(s) - D(s)\vec{y_0}(s-1) = \vec{f}(s) \text{ on } (1,2)$$

where $E_0 = diag(0,1),\ \vec{y_0}(2) = \vec{r}, \vec{y_0}(s) = (y_{01}(s), y_{02}(s))^T$

A boundary layer of width $O(\varepsilon)$ is expected near $s = 0$ in the solution component y_1 if $y_1(0) \neq y_{01}(0)$. In addition to that, the solution exhibits interior layer of width $O(\varepsilon)$ at $s = 1$.

2 Analytical Results

Lemma 2.1. Maximum principle*: Let the conditions (1.3)-(1.5) hold. Let $\vec{\mathfrak{H}} = (\mathfrak{H}_1, \mathfrak{H}_2)^T$ be any function in the domain of $\vec{L}$ such that $\vec{\mathfrak{H}}(0) \geq \vec{0}$, $\vec{\mathfrak{H}}(2) \geq \vec{0}$, $\vec{L}_1\vec{\mathfrak{H}} \leq \vec{0}$ on $(0,1)$, $\vec{L}_2\vec{\mathfrak{H}} \leq \vec{0}$ on $(1,2)$, $[\vec{\mathfrak{H}}](1) = \vec{0}$ and $[\vec{\mathfrak{H}}'](1) \leq \vec{0}$ then $\vec{\mathfrak{H}} \geq \vec{0}$ on [0,2].*

Proof. Let i^* , s^* be such that $\mathfrak{H}_{i^*}(s^*) = \min_{s\in[0,2],\, i=1,2} \mathfrak{H}_i(s)$
If $\mathfrak{H}_{i^*}(s^*) \geq 0$, then there is nothing to prove.
Suppose $\mathfrak{H}_{i^*}(s^*) < 0$ then $s^* \notin \{0,2\}$ and $\mathfrak{H}''_{i^*}(s^*) \geq 0$, $\mathfrak{H}'_{i^*}(s^*) = 0$.
Define

$$\varepsilon_{i^*} = \begin{cases} \varepsilon, & \text{if } i^* = 1 \\ 1 & \text{if } i^* = 2 \end{cases}$$

If $s^* \in (0,1)$, then

$$\begin{aligned}(\vec{L}_1\vec{\mathfrak{H}})_{i^*}(s^*) &= \varepsilon_{i^*}\mathfrak{H}''_{i^*}(s^*) + a_{i^*}(s^*)\mathfrak{H}'_{i^*}(s^*) - \sum_{j=1}^{2}[b_{i^*j}(s^*)\mathfrak{H}_j(s^*)] \\ &= \varepsilon_{i^*}\mathfrak{H}''_{i^*}(s^*) + a_{i^*}(s^*)\mathfrak{H}'_{i^*}(s^*) - b_{i^*i^*}(s^*)\mathfrak{H}_{i^*}(s^*) - \sum_{j=1, j\neq i^*}^{2} [b_{i^*j}(s^*)\mathfrak{H}_j(s^*)] \\ &\geq \varepsilon_{i^*}\mathfrak{H}''_{i^*}(s^*) + a_{i^*}(s^*)\mathfrak{H}'_{i^*}(s^*) - b_{i^*i^*}(s^*)\mathfrak{H}_{i^*}(s^*) - \sum_{j=1, j\neq i^*}^{2} b_{i^*j}(s^*)\mathfrak{H}_{i^*}(s^*) \\ &> 0, \text{which is a contradiction.}\end{aligned}$$

And if $s^* \in (1,2)$ then

$$\begin{aligned}(\vec{L}_2\vec{\mathfrak{H}})_{i^*}(s^*) &= \varepsilon_{i^*}\mathfrak{H}''_{i^*}(s^*) + a_{i^*}(s^*)\mathfrak{H}'_{i^*}(s^*) - \sum_{j=1}^{2}[b_{i^*j}(s^*)\mathfrak{H}_j(s^*)] - d_{i^*}(s^*)\mathfrak{H}_{i^*}(s^*-1) \\ &= \varepsilon_{i^*}\mathfrak{H}''_{i^*}(s^*) + a_{i^*}(s^*)\mathfrak{H}'_{i^*}(s^*) - b_{i^*i^*}(s^*)\mathfrak{H}_{i^*}(s^*) - \sum_{j=1, j\neq i^*}^{2} [b_{i^*j}(s^*)\mathfrak{H}_j(s^*)] \\ &\quad - d_{i^*}(x^*)\mathfrak{H}_{i^*}(s^*-1) \\ &\geq \varepsilon_{i^*}\mathfrak{H}''_{i^*}(s^*) + a_{i^*}(s^*)\mathfrak{H}'_{i^*}(s^*) - b_{i^*i^*}(s^*)\mathfrak{H}_{i^*}(s^*) - \sum_{j=1, j\neq i^*}^{2} b_{i^*j}(s^*)\mathfrak{H}_{i^*}(s^*) \\ &\quad - d_{i^*}(s^*)\mathfrak{H}_{i^*}(s^*) \text{ as } \mathfrak{H}_{i^*}(s^*) \leq \mathfrak{H}_j(s^*) \text{ and } \mathfrak{H}_{i^*}(s^*) \leq \mathfrak{H}_{i^*}(s^*-1) \\ &> 0, \text{which is also a contradiction.}\end{aligned}$$

When $s^* = 1$. In this case we discuss about the differentiability of $\mathfrak{H}_{i^*}$ at $s = 1$. If $\mathfrak{H}'_{i^*}(1)$ does not exist then $[\mathfrak{H}'_{i^*}](1) = \mathfrak{H}'_{i^*}(1+) - \mathfrak{H}'_{i^*}(1-) > 0$, which is a contradiction to $[\mathfrak{H}'_{i^*}](1) \leq 0$. If $\mathfrak{H}_{i^*}$ is differentiable at $s = 1$, then

$$\begin{aligned}a_{i^*}(1)\mathfrak{H}'_{i^*}(1) - \sum_{j=1}^{2} b_{i^*j}(1)\mathfrak{H}_j(1) &= -b_{i^*i^*}(1)\mathfrak{H}_{i^*}(1) - \sum_{j=1, j\neq i^*}^{2} b_{i^*j}(1)\mathfrak{H}_j(1) \\ &\geq -b_{i^*i^*}(1)\mathfrak{H}_{i^*}(1) - \sum_{j=1, j\neq i^*}^{2} b_{i^*j}(1)\mathfrak{H}_{i^*}(1) \\ &= -(\sum_{j=1}^{2} b_{i^*j}(1))\mathfrak{H}_{i^*}(1) > 0\end{aligned}$$

$a_{i^*}(1)\mathfrak{H}'_{i^*}(1) - \sum_{j=1}^{2} b_{i^*j}(1)\mathfrak{H}_j(1) > 0$ and all the entries of $A(s)$, $B(s)$ and $\mathfrak{H}_j(s)$ are in $C([0,2])$, there exist an interval $[1-h, 1)$ on which $a_{i^*}(s)\mathfrak{H}'_{i^*}(s) - \sum_{j=1}^{2} b_{i^*j}(s)\mathfrak{H}_j(s) > 0$.
If $\mathfrak{H}''_{i^*}(\hat{x}) \geq 0$ at any point $\hat{x} \in [1-h, 1)$ then $(\vec{L}_1\vec{\mathfrak{H}})_{i^*}(\hat{x}) \geq 0$, which is a contradiction. Thus

we can assume that $\mathfrak{H}''_{i^*}(s) < 0$ on $[1-h, 1)$ which implies that $\mathfrak{H}'_{i^*}(s)$ is strictly decreasing on $[1-h,1)$ and since $\mathfrak{H}'_{i^*}(1) = 0$, $\mathfrak{H}'_{i^*} \in C((0,2))$, thus $\mathfrak{H}'_{i^*}(s) > 0$ on $[1-h,1)$, consequently the continuous function $\mathfrak{H}_{i^*}(s)$ cannot have minimum at $s = 1$, which contradicts the assumption that $s^* = 1$. Hence $\vec{\mathfrak{H}} \geq \vec{0}$ on $[0,2]$. □

