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# LATTICE STRUCTURES OF PYTHAGOREAN FUZZY GRAPH DILATIONS AND EROSIONS

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**Abstract** This paper introduces Pythagorean fuzzy graph (PFG) mathematical morphology using $\beta_n$-adjacency of vertices and edges, grounded in lattice theory. We define the concepts of vertex dilation, edge dilation, vertex erosion, and edge erosion in PFGs and present an algorithm to compute these operations with an illustrative example. Additionally, we explore various properties of dilation and erosion in PFGs and introduce the concept of morphological dilation-induced strong products of dilated PFGs. The primary objective of this study is to extend the general theory of mathematical morphology on graphs within the framework of fuzzy set theory. Finally, we propose an algorithm for predicting and managing the spread of infectious diseases through vertex and edge dilation in PFG-based decision-making.

## 1 Introduction

### 1.1 Fuzzy Mathematical Morphology

Mathematical Morphology (MM) was developed based on set theory by George Matheron and his student Jean Serra [5, 6] in 1964 and generated many operators for quantification of mineral characteristics. Main idea of MM is a study and analysis of geometric structures by superposing with small patterns, called structuring element. Then MM was generally defined based on lattice theory.

Let $P$ and $Q$ be two complete lattices with supremum $\vee$ and infimum $\vee$. Let $\epsilon$ be an operator from $P$ to $Q$ and $\delta$ be an operator from $Q$ to $P$. The pair $(\epsilon, \delta)$ is an adjuntion between $P$ and $Q$ if $\delta(Y) \leq X \Leftrightarrow Y \leq \epsilon(X)$ for every $X \in P$, $Y \in Q$. Here $\delta$ is called dilation and $\epsilon$ is called erosion.

**Some Properties of Erosion and Dilation**

(i) $\delta(\emptyset) = \emptyset$, $\epsilon(U) = U$, where $\emptyset$ and $U$ are least and greatest elements

(ii) $\vee_i \delta(Y_i) = \delta(\vee_i Y_i)$ for every $Y_i \in P$.
That is dilation is distributive over supremum.

(iii) $\wedge_i \epsilon(X_i) = \epsilon(\wedge_i X_i)$ for every $X_i \in P$.
That is erosion is distributive over infimum.

(iv) Dilation is extensive. i.e., $\delta(X) \geq X$ for every $X$ in $Q$.

(v) Erosion is anti-extensive. i.e., $\epsilon(X) \leq X$ for every $X$ in $P$.

(vi) Erosion and dilation are increasing.

Authors like Di Gesu, De Baets, Bloch, Maitre and Nachtegeal [25–32] did extension of binary morphology to grayscale morphology based on fuzzy set theory, bipolar fuzzy set theory, intuitionistic fuzzy set theory and interval valued fuzzy set theory.

Mathematical morphology (MM) on graphs was introduced by Vincent [21] based on the frame work of complete lattice. Rosenfeld's Fuzzy graph was extended by Shannon and Atanassov [33] and introduced Intuitionistic Fuzzy Graph (IFG). In fuzzy graph, each vertex and edge has membership functions which lie in the interval $[0, 1]$ and sum of membership function and its complement of each vertex and edge is unity, whereas in IFG, sum of membership function and non-membership function of each vertices and edges in IFG should lie in the interval $[0, 1]$. So there is a space for indetermination. However, IFG cannot handle the situations when this sum is greater than unity. In particular, if membership function of a vertex is and its non-membership function is 0.5, then their sum $= 0.6 + 0.5 = 1.1$ which is greater than one. Such problems can not handle with IFG.

Recently Yagar [22] introduced Pythagorean fuzzy sets to handle uncertainty situations. As a continuation of this, Rajkumar [13] defined Pythagorean fuzzy graphs (PFG). In PFG, sum of squares of membership function and non-membership function of each vertices and edges in PFG should lie in the interval $[0, 1]$. The main advantage of this membership functions is that this involves all Intuitionistic fuzzy membership functions. But every Pythagorean membership functions need not be intuitionistic fuzzy membership functions. So this paper aims the theoretical developments of basic operations on Pythagorean fuzzy graph morphology using $\beta_n$-adjacency of vertices and edges. $\beta_n$-adjacency of a vertex or edge helps to adjust the size of the neighbourhood of each vertex or edge.

### 1.2 Pythagorean Fuzzy Graph (PFG)

**Definition 1.1.** [13] A Pythagorean fuzzy graph (PFG) is of the form $G = (G, G^\times, \mu_1, \gamma_1, \mu_2, \gamma_2)$, where

(i) $G = \{v_1, v_2, \ldots, v_n\}$ such that $\mu_1 : G \to [0, 1]$ and $\gamma_1 : G \to [0, 1]$, the membership function and non membership function of the element $v_i \in G$ respectively and $0 \leq [\mu_1(v_i)]^2 + [\gamma_1(v_i)]^2 \leq 1$ for every $v_i \in G$, $i = 1, 2, \ldots, n$.

(ii) $G^\times \subseteq G \times G$ where $\mu_2 : G^\times \to [0, 1]$ and $\gamma_2 : G^\times \to [0, 1]$ are such that
(a) $\mu_2(e_{v_iv_j}) \leq \min(\mu_1(v_i), \mu_1(v_j))$
(b) $\gamma_2(e_{v_iv_j}) \leq \max(\gamma_1(v_i), \gamma_1(v_j))$
(c) $0 \leq [\mu_2(e_{v_iv_j})]^2 + [\gamma_2(e_{v_iv_j})]^2 \leq 1$ for every edges $e_{v_iv_j} \in G^\times$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$.

$G$ is the Pythagorean fuzzy vertex set and $G^\times$ is the Pythagorean fuzzy edge set. For convenience, $(\mu_i, \gamma_i)$ is called Pythagorean Fuzzy Number (PFN).

Adjacency Matrix $A(G)$ of Pythagorean Fuzzy Graph $G$ is defined as an $n \times n$ matrix $A(G) =$, where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$ and $\gamma_2(e_{v_iv_j})$ indicate the relationship strength and non-relationship strength of the vertices $v_i$ and $v_j$ respectively.

**Example 1.2.** Let $G_i = (V, E)$ be a simple graph. Let $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{e_{v_1v_2}, e_{v_2v_3}, e_{v_3v_4}, e_{v_4v_1}\}$. Then $G = (G, G^\times, \mu_1, \gamma_1, \mu_2, \gamma_2)$ in Figure 1 is a Pythagorean fuzzy graph where $G$ and $G^\times$ are fuzzy vertex set and fuzzy edge set respectively.

