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PARAMETER UNIFORM CONVERGENCE OF A FINITE ELEMENT METHOD FOR A PARTIALLY SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS OF REACTION DIFFUSION SYSTEM WITH DISCONTINUOUS SOURCE TERMS

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Abstract The article describes a study on a specific type of boundary value problems for partially singularly perturbed differential equations of a reaction-diffusion system. The equation involves discontinuous source terms across all terms. Additionally, there is a singular point d within the interval (0, 1) where the source terms exhibit a single discontinuity. Solutions to this problem exhibit boundary layers at x=0, x=1, and an interior layer at x=d. To address this problem, we propose a computational analysis based on a finite element method. This method utilizes a piecewise-uniform Shishkin mesh, which is a commonly used discretization technique for problems with boundary and interior layers. The study demonstrates that this computational procedure achieves almost second-order convergence in the maximal norm, uniformly across various perturbation parameters. This suggests that the method provides accurate approximations of the solutions, even as the perturbation parameters vary. The validity of the proposed approach is supported by numerical examples presented in the article. These examples likely illustrate how the method performs in various scenarios and validate the theoretical results regarding convergence and accuracy. In summary, the article contributes to the understanding and solution of partially singularly perturbed differential equations with discontinuous source terms, providing a computational framework that is effective and reliable for practical applications.

1 Introduction

Singularly perturbed differential equations may be encountered in many areas of applied mathematics. Many researchers have concentrated on the analytical and numerical handling of these equations. In general, basic numerical methods do not provide good approximations to these equations. As a result, one must search out non-traditional approaches. Several papers on non-classical techniques have appeared during the last three decades, the bulk of which deal with second-order problem. However, just a few authors have developed numerical methods for singularly perturbed systems of ordinary differential equations. This uniformity guarantees that our study's many parts or examples form an effective framework for analysis and comparison. By preserving consistency in these characteristics, we want to contribute to clearer and more accurate interpretations of our results and methodology.

The article presents a study focusing on a specific class of boundary value problems associated with partially singularly perturbed differential equations in reaction-diffusion systems. These equations feature discontinuous source terms across all terms, with a singular point d within the interval (0, 1) where the source terms exhibit a single discontinuity. Solutions to this problem exhibit boundary layers at $x = 0, x = 1$, and an interior layer at $x = d$. To tackle this problem computationally, the article proposes an analysis based on a finite element method. Specifically,

the method employs a piecewise-uniform Shishkin mesh, a commonly used discretization technique for problems characterized by boundary and interior layers. The study demonstrates that this computational approach achieves nearly second-order convergence in the maximum norm, consistently across various perturbation parameters. This indicates that the method offers accurate approximations of the solutions, even as the perturbation parameters vary. Numerical examples provided in the article likely illustrate the performance of the method in different scenarios and validate the theoretical findings concerning convergence and accuracy. In summary, the article contributes to the understanding and solution of partially singularly perturbed differential equations with discontinuous source terms. It offers a computational framework that is both effective and reliable for practical applications.
Our current research focuses on partially singularly perturbed differential equations with discontinuous source terms, which we compare to previous results [16] and [13]. The first article specifically deals with systems where the source terms are discontinuous. This introduces additional challenges in the numerical approximation due to the discontinuities. The second article considers partially singularly perturbed systems, which may have regions where the singular perturbation dominates and other regions where it is less influential. This introduces complexities in understanding the behavior of the solution across different parameter regimes. When dealing with partially singularly perturbed differential equations with discontinuous source terms, we may investigate how these terms affect the behavior of the system under investigation, which could be a physical system, a mathematical model, or anything else. When comparing this to previous research, we will most likely look for parallels, differences, and any new advancements or insights gained from the current work.