Lemma 2.2. Stability result*: Let the conditions (1.3)-(1.5) hold. Let $\vec{\mathfrak{H}}$ be any function in C^2, such that $[\vec{\mathfrak{H}}](1) = \vec{0}$ and $[\vec{\mathfrak{H}}'](1) = \vec{0}$, then for each $i = 1,2$ and $s \in [0,2]$,*

$$|\mathfrak{H}_i(s)| \leq max\{\|\vec{\mathfrak{H}}(0)\|, \|\vec{\mathfrak{H}}(2)\|, \frac{1}{\beta}\|\vec{L}_1\vec{\mathfrak{H}}\|, \frac{1}{\gamma}\|\vec{L}_2\vec{\mathfrak{H}}\|\}.$$

Proof. Let $M = max\{\|\vec{\mathfrak{H}}(0)\|, \|\vec{\mathfrak{H}}(2)\|, \frac{1}{\beta}\|\vec{L}_1\vec{\mathfrak{H}}\|, \frac{1}{\gamma}\|\vec{L}_2\vec{\mathfrak{H}}\|\}$.
Define two functions

$$\vec{\theta}^{\pm}(s) = M\vec{e} \pm \vec{\mathfrak{H}}(s),$$

where $\vec{e} = (1,1)^T$.

$$\vec{\theta}^{\pm}(0) = M\vec{e} \pm \vec{\mathfrak{H}}(0) \geq \vec{0},\ \vec{\theta}^{\pm}(2) = M\vec{e} \pm \vec{\mathfrak{H}}(2) \geq \vec{0}$$

For $s \in (0,1)$, $\vec{L}_1\vec{\theta}^{\pm}(s) = \vec{L}_1(M\vec{e} \pm \vec{\mathfrak{H}}(s)) = -B(s)M\vec{e} \pm \vec{L}_1\vec{\mathfrak{H}}(s) \leq \vec{0}$.
For $s \in (1,2)$, $\vec{L}_2\vec{\theta}^{\pm}(s) = \vec{L}_2(M\vec{e} \pm \vec{\mathfrak{H}}(s)) = -B(s)M\vec{e} - D(s)M\vec{e} \pm \vec{L}_2\vec{\mathfrak{H}}(s) \leq \vec{0}$
Moreover $[\vec{\theta^{\pm}}](1) = \pm[\vec{\mathfrak{H}}](1) = \vec{0}$ *and* $[\vec{\theta^{\pm}}'](1) = \pm[\vec{\mathfrak{H}}'](1) = \vec{0}$
From the above lemma it follows that $\vec{\theta}^{\pm}(s) \geq \vec{0}$ on $[0,2]$ and proves the result. □

In the following lemma we prove that the solution of (1)-(2) and its derivatives are bounded.

Lemma 2.3. *Let the conditions (1.3)-(1.5) hold. Let $\vec{y}$ be the solution of (1.1)-(1.2). Then for all $s \in [0,2]$,*

$$|y_i(s)| \leq max(\|\vec{y}(0)\|, \|\vec{y}(2)\|, \|\vec{f}\|),\ for\ i = 1,2$$

$$|y_1^{(k)}(s)| \leq C\varepsilon^{-k}(\|\vec{y}(0)\| + \|\vec{y}(2)\| + \varepsilon\|\vec{f}\|)$$

$$|y_2^{(k)}(s)| \leq C(\|\vec{y}(0)\| + \|\vec{y}(2)\| + \|\vec{f}\|),\ for\ k = 1,2$$

and

$$|y_1^{(3)}(s)| \leq C\varepsilon^{-3}(\|\vec{y}(0)\| + \|\vec{y}(2)\| + \varepsilon\|\vec{f}\|) + \varepsilon^{-1}\|f_1'\|$$

$$|y_2^{(3)}(s)| \leq C\varepsilon^{-1}(\|\vec{y}(0)\| + \|\vec{y}(2)\| + \|\vec{f}\|) + \|f_2'\|$$

Proof. The bound on $\vec{y}$ is an immediate consequence of lemma 1.2 and the differential equation(1.1). Thus for $i = 1,2$ and for $s \in [0,2]$,

$$|y_i(s)| \leq C(\|\vec{y}(0)\|, \|\vec{y}(2)\|, \|\vec{f}\|).$$

To bound $y_1'(s)$ on the interval $[0,1]$. For any $s \in [0,1]$, there exists $a \in [0, 1-\varepsilon]$ such that $s \in N_a = [a, a+\varepsilon] \subset [0,1]$. By the mean value theorem, there exists $t \in (a, a+\varepsilon)$ such that $y_1'(t) = \frac{y_1'(a+\varepsilon) - y_1'(a)}{\varepsilon}$ and hence $|y_1'(t)| \leq \varepsilon^{-1}(|y_1'(a+\varepsilon)| + |y_1'(a)|)$.
Also for any $s \in N_a$,

$$\begin{aligned} y_1'(s) &= y_1'(t) + \int_t^s y_1''(x)\,\mathrm{d}x \\ &= y_1'(t) + \varepsilon^{-1}\int_t^s [f_1(x) - a_1(x)y_1'(x) + \sum_{j=1}^{2} b_{1j}(x)y_1'(x)]\,\mathrm{d}x \end{aligned}$$

$$|y_1'(s)| \leq C\varepsilon^{-1}(\|y_1'\| + \varepsilon\|\vec{f}\|) \leq C\varepsilon^{-1}(\|\vec{y}(0)\| + \|\vec{y}(2)\| + \varepsilon\|\vec{f}\|).$$

To bound $y_1'(s)$ on the interval $(1,2]$. We consider, for $a \in (1, 2-\varepsilon]$ the interval $N_a = [a, a+\varepsilon] \subset (1,2]$. Using the bounds of y_i on $[0,1]$ for $i = 1,2$ and proceeding as before, $|y_1'(s)| \leq$

$C\varepsilon^{-1}(\|\vec{y}(0)\| + \|\vec{y}(2)\| + \varepsilon\|\vec{f}\|)$ and hence for all $s \in [0,2]$. Hence for all $s \in [0,2]$, from the equation (1.1) and using the estimates of y_i, y_1', we have

$$|y_1''(s)| \leq C\,\varepsilon^{-2}(\|\vec{y}(0)\| + \|\vec{y}(2)\| + \varepsilon\|\vec{f}\|).$$

Differentiating (1.1) once and substituting the above bounds lead to

$$|y_1^{(3)}(s)| \leq \varepsilon^{-3}(\|\vec{y}(0)\| + \|\vec{y}(2)\| + \varepsilon\|\vec{f}\|) + \varepsilon^{-1}\|f_1'\|.$$

To bound $y_2'(s)$ on the interval $[0,1]$. For any $s \in [0,1]$, using mean value theorem, there exists $t \in [0,1]$ such that $y_2'(t) = y_2(1) - y_2(0)$ and proceeding as before we have for all $s \in [0,2]$,

$$|y_2^{(k)}(s)| \leq C(\|\vec{y}(0)\| + \|\vec{y}(2)\| + \|\vec{f}\|), \text{ for } k = 1,2$$

and $$|y_2^{(3)}(s)| \leq C\varepsilon^{-1}(\|\vec{y}(0)\| + \|\vec{y}(2)\| + \|\vec{f}\|) + \|f_2'\|.$$ □