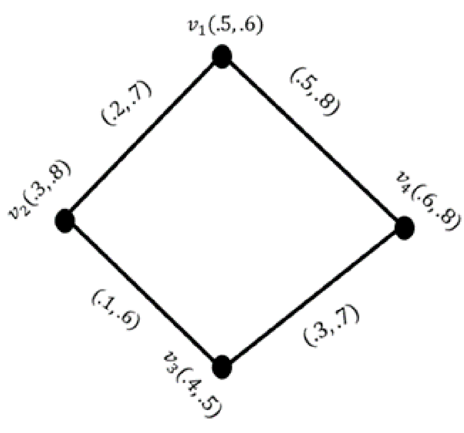


**Figure 1 PFG**

In vertex $v_1(.5, .6)$ in Figure 1

$$\begin{aligned} \mu_1(v_1) &= .5, \quad \gamma_1(v_1) = 0.6 \\ \therefore\ [\mu_1^2(v_1)]^2 + (\gamma_1(e_{v_1}))^2 &= .5^2 + .6^2 \\ &= .61 \le 1. \end{aligned}$$

Similarly condition (i) in definition 1.1 is satisfied for each vertices $v_2, v_3, v_4$ in figure 1.
In edge $e_{v_1v_2}(.2, .7)$ in figure 1.

$$\begin{aligned} \mu_2(e_{v_1v_2}) &= .2,\ \gamma_2(e_{v_1v_2}) = 0.7,\ \mu_1(v_2) = .3,\ \gamma_1(v_2) = .8 \\ \mu_2(e_{v_1v_2}) &= .2 \le \min\{\mu_1(v_1), \mu_1(v_2)\} = \min\{.5, .3\} = .3 \\ \gamma_2(e_{v_1v_2}) &= .7 \le \max\{\gamma_1(v_1), \gamma(v_2)\} \\ &= \max[.6, .8] = .8 \\ [\mu_2(e_{v_1v_2})]^2 + [\gamma_2(e_{v_1v_2}]^2 &= .2^2 + .7^2 \le 1. \end{aligned}$$

$\therefore$ Similarly, condition (ii) in definition 1.1 are satisfied for each edges $e_{v_2v_3}$, $e_{v_3v_4}$ and $e_{v_4v_1}$.
$\therefore G_i = (G, G^\times, \mu_1, \gamma_1, \mu_2, \gamma_2)$ in Figure 1 is a Pythagorean fuzzy graph.

Corresponding Adjacency Matrix $A(G)$ of Pythagorean Fuzzy Graph $G$ in Figure 1 is given as follows:

$$A(G) = \begin{bmatrix} (0,0) & (.2,.7) & (0,0) & (.5,.8) \\ (.2,.7) & (0,0) & (.1,.6) & (0,0) \\ (0,0) & (.1,.6) & (0,0) & (.3,.7) \\ (.5,.8) & (0,0) & (.3,.7) & (0,0) \end{bmatrix}$$

**Definition 1.3.** [13] A Pythagorean fuzzy graph $G_i = (G, G^\times, \mu_{1i}, \gamma_{1i}, \mu_{2i}, \gamma_{2i})$ is said to be PF subgraph of PFG $G_j = (G, G^\times, \mu_{1j}, \gamma_{1j}, \mu_{2j}, \gamma_{2j})$ if $G_i \subseteq G_j$ and $G_i^\times \subseteq G_j^\times$ and

$$\mu_{1i}(v_k) \le \mu_{1j}(v_k), \gamma_{1i}(v_k) \ge \gamma_{1j}(v_k),\ \forall v_k \in G_i$$
$$\mu_{2i}(e_{v_kv_l}) \le \mu_{2j}(e_{v_kv_l}), \gamma_{2i}(e_{v_kv_l}) \ge \gamma_{2j}(e_{v_kv_l}),\ \forall e_{v_kv_l} \in G_i^\times.$$

**Example 1.4.** Consider a PFG $G_j = (G_j, G_j^\times)$ with $G_j = \{v_1, v_2, v_3, v_4\}$ and
$G_j^\times = \{e_{v_1v_2}, e_{v_2v_3}, e_{v_3v_4}, e_{v_4v_1}\}$ in Figure 2 and PFG $G_i = (G_i, G_i^\times)$ with $G_i = \{v_1, v_2, v_3, v_4\}$ and
$G_i^\times = \{e_{v_1v_2}, e_{v_2v_3}, e_{v_3v_4}, e_{v_4v_1}\}$.

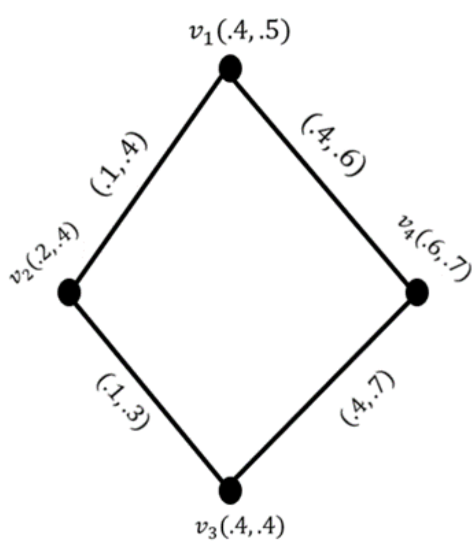


**Figure 2 PF Subgraph**

Here, $G_j \subseteq G_i$ and $G_j^{\times} \subseteq G_i^{\times}$

$$\mu_{1j}(v_k) \leq \mu_{1i}(v_k), \gamma_{1j}(v_k) \geq \gamma_{1i}(v_k),\ \forall v_k \in G_j$$
$$\mu_{2j}(e_{v_k v_l}) \leq \mu_{2i}(e_{v_k v_l}), \gamma_{2j}(e_{v_k v_l}) \geq \gamma_{2i}(e_{v_k v_l}),\ \forall e_{v_k v_l} \in G_j^{\times}.$$

Therefore, $G_j$ in Figure 2 is a PF subgraph of $G_i$ in Figure 1.

In section 2, we introduce lattice structure of Pythagorean fuzzy graph and definitions of vertex dilation, edge dilation, vertex erosion and edge erosion on Pythagorean fuzzy graphs using $\beta_n$-adjacency and illustrate an algorithm to find vertex dilation, edge dilation, vertex erosion and edge erosion on Pythagorean fuzzy graphs with a numerical example. We also define morphological dilation induced strong products of these dilated PFGs. Derivations of some important properties of Pythagorean fuzzy dilation and erosion are done in section 3. An algorithm is proposed to predict and control the spread of infectious diseases in section 4.

## 2 Lattice structure and Pythagorean fuzzy graph dilations and Erosions

Let $G$ be the set of all PF subgraphs $G_i = (G, G^{\times}, \mu_{1i}, \gamma_{1i}, \mu_{2i}, \gamma_{2i})$ defined on $G = (G, G^{\times})$ where each pair satisfies property of subgraph, $G = \{v_1, v_2, \dots, v_n\}$ is the underlying vertex set, $G^{\times} \subseteq G \times G$ is the set of all edges, $\mu_{1i}, \gamma_{1i}$ are membership and non membership functions of vertices $v_k$ in $G_i$ and $\mu_{2i}$ and $\gamma_{2i}$ are membership and non membership functions of edges $e_{v_k v_l}$ in $G_i$. Let 0 be PFG in $G$ with all vertices and edges of membership function 0 and non membership function 1 and 1 be PFG in $G$ with all vertices and edges of membership function 1 and non membership function 0. Smallest element and greatest element of the collection $G$ are the PFGs 0 and 1 respectively. We will be used these notations throughout this paper.

A partial order on $G$ is defined as $G_i \subseteq G_j$ if and only if $(\mu_{1i}(v_i), \gamma_{1i}(v_i)) \leq (\mu_{1j}(v_i), \gamma_{1j}(v_i))$ and $(\mu_{2i}(e_{v_l v_k})), \gamma_{2i}(e_{v_l v_k}) \leq (\mu_{2j}(e_{v_l v_k}), \gamma_{2j}(e_{v_l v_k}))$. It is clear that the relation $\subseteq$ is reflexive, antisymmetric and transitive. Therefore $(G, \subseteq)$ is a poset. Supremum ($\vee$) and infimum ($\wedge$) of Pythagorean fuzzy graphs are defined as follows.