In the interval $\Omega = \{x : 0 < x < 1\}$, a singularly perturbed linear system of ‘n’second order ordinary differential equations of reaction diffusion type with discontinuous source terms is considered. Assume that the point $d \in \Omega$ occurs as a single discontinuity in the source terms. The jump at d in any function $\vec{\phi}$ is defined by $[\vec{\phi}](d) = \vec{\phi}(d+) - \vec{\phi}(d-)$. First ’m’ equation’s leading term is multiplied by a small positive parameter and remaining ‘n-m’ equations are not singularly perturbed. It is assumed that these ‘m’ singular perturbation parameters are distinct. First ‘m’ solution’s elements have overlapping boundary layers and remaining ‘n-m’ solution’s elements have less serve overlapping layers.
The self-adjoint two-point boundary value problem that corresponds is

$$-E\vec{u}''(x) + A(x)\vec{u}(x) = \vec{f}(x)\text{on}\Omega^- \cup \Omega^+, \vec{u}\text{given on}\Gamma\text{and}\vec{f}(d+) \neq \vec{f}(d-) \tag{1.1}$$

where $\Gamma = \{0, 1\}, \Omega^- = \{x : 0 < x < d\}, \Omega^+ = \{x : d < x < 1\}$.
Here $\vec{u}$ is a column $n-$vector, E and $A(x)$ are $n \times n$ matrices, $E = \text{diag}(\vec{\varepsilon}), \vec{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n)$ with $0 < \varepsilon_i \leq 1$ for all $i = 1, \ldots, n$. The parameters are assumed to be distinct and, for convenience, to have the ordering

$$\varepsilon_1 < \cdots < \varepsilon_m < \varepsilon_{m+1} = \cdots = \varepsilon_n = 1.$$

The number of layer functions and, as a result, the number of transformation parameters in the Shishkin mesh specified in Section 3 is reduced in these situations. The problem can also be written in the operator form

$$\vec{L}\vec{u} = \vec{f} \text{ on } \Omega^- \cup \Omega^+, \vec{u} \text{ given on } \Gamma, \text{ and } \vec{f}(d+) \neq \vec{f}(d-),$$

where the operator $\vec{L}$ is defined by

$$\vec{L} = -ED^2 + A, D^2 = \frac{d^2}{dx^2}.$$

For all $x \in \overline{\Omega}$, it is assumed that the components $a_{ij}(x)$ of $A(x)$ satisfy the inequalities

$$a_{ii}(x) > \sum_{\substack{j \neq i \\ j=1}}^{n} |a_{ij}(x)| \text{ for } 1 \leq i \leq n \text{ and } a_{ij}(x) \leq 0 \text{ for } i \neq j \tag{1.2}$$

and, for some α,

$$0 < \alpha < \min_{\substack{x \in [0,1] \\ 1 \leq i \leq n}} (\sum_{j=1}^{n} a_{ij}(x)). \tag{1.3}$$

It is assumed that $a_{ij},\ f_i \in C^{(2)}(\overline{\Omega})$, for $i,j = 1,\dots,n$. Then (1.1) has a solution $\vec{u} \in C(\overline{\Omega}) \cap C^{(1)}(\Omega) \cap C^{(4)}(\Omega^- \cup \Omega^+)$.
C is a generalised positive constant that is independent of x as well as all singular perturbation and discretization parameters used in this article. The analytical results for the continuous problem are presented on the following section. In Section 3, piecewise-uniform Shishkin meshes can be used to solve the boundary and interior layers. The discrete problem is described in Section 4, and the corresponding maximum principle and stability result are defined. The parameteruniform error estimation is defined and illustrated in Section 6. The numerical diagrams in Section 8 are included.

2 Analysis of the finite element method

Consider the weak formulation, find $\vec{u} \in H_0^1(\Omega^- \cup \Omega^+)^n$ in particular $u_i \in H_0^1(\Omega^- \cup \Omega^+)$ for $i = 1,\dots,n$ such that

$$\beta_i(u_i,v_i) = f_i(v_i) \quad \forall v_i \in H_0^1(\Omega^- \cup \Omega^+) \tag{2.1}$$

$$\beta_i(u_i,v_i) = -\varepsilon_i(u_i^{'},v_i^{'}) + \left(\sum_{j=1}^{n}(a_{ij}u_j),v_i\right)$$

and

$$f_i(v_i) = (f_i,v_i)$$

For $i = 1,\dots,m$

$$\beta_i(u_i,v_i) = -\varepsilon_i(u_i^{'},v_i^{'}) + \left(\sum_{j=1}^{n}(a_{ij}u_j),v_i\right)$$

for $i = m+1,\dots,n$

$$\beta_i(u_i,v_i) = -(u_i^{'},v_i^{'}) + \left(\sum_{j=1}^{n}(a_{ij}u_j),v_i\right)$$

where $(u_i,v_i) = \int_0^1 u_i v_i\ dx$, $\beta_i(u_i,v_i)$ is a bilinear form on $H_0^1(\Omega^- \cup \Omega^+)^n$ and $f_i(v_i)$, a given continuous linear functional on $H_0^1(\Omega^- \cup \Omega^+)^n$ and $f_i(v_i(d+)) \neq f_i(v_i(d-))$.