Shishkin decomposition and estimates of the derivatives of smooth and singular components of the solution

The solution $\vec{y}$ of the problem (1.1)-(1.2) can be decomposed into $\vec{y} = \vec{v} + \vec{w}$, where the smooth component $\vec{v} = (v_1, v_2)^T$ is the solution of

$$\vec{L}_1\vec{v}(s) = E\vec{v}''(s) + A(s)\vec{v}'(s) - B(s)\vec{v}(s) = \vec{g}(s) \text{ on } (0,1) \tag{2.1}$$

$\vec{v}(0) = \vec{y}_0(0)$, $\vec{v}(1-) = \vec{y}_0(1-)$.

$$\vec{L}_2\vec{v}(s) = E\vec{v}''(s) + A(s)\vec{v}'(s) - B(s)\vec{v}(s) - D(s)\vec{v}(s-1) = \vec{f}(s) \text{ on } (1,2) \tag{2.2}$$

$$\vec{v}(1+) = \vec{y}_0(1+),\ \vec{v}(2) = \vec{y}_0(2).$$

And the singular component $\vec{w} = (w_1, w_2)^T$ is the solution of

$$\vec{L}_1\vec{w} = \vec{0} \text{ on } (0,1) \tag{2.3}$$

$$\vec{L}_2\vec{w} = \vec{0} \text{ on } (1,2) \tag{2.4}$$

with $\vec{w}(0) = \vec{y}(0) - \vec{v}(0)$, $\vec{w}(2) = \vec{y}(2) - \vec{v}(2)$, $[\vec{w}](1) = -[\vec{v}](1)$, $[\vec{w}'](1) = -[\vec{v}'](1)$.
Now $\vec{v}$ is decomposed into $\vec{v} = \vec{z}_0 + \varepsilon\vec{z}_1 + \varepsilon^2\vec{z}_2$,
where, $\vec{z}_0 = (z_{01}, z_{02})^T$ is the solution of

$$a_1(s)z_{01}'(s) - b_{11}(s)z_{01}(s) - b_{12}(s)z_{02}(s) = g_1(s) \tag{2.5}$$

$$z_{02}''(s) + a_2(s)z_{02}'(s) - b_{21}(s)z_{01}(s) - b_{22}(s)z_{02}(s) = g_2(s) \text{ on } (0,1) \tag{2.6}$$

$$z_{01}(0) = y_0,\ z_{02}(0) = 0. \tag{2.7}$$

And $$a_1(s)z_{01}'(s) - b_{11}(s)z_{01}(s) - b_{12}(s)z_{02}(s) - d_1(s)z_{01}(s-1) = f_1(s) \tag{2.8}$$

$$z_{02}''(s) + a_2(s)z_{02}'(s) - b_{21}(s)z_{01}(s) - b_{22}(s)z_{02}(s) - d_2(s)z_{02}(s-1) = f_2(s) \text{ on } (1,2) \tag{2.9}$$

$$z_{01}(2) = r_1(2),\ z_{02}(2) = r_2(2). \tag{2.10}$$

$\vec{z}_1 = (z_{11}, z_{12})^T$ is the solution of

$$a_1(s)z_{11}'(s) - b_{11}(s)z_{11}(s) - b_{12}(s)z_{12}(s) = -z_{01}''(s) \tag{2.11}$$

$$a_2(s)z_{12}'(s) - b_{21}(s)z_{11}(s) - b_{22}(s)z_{12}(s) = -z_{12}''(s) \text{ on } (0,1) \tag{2.12}$$

$$z_{11}(0) = 0,\ z_{12}(0) = 0. \tag{2.13}$$

And $$a_1(s)z_{11}'(s) - b_{11}(s)z_{11}(s) - b_{12}(s)z_{12}(s) - d_1(s)z_{11}(s-1) = -z_{01}''(s) \tag{2.14}$$

$$a_2(s)z_{12}'(s) - b_{21}(s)z_{11}(s) - b_{22}(s)z_{12}(s) - d_2(s)z_{12}(s-1) = z_{12}''(s) \text{ on } (1,2) \tag{2.15}$$

$$z_{11}(2) = 0,\ z_{12}(2) = 0. \tag{2.16}$$

$\vec{z}_2 = (z_{21}, z_{22})^T$ is the solution of

$$\varepsilon z''_{21}(s) + a_1(s)z'_{21}(s) - b_{11}(s)z_{21}(s) - b_{12}(s)z_{22}(s) = -z''_{11}(s) \tag{2.17}$$

$$z''_{22}(s) + a_2(s)z'_{22}(s) - b_{21}(s)z_{21}(s) - b_{22}(s)z_{22}(s) = 0 \text{ on } (0,1) \tag{2.18}$$

$$z_{21}(0) = p,\ z_{22}(0) = 0. \tag{2.19}$$

And

$$\varepsilon z''_{21}(s) + a_1(s)z'_{21}(s) - b_{11}(s)z_{21}(s) - b_{12}(s)z_{22}(s) - d_1(s)z_{21}(s-1) = -z''_{11}(s) \tag{2.20}$$

$$z''_{22}(s) + a_2(s)z'_{22}(s) - b_{21}(s)z_{21}(s) - b_{22}(s)z_{22}(s) - d_2(s)z_{22}(s-1) = 0 \text{ on } (1,2) \tag{2.21}$$

$$z_{21}(2) = q,\ z_{22}(2) = 0. \tag{2.22}$$

where p, q are the constants chosen in such a way that $|p| \leq C$, $|q| \leq C$.

Estimates of the derivatives of the smooth component

Lemma 2.4. *Let the conditions (1.3)-(1.5) hold. Then for $i = 1, 2$ and for all $s \in [0, 2]$*

$$|v_i^{(k)}(s)| \leq C,\ \textit{for}\ 0 \leq k \leq 2,\ \ |v_1^{(3)}(s)| \leq C\varepsilon^{-1},\quad |v_2^{(3)}(s)| \leq C.$$

Proof. The proof is by the method of steps.
For $s \in [0, 1-]$, since the problems (2.5)-(2.7) and (2.11)-(2.13) are not perturbed and the coefficient functions are assumed to be sufficiently smooth on $[0,1]$, so it is clear that, for $0 \leq k \leq 3$, $\| \vec{z}_0^{(k)} \| \leq C$, $\| \vec{z}_1^{(k)} \| \leq C$.

Applying the procedure adopted in Chapter 4 of [10], the following estimates hold from (2.17)-(2.19), $|\vec{z}_{22}^{(k)}| \leq C$ for $k = 1, 2$, and $|\vec{z}_{21}^{(k)}| \leq C\varepsilon^{-k}$ for $0 \leq k \leq 3$, $|\vec{z}_{22}^{(3)}| \leq C\varepsilon^{-1}$. Hence from the above results, the estimates of the components v_1 and v_2 of $\vec{v}$ on $[0, 1-]$ follow.
Now consider the interval $[1+, 2]$. On this interval $\vec{v}$ satisfies

$$\vec{L}_2\vec{v}(s) = \vec{f}(s) \text{ or } \vec{L}_1\vec{v}(s) = \vec{f}(s) + D(s)\vec{v}(s-1).$$

Using the bounds of $\vec{v}$ and its derivatives on $[0, 1-]$ and the procedure adopted in Chapter 4 of [10] for the operator $\vec{L}_1$, it is not hard to derive the estimates of the derivatives of $\vec{v}$ on $[1+, 2]$. □

Estimates of the derivatives of the singular component
Definition : The layer functions $\mathfrak{B}_1(s)$ and $\mathfrak{B}_2(s)$ are defined as
$\mathfrak{B}_1(s) = exp(\frac{-\alpha s}{\varepsilon})$ on $[0,1]$, $\mathfrak{B}_2(s) = exp(\frac{-\alpha(s-1)}{\varepsilon})$ on $[1,2]$.