**Definition 2.1.** Supremum and infimum of PF subgraphs in $G$ are defined as follows. For $G_i, G_j \in G$,

$$G_i \vee G_j = G_i \cup G_j = (G, G^{\times}, \mu_{1i} \vee \mu_{1j}, \gamma_{1i} \wedge \gamma_{1j}, \mu_{2i} \vee \mu_{2j}, \gamma_{2i} \wedge \gamma_{2j})$$
$$G_i \wedge G_j = G_i \cap G_j = (G, G^{\times}, \mu_{1i} \wedge \mu_{1j}, \gamma_{1i} \vee \gamma_{1j}, \mu_{2i} \wedge \mu_{2j}, \gamma_{2i} \wedge \gamma_{2j})$$

The following proposition proves that the union (intersection) of finite collection of PF subgraphs in $G$ is the suprimum (infimum) of the collection.

**Proposition 2.2.** *Let $F = \{G_1, G_2, \dots, G_n\}$ be a collection of elements in G. Then $F = G_1 \vee G_2 \vee \cdots \vee G_n$ and $\inf F = G_1 \wedge G_2 \wedge \cdots \wedge G_n$.*

*Proof.* Let

$$S = G_1 \vee G2 \vee \cdots \vee G_n = G_1 \cup G_2 \cup \cdots \cup G_n$$
$$= (G, G^{\times}, \vee_{i=1}^{n} \mu_{1i}, \wedge_{i=1}^{n} \gamma_{1i}, \vee_{i=1}^{n} \mu_{2i}, \wedge_{i=1}^{n} \gamma_{2i})$$

Let $G_k = (G, G^{\times}, \mu_{1k}, \gamma_{1k}, \mu_{2k}, \gamma_{2k})$ be any Pythagorean subgraph contained in the collection $F$. To prove that $G_k \subseteq S$.

By definition of suprimum, we have

$$\mu_{1k} \leq \vee_{i=1}^{n} \mu_{1i}, \gamma_{1k} \geq \wedge_{i=1}^{n} \mu_{2i}, \gamma_{2k} \geq \wedge_{i=1}^{n} \gamma_{2i},\ \text{for } k = 1, 2, \dots, n. \tag{2.1}$$

By definition of PF subgraph and equation (2.1), we get $G_k \subseteq S$ which proves any element in $F$ is a subgraph of $S$.

Let

$$G_k \subseteq T \text{ for all } k = 1, 2, \dots n \wedge \text{ where } T = (G, G^{\times}, \mu_{1t}, \gamma_{1t}, \mu_{2t}, \gamma_{2t}). \tag{2.2}$$

We have to prove $S \subseteq T$

$$\begin{aligned}(2.2) &\Rightarrow \mu_{1k} \leq \mu_{1t}, \gamma_{1k} \geq \gamma_{1t}, \gamma_{2k} \leq \gamma_{2t}, \gamma_{2k} \geq \gamma_{2t} \; \forall k = 1, 2, \ldots, n \\ &\Rightarrow \vee_{k=1}^{n} \mu_{1k} \leq \mu_{1t}, \; \wedge_{k=1}^{n} \gamma_{1k} \geq \gamma_{1t}, \; \vee_{k=1}^{n} \mu_{2k} \leq \mu_{2t}, \; \wedge_{k=1}^{n} \gamma_{2k} \geq \gamma_{2t}, \\ &\Rightarrow S \subseteq T \text{ by definition of } S. \\ \therefore^{F} &= S = G_1 \vee G_2 \vee \cdots \vee G_n.\end{aligned}$$

In similar manner we can prove that $\inf F = G_1 \wedge G_2 \wedge \cdots \wedge G_n$. □

**Theorem 2.3.** $(G, \wedge, \vee, 0, 1)$ *is a complete lattice.*

*Proof.* Proof follows from Proposition 2.2, Definition 1.3. □

Neighbourhood of vertex and edge is essential to define dilation and erosion on PFG. The $\beta_n$ adjacency helps us to select the size of neighbourhood. Neighbourhood of a vertex $u$ is the set of all vertices which are connected to $u$ by one edge and it is called $\beta_1$ adjacency vertices of $u$. In the same manner, $\beta_n$ adjacency vertices are defined as follows.

**Definition 2.4.** Two vertices $v_l$ and $v_k$ in $G_i$ in $G$ are $\beta_n$ adjacency vertices ($n$ path adjacency vertices) if they are connected by almost $n$ edges. We represent it as $v_l \beta_n \text{adj}\, v_k$.

Similarly $\beta_n$ adjacency edges are defined as follows.

**Definition 2.5.** Two edges $e_{v_l v_k}$ and $e_{v_m v_n}$ in $G_i$ in $G$ are $\beta_n$ adjacency edges if either $v_l$ or $v_k$ is connected to $v_m$ or $v_n$ by at most $n$ edges. It is illustrated in Example 2.8.

**Definition 2.6.** Dilation and erosion of vertices $v_l$ in $G_i$ are respectively denoted by $\delta_{1i}(v_l)$ and $\epsilon_{1i}(v_l)$ and are defined as
for each vertices $v_l$ in $G$, $\delta_{1i}, \epsilon_{1i} : G \to [0, 1]$ by

$$\delta_{1i}(v_l) = (\sup_{v_k} \mu_{1i}(v_k), \inf_{v_k} \gamma_{1i}(v_k)) \tag{2.3}$$

and

$$\epsilon_{1i}(v_l) = (\inf_{v_k} \mu_{1i}(v_k), \sup_{v_k} \gamma_{1i}(v_k)) \tag{2.4}$$

where $v_k$ is $v_l$ itself or $v_k \beta_n$ adj $v_l$.

Equation (2.3) & (2.4) respectively define Pythagorean Fuzzy graphs called vertex dilated PFG and vertex eroded PFG and denote them as $G_i^{vd}$ or $\delta_{1i}^{v}$ and $G_i^{ve}$ or $\delta_{1i}^{e}$.

**Definition 2.7.** Dilation and erosion of edges $e_{v_l v_k}$ in $G^{\times}$ are respectively denoted by $\delta_{2i}(e_{v_l v_k})$ and $\epsilon_{2i}(e_{v_l v_k})$ and are defined by
for each edges $e_{v_l v_k}$ in $G^{\times}$ dilation $\delta_{2i}, \epsilon_{2i} : G^{\times} \to [0, 1]$ by

$$\delta_{2i}(e_{v_l v_k}) = (\sup_{e_{v_r v_s}} \mu_{2i}(e_{v_r v_s}), \inf_{e_{v_r v_s}} \gamma_{2i}(e_{v_r v_s})) \tag{2.5}$$

and

$$\epsilon_{2i}(e_{v_l v_k}) = (\inf_{e_{v_r v_s}} \mu_{2i}(e_{v_r v_s}), \sup_{e_{v_r v_s}} \gamma_{2i}(e_{v_r v_s})) \tag{2.6}$$

where $e_{v_r v_s}$ is $e_{v_l v_k}$ itself or $e_{v_r v_s} \beta_n$ adjacency edge to $e_{v_l v_k}$.
Equation (2.5) & (2.6) respectively define Pythagorean Fuzzy graphs called edge dilated PFG and edge eroded PFG and denote them as $G_i^{ed}$ or $\delta_{2i}^{e}$ and $G_i^{ee}$ or $\delta_{2i}^{e}$.