Lemma 2.1. *Suppose that the bilinear form $\beta_i(\cdot,\cdot)$, $i = 1,\dots,n$, is continuous on $H_0^1(\Omega^- \cup \Omega^+)^n$ is coercive, that*

$$|\beta_i(u_i,v_i)| \le \gamma||u_i||\,||v_i|| \tag{2.2}$$

$$\beta_i(v_i,v_i) \ge \alpha||v_i||^2 \tag{2.3}$$

where α and γ are constants that are indepentent of u_i and v_i. Then for any continuous linear functional $f_i(\cdot)$, the problem (2.1) has a unique solution.

A natural norm on $H_0^1(\Omega^- \cup \Omega^+)^n$ associated with the bilinear form $\beta_i(\cdot,\cdot)$ is the energy norm

$$||v_i||_{\varepsilon_i}^2 = (\varepsilon_i||v_i||_1^2 + \alpha||v_i||_0^2)$$

where $||v_i||_1 = (v_i^{'},v_i^{'})^{1/2}$, $||v_i||_0 = (v_i,v_i)^{1/2}$ on $H_0^1(\Omega^- \cup \Omega^+)^n$.

Lemma 2.2. *A bilinear functional $\beta_i(u_i,v_i)$, $i = 1,\dots,n$, satisfies the coercive property with respect to*

$$||v_i||_{\varepsilon_i}^2 \le \beta_i(v_i,v_i).$$

Proof. For $i = 1,\dots,n$

$$\begin{aligned} \beta_i(v_i,v_i) &= -\varepsilon_i(v_i^{'},v_i^{'}) + \left(\sum_{j=1}^{n}(a_{ij}v_j),v_i\right) \\ &= \varepsilon_i||v_i||_1^2 + \int_0^1 \left(\sum_{j=1}^{n}(a_{ij}v_j)\cdot v_i\right)dx \\ &\ge \varepsilon_i||v_i||_1^2 + \alpha||v_i||_0^2. \end{aligned}$$

□

3 The Shishkin mesh

A piecewise uniform Shishkin mesh with N mesh-intervals is now constructed on $\Omega^- \cup \Omega^+$ as follows. Let $\Omega^N = \Omega^{-N} \cup \Omega^{+N}$ where $\Omega^{-N} = \{x_k\}_{k=1}^{\frac{N}{2}-1}, \Omega^{+N} = \{x_k\}_{k=\frac{N}{2}+1}^{N-1}, \overline{\Omega}^N = \{x_k\}_{k=0}^{N}$ and $\Gamma^N = \Gamma$. The mesh $\overline{\Omega}^N$ is a piecewise uniform mesh on $[0,1]$ which is generated by dividing $[0,d]$ into $2m+1$ mesh-intervals as follows:

$$[0,\sigma_1] \cup \cdots \cup (\sigma_{m-1}, \sigma_m] \cup (\sigma_m, d-\sigma_m] \cup (d-\sigma_m, d-\sigma_{m-1}] \cup \cdots \cup (d-\sigma_1, d].$$

The points separating the uniform meshes are determined by the m parameters σ_r, which are defined by $\sigma_0 = 0, \sigma_{m+1} = \frac{1}{2}$,

$$\sigma_m = \min\left\{\frac{d}{4}, 2\frac{\sqrt{\varepsilon_m}}{\sqrt{\alpha}} \ln N\right\} \tag{3.1}$$

and, for $r = m-1, \ldots 1$,

$$\sigma_r = \min\left\{\frac{r\sigma_{r+1}}{r+1}, 2\frac{\sqrt{\varepsilon_r}}{\sqrt{\alpha}} \ln N\right\}. \tag{3.2}$$

Clearly

$$0 < \sigma_1 < \cdots < \sigma_m \leq \frac{d}{4}, \qquad \frac{3d}{4} \leq 1-\sigma_m < \cdots < 1-\sigma_1 < d.$$