Theorem 2.5. *Let the conditions (1.3)-(1.5) hold and let $\vec{w} = (w_1, w_2)^T$ be the solution of (2.3), (2.4). Then for proper choices of the constants C, C_1, C_2 and C_3, the following estimates hold. For $s \in [0,1]$,*

$$|w_1(s)| \leq C\mathfrak{B}_1(s) + C\varepsilon(1-s) + C\varepsilon^2(1 - \mathfrak{B}_1(s)),$$

$$|w_1^{(k)}(s)| \leq C\varepsilon^{-k}\mathfrak{B}_1(s),\ \textit{for}\ k = 1, 2, 3,$$

$$|w_2(s)| \leq C\varepsilon(1-s) + C\varepsilon^2(1 - \mathfrak{B}_1(s)),$$

$$|w'_2(s)| \leq C\varepsilon,\ |w''_2(s)| \leq C(\varepsilon + \mathfrak{B}_1(s)),\ |w_2^{(3)}(s)| \leq C\varepsilon^{-1}\mathfrak{B}_1(s).$$

For $s \in [1,2]$,

$$|w_1(s)| \leq C\mathfrak{B}_2(s) + C\varepsilon(2-s) + C\varepsilon^2(1 - \mathfrak{B}_2(s)),$$

$$|w_1^{(k)}(s)| \leq C\varepsilon^{-k}\mathfrak{B}_2(s),\ \textit{for}\ k = 1, 2, 3,$$

$$|w_2(s)| \leq C\varepsilon(2-s) + C\varepsilon^2(1 - \mathfrak{B}_2(s)),$$

$$|w'_2(s)| \leq C\varepsilon,\ |w''_2(s)| \leq C(\varepsilon + \mathfrak{B}_2(s)),\ |w_2^{(3)}(s)| \leq C\varepsilon^{-1}\mathfrak{B}_2(s).$$

Proof. The proof is by the method of steps. First, the bounds of $\vec{w}$ and its derivatives are estimated in $[0,1]$. Next these bounds of $\vec{w}$ and its derivatives are used to get the estimates in $[1,2]$. Let $s \in [0,1]$. Consider the barrier functions

$\vec{\mathfrak{Q}}^{\pm}(s) = \vec{\mathfrak{T}}(s) \pm \vec{w}(s)$, where $\vec{\mathfrak{T}} = (\mathfrak{T}_1, \mathfrak{T}_2)^T$, with
$\mathfrak{T}_1 = C_1\mathfrak{B}_1(s) + C_2\varepsilon(1-s) + C_3\varepsilon^2(1-\mathfrak{B}_1(s))$, $\mathfrak{T}_2 = C_2\varepsilon(1-s) + C_3\varepsilon^2(1-\mathfrak{B}_1(s))$.
Then,

$$\begin{aligned}(\vec{L}_1\vec{\mathfrak{T}})_1 &= C_1\alpha^4\varepsilon^{-1}\mathfrak{B}_1(s) - C_3\alpha^2\varepsilon\mathfrak{B}_1(s) - C_1\alpha^3\varepsilon^{-1}a_1(s)\mathfrak{B}_1(s)\\ &\quad - C_2\varepsilon a_1(s) + C_3\alpha\varepsilon a_1(s)\mathfrak{B}_1(s) - b_{11}(s)C_1\alpha^2\mathfrak{B}_1(s)\\ &\quad - (b_{11}(s)+b_{12}(s))[C_2\varepsilon(1-s) + C_3\varepsilon^2(1-\mathfrak{B}_1(s))]\\ &\le C_1\alpha^3\varepsilon^{-1}(\alpha - a_1(s))\mathfrak{B}_1(s) - C_2\varepsilon a_1(s) + C_3\alpha\varepsilon a_1(s)\mathfrak{B}_1(s)\\ &\quad - (b_{11}(s)+b_{12}(s))[C_2\varepsilon(1-s) + C_3\varepsilon^2(1-\mathfrak{B}_1(s))]\end{aligned}$$

and

$$\begin{aligned}(\vec{L}_1\vec{\mathfrak{T}})_2 &= -C_3\alpha^2\mathfrak{B}_1(s) - C_2\varepsilon a_2(s) + C_3\alpha a_2(s)\varepsilon\mathfrak{B}_1(s)\\ &\quad - b_{21}(s)C_1\alpha^2\mathfrak{B}_1(s)b_{21}(s)C_2\varepsilon(1-s) - b_{21}(s)C_3\varepsilon^2(1-\mathfrak{B}_1(s))\\ &\quad - b_{22}(s)C_2\varepsilon(1-s) - b_{22}(s)C_3\varepsilon^2(1-\mathfrak{B}_1(s))\end{aligned}$$

For proper choices of the constants C_1, C_2 and C_3, we have $\vec{\mathfrak{T}}^{\pm} \le \vec{0}$ and hence $L\vec{w} = \vec{0}$ implies that $\vec{L}_1\vec{\mathfrak{Q}}^{\pm} \le \vec{0}$. Also $\vec{\mathfrak{Q}}^{\pm}(0) \ge \vec{0}$ and $\vec{\mathfrak{Q}}^{\pm}(1) \ge \vec{0}$ on $(0,1)$. Then using the maximum principle in Chapter 4 of [10], to the operator $\vec{L}_1$, the bounds on w_1 and w_2 follows. The bounds on $w_i^{(k)}$ for $k = 1,2,3$ and $i = 1.2$ can be obtained by using the similar arguments dicussed in Chapter 2 of [10].
By using the same techniques and the bounds of $\vec{w}$ and its derivatives on $[0,1]$, the bounds on $\vec{w}$ and its derivatives are derived on $[1,2]$. □

3 Numerical Method

Spatial Discretization

To resolve boundary and interior layers, the solution space $[0,2]$ of the problem is discretized using a piecewise uniform Shishkin mesh with N mesh intervals as follows. Let $\Omega^N = \Omega_1^N \cup \Omega_2^N$ where $\Omega_1^N = \{s_j\}_{j=1}^{\frac{N}{2}-1}$, $\Omega_2^N = \{s_j\}_{j=\frac{N}{2}+1}^{N-1}$ and $s_{\frac{N}{2}} = 1$. Then $\bar{\Omega}_1{}^N = \{s_j\}_{j=1}^{\frac{N}{2}}$, $\bar{\Omega}_2^N = \{s_j\}_{j=\frac{N}{2}}^{N}$, $\bar{\Omega}_1^N \cup \bar{\Omega}_2^N = \bar{\Omega}^N = \{s_j\}_{j=0}^{N}$ and $\Gamma^N = \{0,2\}$ and N, the number of mesh elements is taken as a multiple of 4. The interval $[0,1]$ is divided into two subintervals $[0,\tau]$, $[\tau,1]$, where τ is a transition parameter defined by

$$\tau = \min\{\frac{1}{2}, \frac{2\varepsilon}{\gamma} lnN\}.$$

In each of the subinterval $[0,\tau]$, $[\tau,1]$, $\frac{N}{4}$ mesh points are placed, so that the mesh is piecewise uniform. When $\tau = \frac{1}{2}$, the mesh becomes uniform. Let H_1 and H_2 denote the step sizes in the intervals $[0,\tau]$ and $(\tau,1]$ respectively. Thus $H_1 = \frac{4\tau}{N}$, and $H_2 = \frac{4(1-\tau)}{N}$. Therefore, the possible two Shiskin meshes are represented by $\bar{\Omega}_1{}^N = \{s_j\}_{j=0}^{\frac{N}{2}}$, where

$$s_j = \begin{cases} jH_1, & \text{if} \quad 0 \le j \le N/4\\ \tau + (j - \frac{N}{4})H_2, & \text{if} \quad N/4 \le j \le N/2.\end{cases}$$

Similarly the interval $(1,2]$ is also divided into two subintervals $[1,1+\tau]$ and $(1+\tau,2]$ by using the same parameter τ.