Combination of (2.3) and (2.5) define Pythagorean Fuzzy Graph called dilated PFG $G_i^{D}$ or $(\delta_{1i}^{v}, \delta_{2i}^{e})$ and Combination of (2.4) and (2.6) define an eroded Pythagorean Fuzzy Graph $G_i^{E}$ or $(\epsilon_{1i}^{v}, \epsilon_{2i}^{e})$.

**Algorithm to find Vertex Dilated PFG $G_i^{vd}$ or $\delta_{1i}^v$, Edge Dilated PFG $G_i^{ed}$ or $\delta_{2i}^e$, Dilated PFG $G_i^D$ or $(\delta_{1i}^v, \delta_{2i}^e)$ using $\beta_1$ adjacency vertices and edges.**

**Input:** Given PFG $G_i = (G, G^{\times}, \mu_{1i}, \gamma_{1i}, \mu_{2i}, \gamma_{2i})$ with corresponding membership and non-membership functions $\mu_{1i}$ and $\gamma_{1i}$ of each vertices $v_l$ and corresponding membership and non-membership functions $\mu_{2i}$ and $\gamma_{2i}$ of each edges $e_{v_l v_k}$.

**Output:**

(i) Vertex Dilated PFG $G_i^{vd}$ or $\delta_{1i}^v$.
1. Find adjacency vertices of each vertices $v_l$
2. Find dilation of each vertex using formula (2.3)
3. Replace membership and non-membership functions $\mu_{1i}$ and $\gamma_{1i}$ of each vertices $v_l$ from given PFG $G_i$ by corresponding membership and non-membership functions of dilated vertices so that it results Vertex Dilated PFG $G_i^{vd}$ or $\delta_{1i}^v$.

(ii) Edge Dilated PFG $G_i^{ed}$ or $\delta_{2i}^e$
1. Find $\beta_1$ adjacency edges of each edges $e_{v_l v_k}$
2. Find dilation of each vertex using formula (2.4)
3. Replace membership and non-membership functions $\mu_{2i}$ and $\gamma_{2i}$ of each edges $e_{v_l v_k}$ from given PFG $G_1$ by corresponding membership and non-membership functions of dilated edges so that it results Edge Dilated PFG $G_i^{ed}$ or $\delta_{2i}^e$.

(iii) Dilated PFG $G_i^D = (\delta_{1i}^v, \delta_{2i}^e)$.

Combining Vertex Dilation and Edge Dilation, obtain Dilated PFG $G_i^D$ or $(\delta_{1i}^v, \delta_{2i}^e)$.

**Example 2.8.** Consider the PF graph $G_i = (G, G^{\times}, \mu_{1i}, \gamma_{1i}, \mu_{2i}, \gamma_{2i})$ where $G = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $G^{\times} = \{e_{v_1v_2}, e_{v_2v_3}, e_{v_1v_4}, e_{v_3v_4}, e_{v_4v_5}, e_{v_4v_7}, e_{v_5v_6}, e_{v_6v_7}\}$. The Pythagorean fuzzy graph Fig. 3 is given below:

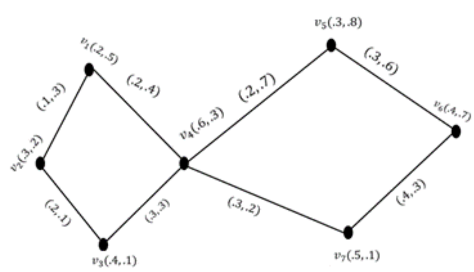


**Figure 3 PFG**

From above Figure 3, $\beta_1$ adjacency of $v_1$ are $v_1, v_2, v_4$, $\beta_1$ adjacency of $e_{v_1v_2}$ are $e_{v_1v_2}, e_{v_1v_4}, e_{v_2v_3}$. Vertex Dilated PFG and Vertex Eroded PFG are given in Figure 4 & 5:

| Dilation and Erosion of Vertices | | | |
|---|---|---|---|
| Vertices | $\beta_1$ Adjacency of Vertex | Dilation of vertex $(v_l)$ $\delta_{G_i}^{vd}(v_l)$ | Erosion of vertex $(v_i)$ $\epsilon_{G_i}^{ve}(v_l)$ |
| $v_1$ | $v_1, v_2, v_4$ | $\delta_{G_i}(v_1) = (.6, .2)$ | $\epsilon_{G_i}(v_1) = (.2, .5)$ |
| $v_2$ | $v_1, v_2, v_3$ | $\delta_{G_i}(v_2) = (.4, .1)$ | $\epsilon_{G_i}(v_2) = (.2, .5)$ |
| $v_3$ | $v_2, v_3, v_4$ | $\delta_{G_i}(v_3) = (.6, .1)$ | $\epsilon_{G_i}(v_3) = (.3, .3)$ |
| $v_4$ | $v_1, v_3, v_4, v_5, v_7$ | $\delta_{G_i}(v_4) = (.6, .1)$ | $\epsilon_{G_i}(v_4) = (.2, .8)$ |
| $v_5$ | $v_4, v_5, v_6$ | $\delta_{G_i}(v_5) = (.6, .3)$ | $\epsilon_{G_i}(v_5) = (.3, .8)$ |
| $v_6$ | $v_5, v_6, v_7$ | $\delta_{G_i}(v_6) = (.5, .1)$ | $\epsilon_{G_i}(v_6) = (.3, .8)$ |
| $v_7$ | $v_4, v_6, v_7$ | $\delta_{G_i}(v_7) = (.6, .1)$ | $\epsilon_{G_i}(v_7) = (.4, .7)$ |

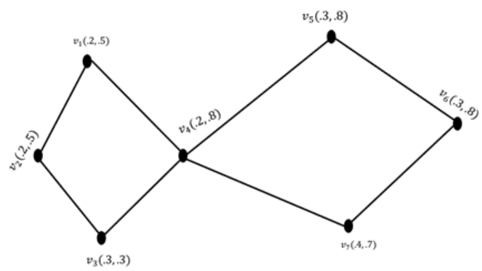


**Figure 5 Vertex Eroded PFG**

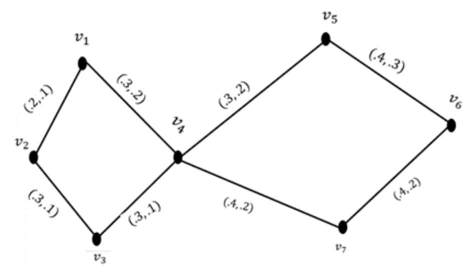


**Figure 6 Edge Dilated PFG**

Edge Dilated PFG and Edge Eroded PFG in Figure 6 & 7 are given.