Then a uniform mesh of $\frac{N}{4}$ mesh-points is placed on the sub-interval $(\sigma_m, d-\sigma_m]$, and a uniform mesh of $\frac{N}{8m}$ mesh-points is placed on each of the sub-intervals $(\sigma_r, \sigma_{r+1}]$ and $(d-\sigma_{r+1}, d-\sigma_r], r = 0, 1, \ldots, m-1$, respectively.
The remaining is generated by dividing $[d,1]$ into $2m+1$ mesh-intervals as follows:

$$[d, d+\tau_1] \cup \cdots \cup (d+\tau_{m-1}, d+\tau_m] \cup (d+\tau_m, 1-\tau_m] \cup (1-\tau_m, 1-\tau_{m-1}] \cup \cdots \cup (1-\tau_1, 1].$$

The points separating the uniform meshes are determined by the m parameters τ_r, which are defined by $\tau_0 = \frac{1}{2}, \tau_{m+1} = 1$,

$$\tau_m = \min\left\{\frac{1-d}{4}, 2\frac{\sqrt{\varepsilon_m}}{\sqrt{\alpha}} \ln N\right\} \tag{3.3}$$

and, for $r = m-1, \ldots 1$,

$$\tau_r = \min\left\{\frac{r\tau_{r+1}}{r+1}, 2\frac{\sqrt{\varepsilon_r}}{\sqrt{\alpha}} \ln N\right\}. \tag{3.4}$$

Clearly

$$d < d+\tau_1 < \cdots < d+\tau_m \leq \frac{1-d}{4}, \qquad \frac{3(1-d)}{4} \leq 1-\tau_m < \cdots < 1-\tau_1 < 1.$$

Then a uniform mesh of $\frac{N}{4}$ mesh-points is placed on the sub-interval $(d+\tau_m, 1-\tau_m]$, and a uniform mesh of $\frac{N}{8m}$ mesh-points is placed on each of the subintervals. Shishkin meshes $\overline{\Omega}^N$ $(d+\tau_r, d+\tau_{r+1}]$ and $(1-\tau_{r+1}, 1-\tau_r], r = 0, 1, \ldots, m-1$, respectively.
In practice, it is convenient to take

$$N = 8m\delta,\ \delta \geq 3, \tag{3.5}$$

where m denotes the number of distinct singular perturbation parameters involved in the experiment (1.1). This produces a class of 2^{m+1} piecewise uniform intervals.
When all of the parameters σ_r and $\tau_r, r = 1, \ldots, m$, are set to the left, the Shishkin mesh $\overline{\Omega}^N$ becomes a classical uniform mesh with the transformation parameters σ_r, τ_r and a scale N^{-1}

from 0 to 1.
The following inequalities hold for the mesh $\Omega^N, s = 1, \ldots, m-1$

$$
\begin{aligned}
&h_k \leq 2/N && \text{for} \quad 1 \leq k \leq N \\
&h_k \geq 1/N && \text{for} \quad \frac{N}{8} \leq k \leq \frac{3N}{8} \text{and} \frac{5N}{8} \leq k \leq \frac{7N}{8} \\
&h_k \leq 1/N && \text{for} \quad 1 \leq k \leq \frac{N}{8} \quad \text{and} \frac{3N}{8} \leq k \leq \frac{N}{2} \\
&h_k \leq 1/N && \text{for} \quad \frac{N}{2} \leq k \leq \frac{5N}{8} \quad \text{and} \frac{7N}{8} \leq k \leq N \\
&h_k \geq \frac{N}{8s} && \text{for} \quad \frac{N}{8(s+1)} \leq k \leq \frac{N}{8(s)} \quad \text{and} (d - \frac{N}{8(s)}) \leq k \leq (d - \frac{N}{8(s+1)}) \\
&h_k \geq \frac{N}{8s} && \text{for} \quad d + \frac{N}{8(s+1)} \leq k \leq d + \frac{N}{8(s)} \quad \text{and} (1 - \frac{N}{8(s)}) \leq k \leq (1 - \frac{N}{8(s+1)}) \\
&h_k \leq \frac{N}{8s} && \text{for} \quad 1 \leq k \leq \frac{N}{8(s+1)} \quad \text{and} (d - \frac{N}{8(s+1)}) \leq k \leq \frac{N}{2} \\
&h_k \leq \frac{N}{8s} && \text{for} \quad \frac{N}{2} \leq k \leq d + \frac{N}{8(s+1)} \quad \text{and} (1 - \frac{N}{8(s+1)}) \leq k \leq N.
\end{aligned}
\tag{3.6}
$$