The discrete problem

To solve the problem (1.1)-(1.2) numerically the following classical finite difference scheme is applied on the mesh $\bar{\Omega}^N$.
The discrete problem is now defined to be

$$\vec{L}^N\vec{Y}(s_j) = E\delta^2\vec{Y}(s_j)+A(s_j)D^+\vec{Y}(s_j)-B(s_j)\vec{Y}(s_j)-D(s_j)\vec{Y}(s_j-1) = \vec{f}(s_j) \text{ on } \Omega^N \quad (3.1)$$

$\vec{Y}(s_j-1) = \vec{\phi}(s_j-1)$ for $s_j \in \Omega_1^N$ and $\vec{Y} = \vec{y}$ on Γ^N.
The problem (3.1) can be written as,

$$\vec{L}_1^N\vec{Y}(s_j) = E\delta^2\vec{Y}(s_j) + A(s_j)D^+\vec{Y}(s_j) - B(s_j)\vec{Y}(s_j) = \vec{g}(s_j) \text{ on } \Omega_1^N \quad (3.2)$$

$$\vec{L}_2^N\vec{Y}(s_j) = E\delta^2\vec{Y}(s_j)+A(s_j)D^+\vec{Y}(s_j)-B(s_j)\vec{Y}(s_j)-D(s_j)\vec{Y}(s_j-1) = \vec{f}(s_j) \text{ on } \Omega_2^N \quad (3.3)$$

$$\vec{Y} = \vec{y} \text{ on } \Gamma^N,\ D^-\vec{Y}(s_{N/2}) = D^+\vec{Y}(s_{N/2})$$

where $\vec{Y}(s_j) = (Y_1(s_j), Y_2(s_j))^T$ and, for $1 \le j \le N-1$,

$$D^+\vec{Y}(s_j) = \frac{\vec{Y}(s_{j+1}) - \vec{Y}(s_j)}{h_{j+1}},\ D^-\vec{Y}(s_j) = \frac{\vec{Y}(s_j) - \vec{Y}(s_{j-1})}{h_j},$$

$$\delta^2\vec{Y}(s_j) = \frac{1}{\bar{h_j}}[D^+\vec{Y}(s_j) - D^-\vec{Y}(s_j)],\ \text{where } h_j = s_j - s_{j-1}, \bar{h_j} = \frac{h_j + h_{j+1}}{2}$$

. The discrete maximum principle and the discrete stability result are analogous to those for the continuous case.

Lemma 3.1. Discrete maximum principle

Let the conditions (1.3)-(1.5) hold. Then, for any mesh function $\vec{\Psi}$, the inequalities $\vec{\Psi} \ge \vec{0}$ on Γ^N, $\vec{L}_1^N\vec{\Psi} \le \vec{0}$ on Ω_1^N , $\vec{L}_2^N\vec{\Psi} \le \vec{0}$ on Ω_2^N and $D^+\vec{\Psi}(s_{N/2}) - D^-\vec{\Psi}(s_{N/2}) \le \vec{0}$ imply that $\vec{\Psi} \ge \vec{0}$ on $\bar{\Omega}^N$.

Proof. Let i^*, j^* be such that $\Psi_{i^*}(s_{j^*}) = \min_{i,j} \Psi_i(s_j)$
If $\Psi_{i^*}(s_{j^*}) \ge 0$ then there is nothing to prove.
Suppose $\Psi_{i^*}(s_{j^*}) < 0$ then $j^* \notin \Gamma^N$.
If $s_{j^*} \in \Omega_1^N$, then $\Psi_{i^*}(s_{j^*}) - \Psi_{i^*}(s_{j^*-1}) \le 0$, $\Psi_{i^*}(s_{j^*+1}) - \Psi_{i^*}(s_{j^*}) \ge 0$,
therefore $\delta^2\Psi_{i^*}(s_{j^*}) \ge 0$.
Define ε_{i^*} as

$$\varepsilon_{i^*} = \begin{cases} \varepsilon, & \text{if } i^* = 1 \\ 1 & \text{if } i^* = 2 \end{cases}$$

It follows that

$$\begin{aligned}(\vec{L}_1^N\vec{\Psi})_{i^*}(s_{j^*}) &= \varepsilon_{i^*}\delta^2\Psi_{i^*}(s_{j^*}) + a_{i^*}(s_{j^*})D^+\Psi_{i^*}(s_{j^*}) - \sum_{j=1}^{2}[b_{i^*j}(s_{j^*})\Psi_j(s_{j^*})] \\ &= \varepsilon_{i^*}\delta^2\Psi_{i^*}(s_{j^*}) + a_{i^*}(s_{j^*})D^+\Psi_{i^*}(s_{j^*}) - b_{i^*i^*}(s_{j^*})\Psi_{i^*}(s_{j^*}) - \\ &\quad \sum_{j=1, j\ne i^*}^{2} [b_{i^*j}(s_{j^*})\Psi_j(s_{j^*})] > 0, \text{which is a contradiction.}\end{aligned}$$

If $s_{j^*} \in \Omega_2^N$, then

$$\begin{aligned}(\vec{L}_2^N\vec{\Psi})_{i^*}(s_{j^*}) &= \varepsilon_{i^*}\delta^2\Psi_{i^*}(s_{j^*}) + a_{i^*}(s_{j^*})D^+\Psi_{i^*}(s_{j^*}) - \sum_{j=1}^{2}[b_{i^*j}(s_{j^*})\Psi_j(s_{j^*})] - d_{i^*}\Psi_{i^*}(s_{j^*}-1) \\ &= \varepsilon_{i^*}\delta^2\Psi_{i^*}(s_{j^*}) + a_{i^*}(s_{j^*})D^+\Psi_{i^*}(s_{j^*}) - b_{i^*i^*}(s_{j^*})\Psi_{i^*}(s_{j^*}) - \\ &\quad \sum_{j=1, j\ne i^*}^{2} [b_{i^*j}(s_{j^*})\Psi_j(s_{j^*})] - d_{i^*}\Psi_{i^*}(s_{j^*}-1) > 0, \text{which is a contradiction.}\end{aligned}$$

Because of the boundary values, the only other possibility is that $s_{j^*} = s_{N/2}$. Then $D^-\Psi_{i^*}(s_{N/2}) \leq 0$ and $D^+\Psi_{i^*}(s_{N/2}) \geq 0$, by hypothesis

$$D^+\Psi_{i^*}(s_{N/2}) - D^-\Psi_{i^*}(s_{N/2}) \leq 0 \; D^-\Psi_{i^*}(s_{N/2}) \leq 0 \leq D^+\Psi_{i^*}(s_{N/2})$$

and $D^+\Psi_{i^*}(s_{N/2}) \leq D^-\Psi_{i^*}(s_{N/2})$
And so $\Psi_{i^*}(s_{N/2-1}) = \Psi_{i^*}(s_{N/2}) = \Psi_{i^*}(s_{N/2+1}) < 0$. Then

$$(\vec{L}_1^N\vec{\Psi})_{i^*}(s_{\frac{N}{2}-1}) = \varepsilon_{i^*}\delta^2\Psi_{i^*}(s_{\frac{N}{2}-1}) + a_{i^*}(s_{\frac{N}{2}-1})\Psi'_{i^*}(s_{\frac{N}{2}-1}) - \sum_{j=1}^{2}[b_{i^*j}(s_{\frac{N}{2}-1})\Psi_j(s_{\frac{N}{2}-1})]$$

$$= \varepsilon_{i^*}\delta^2\Psi_{i^*}(s_{\frac{N}{2}-1}) - \sum_{j=1, j\neq i^*}^{2} b_{i^*j}(s_{\frac{N}{2}-1})\Psi_{i^*}(s_{\frac{N}{2}-1}) > 0, \text{a contradiction.}$$