| Dilation and Erosion of Edges | | | |
|---|---|---|---|
| Edges | $\beta_1$ adjacency of Edge | Dilation of Edge $(e_{v_iv_j})$ $\delta_{G_i}(e_{v_iv_j})$ | Erosion of Edge $(e_{v_iv_j})$ $\epsilon_{G_i}(e_{v_iv_j})$ |
| $e_{v_1v_2}$ | $e_{v_1v_2}, e_{v_2v_3}, e_{v_1v_4}$ | $\delta_{G_i}(e_{v_1v_2}) = (.2, .1)$ | $\epsilon_{G_i}(e_{v_1v_2}) = (.1, .4)$ |
| $e_{v_2v_3}$ | $e_{v_1v_2}, e_{v_2v_3}, e_{v_3v_4}$ | $\delta_{G_i}(e_{v_2v_3}) = (.3, .1)$ | $\epsilon_{G_i}(e_{v_2v_3}) = (.1, .3)$ |
| $e_{v_1v_4}$ | $e_{v_1v_2}, e_{v_4v_5}, e_{v_3v_4}, e_{v_4v_7}, e_{v_1v_4}$ | $\delta_{G_i}(e_{v_1v_4}) = (.3, .2)$ | $\epsilon_{G_i}(e_{v_1v_4}) = (.1, .7)$ |
| $e_{v_3v_4}$ | $e_{v_2v_3}, e_{v_1v_4}, e_{v_3v_4}, e_{v_4v_7}, e_{v_4v_5}$ | $\delta_{G_i}(e_{v_3v_4}) = (.3, .1)$ | $\epsilon_{G_i}(e_{v_3v_4}) = (.2, .7)$ |
| $e_{v_4v_5}$ | $e_{v_1v_4}, e_{v_3v_4}, e_{v_4v_7}, e_{v_4v_5}, e_{v_5v_6}$ | $\delta_{G_i}(e_{v_4v_5}) = (.3, .2)$ | $\epsilon_{G_i}(e_{v_4v_5}) = (.2, .7)$ |
| $e_{v_4v_7}$ | $e_{v_1v_4}, e_{v_3v_4}, e_{v_4v_7}, e_{v_4v_5}, e_{v_6v_7}$ | $\delta_{G_i}(e_{v_4v_7}) = (.4, .2)$ | $\epsilon_{G_i}(e_{v_4v_7}) = (.2, .7)$ |
| $e_{v_5v_6}$ | $e_{v_5v_6}, e_{v_4v_5}, e_{v_6v_7}$ | $\delta_{G_i}(e_{v_5v_6}) = (.4, .3)$ | $\epsilon_{G_i}(e_{v_5v_6}) = (.2, .7)$ |
| $e_{v_6v_7}$ | $e_{v_6v_7}, e_{v_4v_7}, e_{v_5v_6}$ | $\delta_{G_i}(e_{v_6v_7}) = (.4, .2)$ | $\epsilon_{G_i}(e_{v_6v_7}) = (.3, .6)$ |

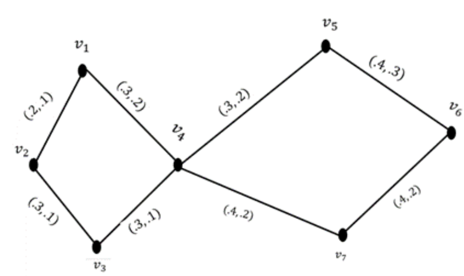


**Figure 6 Edge Dilated PFG**

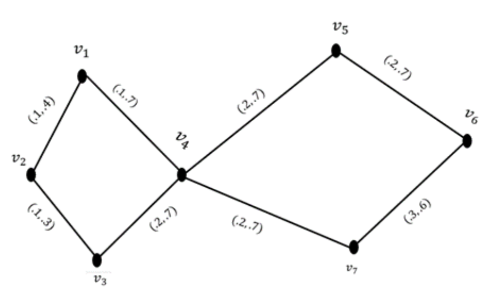


**Figure 7 Edge Eroded PFG**

Combining vertex dilation and edge dilation on PFG results dilated PFG $\delta_{G_i}$ or $G_i^D$ in Figure 8.

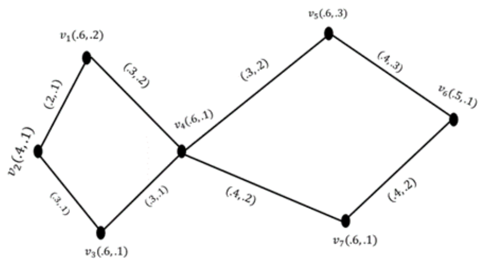


**Figure 8 Dilated PFG**

Combining vertex erosion and edge erosion on PFG results eroded PFG $\epsilon_{G_i}$ or $G_i^E$ in Figure 9.

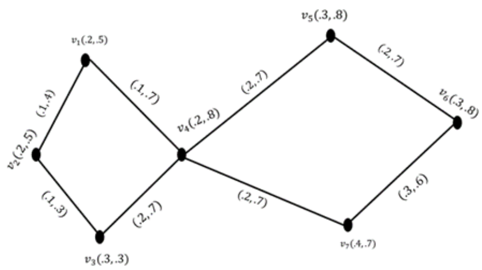


**Figure 9 Eroded PFG**

**Definition 2.9** (Morphological dilation induced strong products of these dilated PFGs). Let $G_i = (V_i, E_i)$ and $G_j = (V_j, E_j)$ be two simple graphs.

Let $G_i^D = (G_i, G_i^{\times}, \mu_{1i}^D, \gamma_{1i}^D, \mu_{2i}^D, \gamma_{2i}^D)$ and $G_j^D = (G_j, G_j^{\times}, \mu_{1j}^D, \gamma_{1j}^D, \mu_{2j}^D, \gamma_{2j}^D)$ be the dilated Pythagorean Fuzzy Graphs of the Pythagorean Fuzzy Graphs $G_i = (G_i, G_i^{\times}, \mu_{1i}, \gamma_{1i}, \mu_{2i}, \gamma_{2i})$ and $G_j = (G_j, G_j^{\times}, \mu_{1j}, \gamma_{1j}, \mu_{2j}, \gamma_{2j})$ respectively. The morphological dilation induced strong products of these dilated PFGs is denoted by $G_i^D \boxtimes G_j^D$ and is defined as

i. $$(\mu_{1i}^D \boxtimes \mu_{1j}^D)(u_1, u_2) = \mu_{1i}^D(u_1) \wedge \mu_{1j}^D(u_2)$$
$$(\gamma_{1i}^D \boxtimes \gamma_{1j}^D)(u_1, u_2) = \gamma_{1i}^D(u_1) \vee \gamma_{1j}^D(u_2) \text{ for all } (u_1, u_2) \in V_i \times V_j.$$

ii. $$(\mu_{2i}^D \boxtimes \mu_{2j}^D)((u, u_2), (u, v_2)) = \mu_{1i}^D(u) \wedge \mu_{2j}^D(e_{u_2 v_2})$$
$$(\gamma_{2i}^D \boxtimes \gamma_{2j}^D)((u, u_2), (u, v_2)) = \gamma_{1i}^D(u) \vee \gamma_{2j}^D(e_{u_2 v_2}) \text{ for all } u \in V_i,\ e_{u_2, v_2} \in E_j.$$

iii. $$(\mu_{2i}^D \boxtimes \mu_{2j}^D)((u_1, w), (v_1, w)) = \mu_{2i}^D(e_{u_1 v_1}) \wedge \mu_{1j}^D(w)$$
$$(\gamma_{2i}^D \boxtimes \gamma_{2j}^D)((u, u_2), (u, v_2)) = \gamma_{2i}^D(u_1) \vee \gamma_{1j}^D(w) \text{ for all } w \in V_j,\ e_{u_1 v_1} \in E_i.$$

iv. $$(\mu_{2i}^D \boxtimes \mu_{2j}^D)((u_1, u_2), (v_1, v_2)) = \mu_{2i}^D(e_{u_{u_1 v_1}}) \wedge \mu_{2j}^D(e_{u_2 v_2}).$$

**Example 2.10.** Let $G_i = (V_i, E_i)$ and $G_j = (V_j, E_j)$ be two simple graphs, where $V_i = \{v_1, v_2, v_3\}$, $V_j = \{u_1, u_2\}$, $E_i = \{e_{v_1 v_2}, e_{v_2 v_3}\}$ and $E_i = \{e_{u_1 u_2}\}$. Consider two Pythagorean Fuzzy Graphs $G_i = (G_i, G_i^{\times}, \mu_{1i}, \gamma_{1i}, \mu_{2i}, \gamma_{2i})$ and $G_j = (G_j, G_j^{\times}, \mu_{1j}, \gamma_{1j}, \mu_{2j}, \gamma_{2j})$ in Figure 10.