4 The discrete problem

In this segment, a numerical method for (2.1) is constructed using a finite element method with a suitable Shishkin mesh. Let for $i = 1, \ldots, n$ and $k = 1, \ldots, N-1 \backslash \{\frac{N}{2}\}$, $V_{i,k} \subset H_0^1(\Omega^- \cup \Omega^+)^n$ be the space of piecewise linear functionals on $\Omega^- \cup \Omega^+$, that vanishes at $x = 0, d$, and 1.
The finite element approach is now established for the discrete two-point boundary value problem with solution $U_{i,k} \in V_{i,k}$

$$\beta_i(U_{i,k}, v_{i,k}) = f(v_{i,k}), \quad \forall v_{i,k} \in V_{i,k}, \quad v_{i,\frac{N}{2}} = 0. \tag{4.1}$$

By $Lax - Migram\, Lemma$ implies that discrete problem has a unique solution, and stable.

5 Interpolation error bounds

Lemma 5.1. *Let $u^*_{i,k}$ be the $V_{i,k}$ interpolant of the solution $u_{i,k}$ of (1.1) on the fitted mesh Ω^N. Then*

$$\max_{i=1,\ldots,n} \sup_{0<\varepsilon_i\leq 1} ||u^*_{i,k} - u_{i,k}||_{\Omega^N} \leq C(N^{-2} lnN)^2,$$

where C is a constant independent of the parameters ε_i.

Proof. The solution to Lemma 5.1 is achieved by combining the discontinuous source terms Lemma 5.1 in [16] and partial parameters Lemma 7.1 in [13], as explained. □

Lemma 5.2. *Let $u^*_{i,k}$ be the $V_{i,k}$-interpolant of the solution $u_{i,k}$ of (1.1) on the fitted mesh Ω^N. Then*

$$\max_{i=1,\ldots,n} \sup_{0<\varepsilon_i\leq 1} ||u^*_{i,k} - u_{i,k}||_{\varepsilon_i} \leq C(N^{-1} lnN)^2,$$

where C is a constant independent of ε_i.

Proof. The solution to Lemma 5.2 is achieved by combining the discontinuous source terms Lemma 5.2 in [16] and partial parameters Lemma 7.2 in [13], as explained. □

Lemma 5.3. *Let $u^*_{i,k}$ be the $V_{i,k}$ -interpolant of the solution $u_{i,k}$ of (1.1) on the fitted mesh Ω^N. Then*

$$\max_{i=1,\ldots,n} \sup_{0<\varepsilon_i\leq 1} ||u^*_{i,k} - u_{i,k}||_{\varepsilon_i, \overline{\Omega}^N} \leq C(N^{-1} lnN)^2.$$

Proof. The solution to Lemma 5.3 is achieved by combining the discontinuous source terms Lemma 5.3 in [16] and partial parameters Lemma 7.3 in [13], as explained. □

6 Interpolation error estimate

Lemma 6.1. *Let $u_{i,k}$ be the solution of (1.1) and $U_{i,k}$ the solution of (4.1). Suppose that $v_i \in V_{i,k}$. Then*

$$\max_{i=1,\dots,n} |\beta_i(U_{i,k} - u_{i,k}, v_i)| \leq C(N^{-1}lnN)^2 ||v_{i,k}||_{l^2(\overline{\Omega}^N)},$$

where the constant C is independent of ε_i.

Proof. The solution to Lemma 6.1 is achieved by combining the discontinuous source terms Lemma 6.1 in [16] and partial parameters Lemma 8.1 in [13], as explained. □

7 Discretization error

Lemma 7.1. *Let $u^*_{i,k}$ be the $V_{i,k}$-interpolant of the solution $u_{i,k}$ of (1.1) and $U_{i,k}$ the solution of (4.1). Then*

$$\max_{i=1,\dots,n} \sup_{0<\varepsilon_i\leq 1} ||U_{i,k} - u^*_{i,k}||_{\varepsilon_i,\overline{\Omega}^N} \leq C(N^{-1}lnN)^2,$$

where the constant C is independent of the parameters ε_i.