This concludes the proof of the lemma. □

Lemma 3.2. Discrete stability result
Let the conditions (1.3)-(1.5) hold. Then, for any mesh function $\vec{\Psi}$, satisfying $D^+\vec{\Psi}(s_{N/2}) = D^-\vec{\Psi}(s_{N/2})$,

$$|\Psi_i(s_j)| \leq max\{\|\vec{\Psi}(s_0)\|, \|\vec{\Psi}(s_N)\|, \frac{1}{\beta}\|\vec{L}_1^N\vec{\Psi}\|_{\Omega_1^N}, \frac{1}{\gamma}\|\vec{L}_2^N\vec{\Psi}\|_{\Omega_2^N}\}, for\, i = 1, 2\, and \quad 0 \leq j \leq N.$$

Proof. Consider the barrier functions,

$$\vec{\Theta}^{\pm}(s_j) = max\{\|\vec{\Psi}(s_0)\|, \|\vec{\Psi}(s_N)\|, \frac{1}{\beta}\|\vec{L}_1^N\vec{\Psi}\|_{\Omega_1^N}, \frac{1}{\gamma}\|\vec{L}_2^N\vec{\Psi}\|_{\Omega_2^N}\}\vec{e} \pm \vec{\Psi}(s_j)$$

where $\vec{e} = (1, 1)^T$. Using the properties of $A(s_j)$, $B(s_j)$ and $D(s_j)$, it is not hard to find that

$$\vec{L}_1^N\vec{\Theta}^{\pm}(s_j) \leq \vec{0}, \text{for } s_j \in \Omega_1^N, \vec{L}_2^N\vec{\Theta}^{\pm}(s_j) \leq \vec{0}, \text{for } s_j \in \Omega_2^N.$$

At $j = \frac{N}{2}$, $D^+\vec{\Theta}^{\pm}(s_{N/2}) - D^-\vec{\Theta}^{\pm}(s_{N/2}) = \pm(D^+\vec{\Psi}(s_{N/2}) - D^-\vec{\Psi}(s_{N/2})) = \vec{0}$.
Therefore by previous lemma $\vec{\Theta}^{\pm} \geq \vec{0}$ on $\bar{\Omega}^N$, which leads the required result.

□

As in the continuous case the discrete solution $\vec{Y}$ can be decomposed into $\vec{V}$ and $\vec{W}$, which are defined to be the solutions of the following discrete problems.

$$\vec{L}_1^N\vec{V}(s_j) = \vec{g}(s_j),\ s_j \in \Omega_1^N,\ \vec{V}(0) = \vec{v}(0),\ \vec{V}(s_{\frac{N}{2}-1}) = \vec{v}(1-)$$

$$\vec{L}_2^N\vec{V}(s_j) = \vec{f}(s_j),\ s_j \in \Omega_2^N,\ \vec{V}(s_{\frac{N}{2}+1}) = \vec{v}(1+),\ \vec{V}(2) = \vec{v}(2),$$

and

$$\vec{L}_1^N\vec{W}(s_j) = \vec{0},\ s_j \in \Omega_1^N,\ \vec{W}(0) = \vec{w}(0),\ \vec{L}_2^N\vec{W}(s_j) = \vec{0},\ s_j \in \Omega_2^N,\ \vec{W}(2) = \vec{w}(2)$$

$$D^-\vec{V}(s_{\frac{N}{2}}) + D^-\vec{W}(s_{\frac{N}{2}}) = D^+\vec{V}(s_{\frac{N}{2}}) + D^+\vec{W}(s_{\frac{N}{2}}).$$

The error at each point $s_j \in \bar{\Omega}^N$ is denoted by $\vec{e}(s_j) = \vec{Y}(s_j) - \vec{y}(s_j)$, then the local truncation error $\vec{L}^N\vec{e}(s_j)$, $j \neq \frac{N}{2}$, has the decomposition $\vec{L}^N\vec{e}(s_j) = \vec{L}^N(\vec{V} - \vec{v})(s_j) + \vec{L}^N(\vec{W} - \vec{w})(s_j)$.

Theorem 3.3. *Let the conditions (1.3)-(1.5) hold. If $\vec{v}, \vec{w}$ are the smooth and singular components of the solution of (1.1)-(1.2) and if $\vec{V}$, $\vec{W}$ are the smooth and singular components of the solution of (3.1), then for $i = 1, 2, j \neq \frac{N}{2}$.*

$$|\vec{L}_1^N(\vec{V} - \vec{v})(s_j)| \leq CN^{-1},\ |\vec{L}_1^N(\vec{W} - \vec{w})(s_j)| \leq CN^{-1}lnN,\ 0 \leq j \leq \frac{N}{2} - 1$$

and

$$|\vec{L}_2^N(\vec{V} - \vec{v})(s_j)| \leq CN^{-1},\ |\vec{L}_2^N(\vec{W} - \vec{w})(s_j)| \leq CN^{-1}lnN,\ \frac{N}{2} + 1 \leq j \leq N.$$

Proof. The estimates for the derivatives of the smooth and singular components are found similar as in [10], the required bounds hold good. □

At $j = \frac{N}{2}$, for $i = 1, 2$. Let $h^* = max(h^-_{\frac{N}{2}}, h^+_{\frac{N}{2}})$.
Then

$$\begin{aligned}
|(D^+ - D^-)e_i(s_{\frac{N}{2}})| = |(D^+ - D^-)(y_i)(s_{\frac{N}{2}})| &\leq |(D^+ - \frac{d}{ds})(y_i)(s_{\frac{N}{2}})| + |(D^- - \frac{d}{ds})(y_i)(s_{\frac{N}{2}})| \\
&\leq \frac{1}{2} h^+_{\frac{N}{2}} |y_i''(\eta_1)|_{\eta_1 \in (1,2)} + \frac{1}{2} h^-_{\frac{N}{2}} |y_i''(\eta_2)|_{\eta_2 \in (0,1)} \\
&\leq C h^* \max_{\eta \in (0,2)} |y_i''(\eta)| \leq C h^* \max_{\eta \in (0,2)} (|v_i''(\eta)| + |w_i''(\eta)|) \\
&\leq C h^* (C + \frac{C\mathfrak{B}_1(\eta)}{\varepsilon_i^2})_{\eta \in (0,1]}
\end{aligned}$$

Using the estimate of $\mathfrak{B}_1$ and using the inequality $t \leq exp(t)$, we have

$$|(D^+ - D^-)e_i(s_{\frac{N}{2}})| \leq C \frac{h^*}{\varepsilon_i}, \text{ with } \varepsilon_1 = \varepsilon, \varepsilon_2 = 1. \tag{3.4}$$

Define a set of discrete barrier functions on $\bar{\Omega}^N$, for $i = 1, 2$ by

$$\Lambda_i(s_j) = \frac{\prod_{k=1}^{j} \left(1 + \frac{\gamma h_k}{\sqrt{2\varepsilon_i}}\right)}{\prod_{k=1}^{\frac{N}{2}} \left(1 + \frac{\gamma h_k}{\sqrt{2\varepsilon_i}}\right)}, \text{if } 0 \leq j \leq \frac{N}{2}, \Lambda_i(s_j) = \frac{\prod_{k=j}^{N-1} \left(1 + \frac{\gamma h_{k+1}}{\sqrt{2\varepsilon_i}}\right)}{\prod_{k=\frac{N}{2}}^{N-1} \left(1 + \frac{\gamma h_{k+1}}{\sqrt{2\varepsilon_i}}\right)}, \text{if } \frac{N}{2} \leq j \leq N \tag{3.5}$$

with ε_i as

$$\varepsilon_i = \begin{cases} \varepsilon, & \text{if } i = 1 \\ 1 & \text{if } i = 2. \end{cases}$$