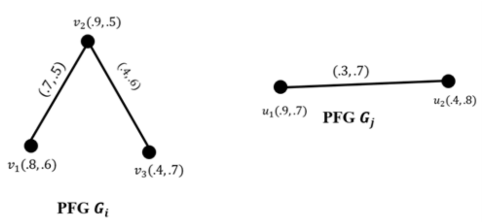


**Figure 10**

Dilated PFGs of Pythagorean Fuzzy Graphs and are shown in Figure 11.

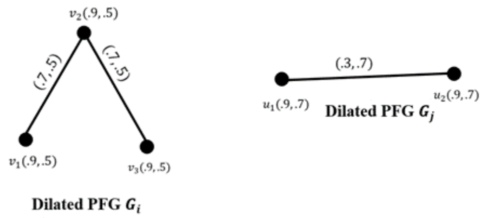


Figure 11

The morphological dilation induced strong products $G_i^D \boxtimes G_j^D$ is shown below in Figure 12.

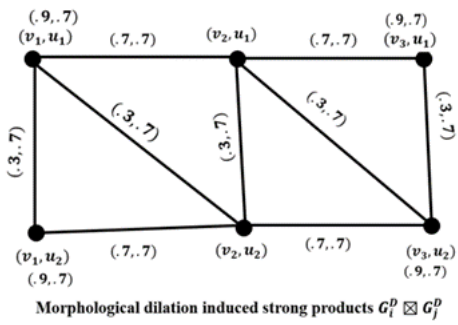


Figure 12

**Proposition 2.11.** *Let $G_i^D$ and $G_j^D$ be two dilated PFGs of Pythagorean Fuzzy Graphs $G_i$ and $G_j$ respectively. The morphological dilation induced strong products $G_i^D \boxtimes G_j^D$ is a PFG.*

Now we derive some properties of dilated PFG and eroded PFG as follows in the next section.

## 3 Properties of Dilated PFG and Eroded PFG

**Theorem 3.1.** *$G_i^D$, dilation of Pythagorean fuzzy graph $G_i$ in G, is a member of G.*

*Proof.* By definition 2.6 and 2.7 for each vertices $v_k$ and edges $e_{v_k v_l}$ in $G_i$ and $G_i^D$

$$\mu_{1i}(v_k) \leq \mu_{1i}^D(v_k), \gamma_{1i}(v_k) \geq \gamma_{1i}^D(v_k), \mu_{2i}(e_{v_k v_l}) \leq \mu_{2i}^D(e_{v_k v_l}), \gamma_{2i}(e_{v_k v_l}) \geq \gamma_{2i}^D(e_{v_k v_l})$$
$$\Rightarrow G_i \subseteq G_i^D \text{ by defintion 1.3 of PF subgraph.}$$

Now we need to prove that dilated Pythagorean fuzzy graph is again Pythagorean fuzzy graph.

Suppose $\mu_{2i}^D(e_{v_k v_l}) > \mu_{1i}^D(v_k) \wedge \mu_{1i}^D(v_l)$.
Then by definition of edge dilation, $\exists$ a $\beta_n$-adjacency edges $e_{v_m v_n}$ in $G_i$ such that

$$\mu_{2i}(e_{v_m v_n}) = \mu_{2i}^D(e_{v_k v_l}). \tag{3.1}$$

By definition of PFG $G_i$,

$$\mu_{2i}(e_{v_m v_n}) \leq \mu_{1i}(v_m) \wedge \mu_{1i}(v_n). \tag{3.2}$$

Note that if an edge $e_{v_m v_n}$ is a $\beta_n$ adjacency edge of $e_{v_k v_l}$, then its vertices $v_m$ or $v_n$ is a $\beta_n$ adjacency vertices of $v_k$ or $v_l$. By definition of vertex dilation, $\mu_{1i}^D(v_k)$ and $\mu_{1i}^D(v_l)$ are suprimum of their $\beta_n$ adjacency vertices which includes $v_m$ or $v_n$.

$$\therefore\ \mu_{1i}(v_m) \wedge \mu_{1i}(v_n) \text{ are less than or equal to } \mu_{1i}^D(v_k) \vee \mu_{1i}^D(v_l)$$
$$\mu_{1i}^D(v_k) \wedge \mu_{1i}^D(v_l) \geq \mu_{1i}(v_m) \wedge \mu_{1i}(v_n) \tag{3.3}$$

By (3.1) and (3.3) $\mu_{2i}(e_{v_m v_n}) > \mu_{1i}(v_m) \wedge \mu_{1i}(v_n)$ which is a contradiction to (3.2)

$$\therefore\ \mu_{2i}^D(e_{v_k v_l}) \leq \mu_{1i}^D(v_k) \wedge \mu_{1i}^D(v_l).$$

Using similar argument, we can prove

$$\gamma_{2i}^D(e_{v_kv_l}) \leq \gamma_{1i}^D(v_k) \vee \gamma_{1i}^D(v_l).$$

Since each membership and non membership grades are in $[0, 1]$,

$$0 \leq \mu_{1i}(v_k) + \gamma_{1i}(v_k) \leq 1$$
$$0 \leq \mu_{2i}(e_{v_kv_l}) + \gamma_{2i}(e_{v_kv_l}) \leq 1$$
$$\therefore G_i^D \text{ is a member of } G.$$

□

**Theorem 3.2.** *$G_i^E$ erosion of Pythagorean fuzzy graph $G_i$, is member of $G$.*

*Proof.* In similar manner in Theorem 3.1, we can prove Theorem 3.2. □

**Remark 3.3.** Dilated PFG $G_i^D$ and eroded PFG $G_i^E$ in figure 8 and 9 of a PFG $G_i$ in example 2.8 are again PFG.

**Theorem 3.4.** *Let $G_i = (G, G^\times, \mu_{1i}, \gamma_{1i}, \mu_{2i}, \gamma_{2i}) \in G$ then $G_i \subseteq G_i^D$ and $G_i^E \subseteq G_i$.*

*Proof.* By Definition 1.3 and 2.5, $G_i \subseteq G_i^D$ and $G_i^E \subseteq G_i$. □

**Remark 3.5.** By Theorem 3.4, dilation on PFG is extensive and erosion in anti-extensive.

**Theorem 3.6.** *Let $G_i = (G, G, \mu_{1i}, \gamma_{1i}, \mu_{2i}, \gamma_{2i}) \in G$; $i = 1, 2$.*

$$G_3 = G_1 \cup G_2 = (G, G^\times, \mu_{13}, \gamma_{13}, \mu_{23}, \gamma_{23})$$
$$G_4 = G_1 \cap G_2 = (G, G^\times, \mu_{14}, \gamma_{13}, \mu_{24}, \gamma_{24}$$

*then*
*1. $G_3^D = G_1^D \cup G_2^D$; dilation is distributive with respect to union*
*2. $G_4^D = G_1^D \cap G_2^D$; erosion is distributive with respect to intersection.*