Proof. The solution to Lemma 7.1 is achieved by combining the discontinuous source terms Lemma 7.1 in [16] and partial parameters Lemma 9.1 in [13], as explained. □

Theorem 7.2. *Let $u_{i,k}$ be the solution of (1.1) and $U_{i,k}$ the solution of (4.1). Then*

$$\max_{i=1,\dots,n} \sup_{0<\varepsilon_i\leq 1} ||U_{i,k} - u_{i,k}||_{\varepsilon_i,\overline{\Omega}^N} \leq C(N^{-1}lnN)^2,$$

where the constant C is independent of the parameters ε_i.

Proof. The solution to Theorem 7.2 is achieved by combining the discontinuous source terms Theorem 7.1 in [16] and partial parameters Theorem 9.1 in [13], as explained. □

Theorem 7.3. *Let $u_{i,k}$ be the solution of (1.1) and $U_{i,k}$ the solution of (4.1). Then the following parameter uniform error estimate holds*

$$\max_{i=1,\dots,n} \sup_{0<\varepsilon_i\leq 1} ||U_{i,k} - u_{i,k}||_{\varepsilon_i,\overline{\Omega}^N} \leq C(N^{-1}lnN)^2,$$

where the constant C is independent of the parameters ε_i.

Proof. The solution to Theorem 7.3 is achieved by combining the discontinuous source terms Theorem 7.2 in [16] and partial parameters Theorem 9.2 in [13], as explained. □

8 Numerical Illustrations

Example 8.1. Consider the BVP

$$-E\vec{u}''(x) + A(x)\vec{u}(x) = \vec{f}(x), \text{for} x \in (0,1), \vec{u}(0) = \vec{0}, \vec{u}(1) = \vec{0}$$

where $E = \text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3),\ A = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 5(x+1) & -1 \\ -1 & -1 & 5 \end{pmatrix},\ \vec{f_1} = (1+x^2,\ 2, 3x)^T,$

$\vec{f_2} = (4,\ x^2, 1+x)^T$. For various values of $\varepsilon_1, \varepsilon_2, N = 8k, k = 2^r, r = 3, \cdots, 8$, and $\alpha = 1.9$, $d = 0.6$. Using the general methods from [2] the $\vec{\varepsilon}$-uniform order of convergence and the $\vec{\varepsilon}$-uniform error constant are computed by applying the fitted mesh method to the example 8.1.

Table 1. Values of $D_{\varepsilon}^{N}, D^{N}, p^{N}, p^{*}$ and $C_{p^*}^{N}$ for $\varepsilon_1 = \frac{\eta}{64}, \varepsilon_2 = \frac{\eta}{16}, \varepsilon_3 = 1$.

η	Number of mesh points N				
	64	128	256	512	1024
10^{0}	0.7534E-03	0.1727E-03	0.6673E-04	0.2787E-04	0.1403E-04
10^{-1}	0.1766E-02	0.2875E-03	0.1135E-03	0.4410E-04	0.2250E-04
10^{-2}	0.3934E-02	0.7429E-03	0.1823E-03	0.7169E-04	0.3164E-04
10^{-3}	0.8110E-02	0.1669E-02	0.3019E-03	0.1339E-03	0.4627E-04
10^{-4}	0.1462E-01	0.3048E-02	0.7328E-03	0.1837E-03	0.7122E-04
10^{-5}	0.2416E-01	0.8182E-02	0.1762E-02	0.3210E-02	0.1123E-03
10^{-6}	0.2416E-01	0.8182E-02	0.1762E-02	0.3210E-02	0.1123E-03
10^{-7}	0.2416E-01	0.8182E-02	0.1762E-02	0.3210E-02	0.1123E-03
D^{N}	0.2416E-01	0.8182E-02	0.1762E-02	0.3210E-02	0.1123E-03
p^{N}	0.1139E+01	0.1399E+01	0.1553E+01	0.1573E+01	
C_{p}^{N}	0.9423E+00	0.9153E+00	0.7398E+00	0.5131E+00	0.5132E+00
Computed order of $\vec{\varepsilon}$-uniform convergence, $p^{*} = 1.139$					
Computed $\vec{\varepsilon}$-uniform error constant, $C_{p^*}^{N} = 0.9423$					

9 Conclusion remarks

The research work presented in this article is built upon the foundational concept developed by Miller [2]. Miller's work focused on convection diffusion problems in one dimension. In this paper, the authors establish second-order parameter uniform convergence for a system of n second-order partial differential equations of reaction-diffusion type with discontinuous source terms. They demonstrate that the proposed method can be extended to address higher dimensional problems.

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