From the definition of the barrier functions

$$\Lambda_i(0) = \Lambda_i(2) = 0, \quad \Lambda_i(1) = 1 \text{ for } i = 1, 2. \tag{3.6}$$

$$\text{Also using the assumption that } \varepsilon << 1, \Lambda_1(s_j) < \Lambda_2(s_j), \text{ for } 0 \leq j \leq N. \tag{3.7}$$

$$\text{Therefore, for } 0 \leq j \leq N,\ 0 \leq \Lambda_1(s_j) < \Lambda_2(s_j) \leq 1. \tag{3.8}$$

For $s_j \in \bar{\Omega}_1^N$

$$\begin{aligned}
D^+\Lambda_1(s_j) &= \frac{\Lambda_1(s_{j+1}) - \Lambda_1(s_j)}{h_{j+1}} \\
&= \frac{1}{h_{j+1}} \left[\frac{\prod_{k=1}^{j+1} \left(1 + \frac{\gamma h_k}{\sqrt{2\varepsilon}}\right)}{\prod_{k=1}^{\frac{N}{2}} \left(1 + \frac{\gamma h_k}{\sqrt{2\varepsilon}}\right)} - \frac{\prod_{k=1}^{j} \left(1 + \frac{\gamma h_k}{\sqrt{2\varepsilon}}\right)}{\prod_{k=1}^{\frac{N}{2}} \left(1 + \frac{\gamma h_k}{\sqrt{2\varepsilon}}\right)} \right] \\
&= \frac{1}{h_{j+1}} \frac{\prod_{k=1}^{j} \left(1 + \frac{\gamma h_k}{\sqrt{2\varepsilon}}\right)}{\prod_{k=1}^{\frac{N}{2}} \left(1 + \frac{\gamma h_k}{\sqrt{2\varepsilon}}\right)} \left[1 + (\gamma h_{j+1}/\sqrt{2\varepsilon}) - 1\right]
\end{aligned}$$

$$D^+\Lambda_1(s_j) = \frac{\gamma \Lambda_1(s_j)}{\sqrt{2\varepsilon}}, \text{ similarly } D^+\Lambda_2(s_j) = \frac{\gamma \Lambda_2(s_j)}{\sqrt{2}}$$

$$\text{Since } \varepsilon < 1, \text{therefore } D^+\Lambda_i(s_j) \leq \frac{\gamma\Lambda_i(s_j)}{\sqrt{2\varepsilon}} \text{ for } i = 1, 2. \tag{3.9}$$

And using the similar procedure, we have

$$D^-\Lambda_1(s_j) = \frac{\gamma\Lambda_1(s_j)}{\sqrt{2\varepsilon}(1 + (\gamma h_j/\sqrt{2\varepsilon}))}, \; D^-\Lambda_2(s_j) = \frac{\gamma\Lambda_2(s_j)}{\sqrt{2}(1 + (\gamma h_j/\sqrt{2}))}. \tag{3.10}$$

$$\begin{aligned}\text{and } \delta^2\Lambda_1(s_j) &= \frac{D^+\Lambda_1(s_j) - D^-\Lambda_1(s_j)}{\bar{h}_j} \\ &= \frac{1}{\bar{h}_j}\left[\frac{\gamma}{\sqrt{2\varepsilon}} - \frac{\gamma}{\sqrt{2\varepsilon}(1 + \gamma h_j/\sqrt{2\varepsilon})}\right]\Lambda_1(s_j)\end{aligned}$$

$$\delta^2\Lambda_1(s_j) \leq \frac{\gamma^2\Lambda_1(s_j)}{\varepsilon}, \text{ similarly } \delta^2\Lambda_2(s_j) \leq \gamma^2\Lambda_2(s_j)$$

$$\text{Since } \varepsilon < 1, \text{therefore } \delta^2\Lambda_i(s_j) \leq \frac{\gamma^2\Lambda_i(s_j)}{\varepsilon} \text{ for } i = 1, 2. \tag{3.11}$$

Proceeding in a similar way for $i = 1, 2$ and $s_j \in \bar{\Omega}_2^N$, we have

$$D^+\Lambda_1(s_j) = -\frac{\gamma\Lambda_1(s_j)}{\sqrt{2\varepsilon}(1 + \gamma h_{j+1}/\sqrt{2\varepsilon})}, \; D^+\Lambda_2(s_j) = -\frac{\gamma\Lambda_2(s_j)}{\sqrt{2}(1 + \gamma h_{j+1}/\sqrt{2})}, \tag{3.12}$$

$$D^-\Lambda_i(s_j) \leq -\frac{\gamma\Lambda_i(s_j)}{\sqrt{2\varepsilon}}, \delta^2\Lambda_i(s_j) \leq \frac{\gamma^2\Lambda_i(s_j)}{\varepsilon} \tag{3.13}$$

$$\text{Therefore from (3.11) and (3.13), } \; \varepsilon\delta^2\Lambda_i(s_j) \leq \gamma^2\Lambda_i(s_j) \tag{3.14}$$

Hence with $\varepsilon_i = \varepsilon$ for $i = 1$ and $\varepsilon_i = 1$ for $i = 2$,

$$\begin{aligned}(\vec{L}_1^N\vec{\Lambda})_i(s_j) &= \varepsilon_i\delta^2\Lambda_i(s_j) + a_i(s_j)D^+\Lambda_i(s_j) - \sum_{r=1}^{2} b_{ir}(s_j)\Lambda_r(s_j) \\ &\leq \gamma^2 + a_i(s_j)\frac{\gamma}{\sqrt{2\varepsilon_i}} - \sum_{r=1}^{2} b_{ir}(s_j)\Lambda_r(s_j) \text{ using (3.8)}\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}(\vec{L}_2^N\vec{\Lambda})_i(s_j) &= \varepsilon_i\delta^2\Lambda_i(s_j) + a_i(s_j)D^+\Lambda_i(s_j) - \sum_{r=1}^{2} b_{ir}(s_j)\Lambda_r(s_j) - d_i(s_j)\Lambda_i(s_j - 1) \\ &\leq \gamma^2 - a_i(s_j)\frac{\gamma\Lambda_i(s_j)}{\sqrt{2\varepsilon_i}(1 + \frac{\gamma h_{j+1}}{\sqrt{2\varepsilon_i}})} - \sum_{r=1}^{2} b_{ir}(s_j)\Lambda_r(s_j) - d_i(s_j)\Lambda_i(s_j - 1)\end{aligned} \tag{3.16}$$

From equations (3.10), (3.12) and (3.8),

$$(D^+ - D^-)\Lambda_i(s_j) \leq -\frac{C}{\sqrt{\varepsilon_i}}, \text{for } j = N/2. \tag{3.17}$$

The following theorem gives the first order parameter-uniform error estimate .