*Proof.* (i) Let $v_k$ be any vertex in $G_3$, then

$$\mu_{13}(v_k) = \mu_{11}(v_k) \vee \mu_{12}(v_k)$$
$$\gamma_{13}(v_k) = \gamma_{11}(v_k) \wedge \gamma_{12}(v_k)$$
$$\mu_{23}(e_{v_kv_l}) = \mu_{21}(e_{v_kv_l}) \vee \mu_{22}(e_{v_kv_l})$$
$$\gamma_{23}(e_{v_kv_l}) = \gamma_{21}(e_{v_kv_l}) \wedge \mu_{22}(e_{v_kv_l}).$$

Now

$$\begin{aligned}\mu_{13}^D(v_k) &= \vee_{m=1}^n \mu_{13}(v_m); \text{ where } v_m \text{ is a } \beta_n\text{-adjacency vertex of } v_k \\ &\leq \vee_{m=1}^n [\mu_{11}(v_m) \vee \mu_{12}(v_m)] \\ &\leq [\vee_{m=1}^n \mu_{11}(v_m)] \vee [\vee_{n=1}^n \mu_{12}(v_m)] \\ &\leq \mu_{11}^D(v_k) \vee \mu_{12}^D(v_k)\end{aligned}$$

and $\gamma_{13}^D(v_k) = \wedge_{m=1}^n \gamma_{13}(v_m)$; where $v_m$ is a $\beta_n$-adjacency vertex of $v_k$

$$\begin{aligned}&\geq \wedge_{m=1}^n [\gamma_{11}(v_m) \wedge \gamma_{12}(v_m)] \\ &\geq [\wedge_{m=1}^n \gamma_{11}(v_m)] \wedge [\wedge_{m=1}^n \gamma_{12}(v_m)] \\ &\geq \gamma_{11}^D(v_k) \wedge \gamma_{12}^D(v_k).\end{aligned}$$

Similarly we can prove that for each edges $e_{v_kv_l}$ in $G_3$,

$$\mu_{23}^D(e_{v_kv_l}) = \mu_{21}^D(e_{v_kv_l}) \vee \mu_{22}^D(e_{v_kv_l})$$
$$\mu_{23}^D(e_{v_kv_l}) = \mu_{21}^D(e_{v_kv_l}) \wedge \mu_{22}^D(e_{v_kv_l})$$
$$\therefore \; G_3^D = G_1^D \cup G_2^D.$$

(ii) In similar way, we can prove $G_4^D = G_1^D \cap G_2^D$.

□

**Theorem 3.7.** *Let* $G_i = (G, G^{\times}, \mu_{1i}, \gamma_{1i}, \mu_{2i}, \gamma_{2i})$ *and* $G_j = (G, G^{\times}, \mu_{1j}, \gamma_{1j}, \mu_{2j}, \gamma_{2j})$, *are members of* $G$ *then*
*1.* $G_i \subseteq G_j \Rightarrow G_i^D \subseteq G_j^D$ *(Dilation is increasing)*
*2.* $G_i \subseteq G_j \Rightarrow G_i^E \subseteq G_j^E$ *(Erosion is increasing)*

*Proof.* (i) Suppose $G_i \subseteq G_j$. Then for any vertex $v_k$ and edge $e_{v_k v_l}$,

$$\mu_{1i}(v_k) \leq \mu_{1j}(v_k), \gamma_{1i}(v_k) \geq \gamma_{1j}(v_k)$$
$$\mu_{2i}(e_{v_k v_l}) \leq \mu_{2j}(e_{v_k v_l}), \gamma_{2i}(e_{v_k v_l}) \geq \gamma_{2j}(e_{v_k v_l}).$$

If possible let $G_i^D \not\subseteq G_j^D$. Then atleast one of the following will not be true.

$$\mu_{1i}^D(v_k) \leq \mu_{1j}^D(v_k), \gamma_{1i}^D(v_k) \geq \gamma_{1j}^D(v_k)$$
$$\mu_{2i}^D(e_{v_k v_l}) \leq \mu_{2i}^D(e_{v_k v_l}), \gamma_{2i}^D(e_{v_k v_l}) \geq \gamma_{2j}^D(e_{v_k v_l}).$$

Suppose
$$\mu_{1i}^D(v_k) > \mu_{1j}^D(v_k) \tag{3.4}$$
Then $\exists$ a $\beta_n$ adjacency vertex $v_m$ in $G_i$ and $V_n$ in $G_j$ such that
$$\mu_{1i}(v_m) = \mu_{1i}^D(v_k) \tag{3.5}$$
and
$$\mu_{1i}(v_n) = \mu_{1j}^D(v_k). \tag{3.6}$$
If the vertices $v_m$ in $G_i$ and $V_n$ in $G_j$ are corresponding vertices, $\mu_{1i}(v_m) \leq \mu_{1j}(v_n)$ since $G_i \subseteq G_j$. But by equations (3.4), (3.5) and (3.6), $\mu_{1i}(v_m) > \mu_{ij}(v_n)$ which is a contradiction to $G_i \subseteq G_j$. If $v_m$ in $G_i$ and $v_n$ in $G_j$ are not corresponding vertices, then suppose $v_r$ in $G_j$ corresponds to $v_n$ in $G_i$. Then $\mu_{1j}(v_r) \leq \mu_{1j}(v_n) = \mu_{1j}^D(v_k)$. By (3.5),

$$\mu_{1i}(v_m) = \mu_{1i}^D(v_k) > \mu_{1j}^D(v_k) = \mu_{1j}(v_n) > \mu_{1j}(v_r)$$
$$\Rightarrow \mu_{1i}(v_m) > \mu_{1j}(v_r) \text{ which is a contradiction}$$
$$\therefore\ G_i^D \subseteq G_j^D.$$

(ii) In similar manner in case 1, we can prove case 2.

□

**Theorem 3.8.** *Let* $G_i = G, G^{\times}, \mu_{1i}, \gamma_{1i}, \mu_{2i}, \gamma_{2i}) \in G$, $i = 1, 2$ *then* $G_1^D \subseteq G_2 \Leftrightarrow G_1 \subseteq G_2^E$.

*Proof.* $G_1^D \subseteq G_2 \Leftrightarrow$ By definition 1.3 of PF subgraph, for any vertex $v_k$ and edge $e_{v_k v_l}$,

$$\mu_{11}^D(v_k) \leq \mu_{12}(v_k),\ \gamma_{11}^D(v_k) \geq \mu_{12}(v_k)\mu_{21}^D(e_{v_k v_l}) \leq \mu_{22}^D(e_{v_k v_l}),$$
$$\gamma_{11}^D(e_{v_k v_l}) \geq \gamma_{22}^D(e_{v_k v_l}).$$
$$\Leftrightarrow \vee\mu_{11}(v_m) \leq \mu_{12}(v_k),\ \wedge\gamma_{11}(v_m)) \geq \gamma_{12}(v_m) \geq \gamma_{12}(v_k),$$
$$\vee\, \mu_{21}(e_{v_m v_n}) \leq \mu_{21}(e_{v_k v_l}),\ \wedge\gamma_{21}(e_{v_m v_n}) \geq \gamma_{22}(e_{v_k v_l})$$

where $v_m$ is the $\beta_n$ adjacency vertex of $v_k$ and $e_{v_m v_n}$ is the $\beta_n$ adjacency edge of $e_{v_k v_l}$

$$\Leftrightarrow \mu_{11}(v_m) \leq \mu_{12}(v_k), \gamma_{11}(v_m) \geq \gamma_{12}(v_k),$$
$$\mu_{21}(e_{v_m v_n}) \leq \mu_{22}(e_{v_k v_l}),\ \gamma_{22}^D(e_{v_k v_l})$$

for each vertices $v_k$ and each edges $e_{v_k v_l}$.