Theorem 3.4. *Let $\vec{y}(s_j)$ be the solution of the problem (1.1)-(1.2) and $\vec{Y}(s_j)$ be the solution of the problem (3.1). Then,*

$$\|\vec{Y}(s_j) - \vec{y}(s_j)\| \leq CN^{-1}lnN, \; 0 \leq j \leq N$$

Proof. Consider the mesh function $\vec{\eth}$ given by

$$\eth_i(s_j) = C_1 N^{-1} lnN + C_2 \frac{h^*}{\sqrt{\varepsilon_i}} \Lambda_i(s_j) \pm e_i(s_j), i = 1, 2,\ 0 \leq j \leq N.$$

where C_1 and C_2 are constants suitably chosen. And $\varepsilon_i = \varepsilon$ for $i = 1$, $\varepsilon_i = 1$ for $i = 2$. Then, $\eth_i(0) = \eth_i(2) = C_1 N^{-1} lnN \geq 0$
And using (3.15) and theorem 3.3

$$\begin{aligned}
(\vec{L}_1^N \vec{\eth})_i(s_j) &= -\sum_{r=1}^{2} b_{ir}(s_j) C_1 N^{-1} lnN + C_2 \frac{h^*}{\sqrt{\varepsilon_i}} (\vec{L}_1^N \vec{\Lambda})_i(s_j) \pm (\vec{L}_1^N \vec{e})_i(s_j) \\
&\leq -C_1 \sum_{r=1}^{2} b_{ir}(s_j) N^{-1} lnN + C_2 \frac{h^*}{\sqrt{\varepsilon_i}} \frac{a_i(s_j)\gamma}{\sqrt{2\varepsilon_i}} + C_2 \frac{h^*}{\sqrt{\varepsilon_i}} \left[\gamma^2 - \sum_{r=1}^{2} b_{ir}(s_j) \Lambda_r(s_j) \right] \\
&\quad \pm CN^{-1} lnN \\
&= -C_1 \sum_{r=1}^{2} b_{ir}(s_j) N^{-1} lnN + C_2 K_1 + C_2 \frac{h^*}{\sqrt{\varepsilon_i}} \left[\gamma^2 - \sum_{r=1}^{2} b_{ir}(s_j) \Lambda_r(s_j) \right] \pm CN^{-1} lnN
\end{aligned}$$

where K_1 is a constant. Let $\mu_i(s_j) = \left[\gamma^2 - \sum_{r=1}^{2} b_{ir}(s_j) \Lambda_r(s_j) \right]$, for $i = 1, 2$.
Then choosing $C_1 > C_2(\|\vec{\mu}\| + K_1) + C$, we have $(\vec{L}_1^N \vec{\eth})_i(s_j) \leq 0$, on Ω_1^N, for $i = 1, 2$
And using (3.16) and theorem 3.3

$$\begin{aligned}
(\vec{L}_2^N \vec{\eth})_i(s_j) &= -\sum_{r=1}^{2} b_{ir}(s_j) C_1 N^{-1} lnN - d_i(s_j) C_1 N^{-1} lnN + C_2 \frac{h^*}{\sqrt{\varepsilon_i}} (\vec{L}_2^N \vec{\Lambda})_i(s_j) \pm (\vec{L}_2^N \vec{e})_i(s_j) \\
&\leq -C_1 (\sum_{r=1}^{2} b_{ir}(s_j) + d_i(s_j)) N^{-1} lnN - C_2 \frac{h^*}{\sqrt{\varepsilon_i}} \frac{a_i(s_j)\gamma}{\sqrt{2\varepsilon_i}(1 + \frac{\gamma h_{j+1}}{\sqrt{2\varepsilon_i}})} \Lambda_i(s_j) + C_2 \frac{h^*}{\sqrt{\varepsilon_i}} \Bigg[\gamma^2 \\
&\quad - \sum_{r=1}^{2} b_{ir}(s_j) \Lambda_r(s_j) - d_i(s_j) \Lambda_i(s_j - 1) \Bigg] \pm CN^{-1} lnN \\
&= -C_1 (\sum_{r=1}^{2} b_{ir}(s_j) + d_i(s_j)) N^{-1} lnN + C_2 K_2 + C_2 \frac{h^*}{\sqrt{\varepsilon_i}} \Bigg[\gamma^2 - \sum_{r=1}^{2} b_{ir}(s_j) \Lambda_r(s_j) \\
&\quad - d_i(s_j) \Lambda_i(s_j - 1) \Bigg] \pm CN^{-1} lnN
\end{aligned}$$

where K_2 is a constant. Let $\eta_i(s_j) = \gamma^2 - \sum_{r=1}^{2} b_{ir}(s_j) \Lambda_r(s_j) - d_i(s_j) \Lambda_i(s_j - 1)$.
Then choosing $C_1 > C_2(\|\vec{\eta}\| + |K_2|) + C$, $(\vec{L}_2^N \vec{\eth})_i(s_j) \leq 0$, on Ω_2^N, for $i = 1, 2$.
Also,

$$\begin{aligned}
D^+ \eth_i(1) - D^- \eth_i(1) &= \frac{h^*}{\sqrt{\varepsilon_i}} C_2 (D^+ - D^-) \Lambda_i(1) \pm (D^+ - D^-) e_i(s_{N/2}) \\
&\leq -\frac{h^*}{\sqrt{\varepsilon_i}} C_2 \frac{C}{\sqrt{\varepsilon_i}} \pm \frac{Ch^*}{\varepsilon_i} \leq 0
\end{aligned}$$

Hence using Lemma 3.1 for $\vec{\eth}$, it follows that $\eth_i(s_j) \geq 0$ for $i = 1, 2,\ 0 \leq j \leq N$. Also from (3.8), $\Lambda_i(s_j) \leq 1$ for $i = 1, 2,\ 0 \leq j \leq N$, therefore for sufficiently large N,

$$\|(\vec{Y} - \vec{y})(s_j)\| \leq CN^{-1} lnN,$$

which completes the proof. □

4 Numerical Illustration

Here we present an example to verify the efficacy of the above proposed numerical method. Consider the following system of two partially singularly perturbed delay differential equations

$$\vec{L}\vec{y}(s) = E\vec{y}''(s) + A(s)\vec{y}'(s) - B(s)\vec{y}(s) - D(s)\vec{y}(s-1) = \vec{f}(s)\; on(0,2)$$

$$with\, \vec{y} = (1+s, 2s)^T\; on\,[-1,0]\; and\, \vec{y}(2) = \vec{0}$$

where $\vec{f}(s) = (s^2, 2s)^T$. $E = diag(\varepsilon, 1)$, $A(s) = diag(3,2)$, $B(s) = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$ and $D(s) = \begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix}$

The maximum pointwise errors and the rate of convergence are calculated and are presented in Table 1.

Table 1.
Maximum pointwise errors D_ε^N, D^N, p^N, p^* and $C_{p^*}^N$ generated for the example.

η	Number of mesh points N			
	64	128	256	512
0.500E+00	0.155E-01	0.110E-01	0.712E-02	0.409E-02
0.250E+00	0.171E-01	0.127E-01	0.839E-02	0.527E-02
0.125E+00	0.179E-01	0.135E-01	0.911E-02	0.586E-02
0.625E-01	0.184E-01	0.139E-01	0.947E-02	0.616E-02
0.312E-01	0.186E-01	0.141E-01	0.965E-02	0.630E-02
D^N	0.186E-01	0.141E-01	0.965E-02	0.630E-02
p^N	0.401E+00	0.548E+00	0.614E+00	
C_p^N	0.407E+00	0.407E+00	0.367E+00	0.317E+00
The order of $\vec{\varepsilon}$-uniform convergence $p^* = 0.4007015E+00$				
Computed $\vec{\varepsilon}$-uniform error constant, $C_{p^*}^N = 0.4065366E+00$				

5 Conclusion

A weakly coupled system of PSPDDEs in which one equation involves a singular perturbation parameter ε and each equation has a term with delay at an interior of the domain has been considered. A finite difference numerical scheme with a piecewise uniform Shishkin mesh, which is fitted to resolve the boundary and interior layers was constructed. First order parameter uniform convergence has been obtained in the maximum norm.

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