$$\Leftrightarrow \mu_{11}(v_m) \leq \wedge\mu_{12}(v_k), \gamma_{11}(v_m) \geq \vee\gamma_{12}(v_k),$$

$$\mu_{21}(e_{v_m v_n}) \leq \mu_{22}(e_{v_k v_l}),\ \gamma_{21}^{D}(e_{v_m v_n}) \geq \vee\gamma_{21}^{D}(e_{v_k v_l}),$$

where $v_k$ are the $\beta_n$-adjacency vertices of $v_m$ and $e_{v_k v_l}$ are the $\beta_n$ adjacency edges of $e_{v_m v_n}$.

$$\begin{aligned}
&\Leftrightarrow \mu_{11}(v_m) \leq \mu_{12}^{E}(v_m), \gamma_{11}(v_m) \geq \vee\gamma_{12}^{E}(v_m),\\
&\mu_{21}(e_{v_m v_n}) \leq \mu_{22}^{E}(e_{v_m v_n}),\ \gamma_{21}^{D}(e_{v_m v_n}) \geq \gamma_{22}^{E}(e_{v_m v_n})\\
&\Leftrightarrow G_1 \subseteq G_{12}^{E}.
\end{aligned}$$

This completes the proof. □

**Remark 3.9.** The above theorem proves an adjunction between dilation and erosion in Pythagorean fuzzy graph.

## 4 Application: Spread of a pandemic: Prediction and Control

It is a great challenge to predict and control the spread of a disease in a school. Construction of a disease spread prediction model by considering the infection patterns of neighboring areas is explained in this section using an idea of vertex dilation and edge dilation in a Pythagorean fuzzy graph.

**Algorithm for Disease Spread Prediction using Vertex Dilation:**

**Construction of PFG:**

Construct a Pythagorean fuzzy graph $G_i = (G, G^{\times}, \mu_{1i}, \gamma_{1i}, \mu_{2i}, \gamma_{2i})$ wheres $\mu_{1i}$ and $\gamma_{1i}$ are membership and non-membership functions of each vertices $\mu_{2i}$ and $\gamma_{2i}$ are membership and non-membership functions of each edges $e_{v_l v_k}$. There are different co-curricular activity groups in a school. Each student should be member of at least one group. These groups are represented as vertices and the edges are represented as social connections of members of the groups. Students from different groups may not be from same class. They may interact in lunch time or in assembly or at the time of different events in the school. We, in this example, consider six groups (vertices): Groups A-Sports Group, Group B-Book club, Group C-Study group, Group D-Music club, Group E-Nature Club and Group F-NSS. Due to large size of the team and high contact frequency, transmission rate is high for sports like football and volleyball and music club. Interactions are limited in Book group and Nature club and they are small in size so that transmission rate is low. However, medium size and regular meetings in a room create a moderate transmission in study group. NSS programs also creates medium transmission. Health status of a student (body mass index, vaccination, presences of illness) also causes spreading of diseases.

| Group | Characteristics |
|---|---|
| A | Large size, contact frequency is high, Transmission rate is high |
| B | Small size, contact frequency is too limited, Transmission rate is low |
| C | Medium size, contact frequency is not high, Transmission rate is moderate |
| D | Large size, contact frequency is high, Transmission rate is high |
| E | Small size, contact frequency is limited, Transmission rate is low |
| F | Medium size, contact frequency is not limited, Transmission rate moderate |

Membership functions of each vertices represent the likelihood of transmitting the disease, whereas non-membership functions represent the likelihood of not transmitting the disease. The values of membership and non-membership functions of vertices of PFG are influenced by the factors like size, health status, and contact frequency.

**Assign membership functions for the factors size and health status:**

| Size | Approximate Score |
|---|---|
| Less than 20 students | 0.3 |
| 20–30 students | 0.6 |
| Greater than 30 students | 0.9 |

| Health-Vaccination | Approximate Score |
|---|---|
| Less than 50% vaccination | 0.3 |
| 50–75% vaccination | 0.6 |
| Greater than or equal to 75% | 0.8 |

**Calculation of pythagorean membership $(\mu_{1i})$ and pythagorean non-membership functions $(\gamma_{1i})$:**
Use the maximum (supremum) operation to combine the membership values for each factor:

$$\mu_{1i} = \text{maximum (Membership-Size, Membership-HealthStatus, Membership-SpreadingTendency)}$$

Use the minimum (infimum) operation to combine the non-membership values for each factor:

$$\gamma_1 = \text{minimum (NonMembership-size, NonMembership-HealthStatus, NonMembership-SpreadingTendency)}$$

| Group | Factors | | | Membership function $(\mu_{1i})$ |
|---|---|---|---|---|
| | Size | Contact Frequency | Health | |
| A | 0.6 | 0.8 | 0.5 | 0.8 |
| B | 0.3 | 0.1 | 0.2 | 0.3 |
| C | 0.4 | 0.3 | 0.5 | 0.5 |
| D | 0.7 | 0.8 | 0.4 | 0.8 |
| E | 0.2 | 0.1 | 0.4 | 0.4 |
| F | 0.4 | 0.3 | 0.4 | 0.4 |

| Group | Factors | | | Non-membership function $(\gamma_{1i})$ |
|---|---|---|---|---|
| | Size | Contact Frequency | Health | |
| A | 0.1 | 0.2 | 0.5 | 0.1 |
| B | 0.8 | 0.9 | 0.6 | 0.6 |
| C | 0.5 | 0.7 | 0.5 | 0.5 |
| D | 0.3 | 0.2 | 0.3 | 0.2 |
| E | 0.9 | 0.9 | 0.5 | 0.5 |
| F | 0.7 | 0.6 | 0.5 | 0.5 |

Interaction among the students of these groups A, B, C, D, E and F in different activities and sharing of space and equipment cause transmission of disease within the school. At what extent, this interaction among groups happens is taken to be membership degree of an edge connecting these vertices. This can be calculated based on the observation.

Possible edges with membership $(\mu_{2i})$ and non-membership functions $(\gamma_{2i})$ are listed below:

| Edges | Membership and non-membership degrees |
|---|---|
| AC | (0.5,0.6) |
| AD | (0.8,0.3) |
| AF | (0.2,0.7) |
| BC | (0.2,0.6) |
| BD | (0.2,0.7) |
| BE | (0.3,0.7) |
| CD | (0.3,0.5) |
| CE | (0.4,0.7) |
| CF | (0.2,0.7) |
| DF | (0.3,0.9) |
| EF | (0.2,0.6) |

Thus, a Pythagorean Fuzzy Graph (Figure 13) which represent the above table is given below:

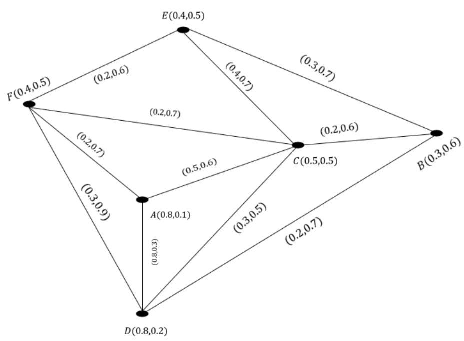


**Figure 13 PFG.**

**Dilation of vertices and edges & Measurement of risk.**
Analysis of spread of disease can be done using vertex dilation and edge dilation of Pythagorean fuzzy graph. Vertex dilation helps to identify at what extend a group spread disease and edge dilation indicate at what extend the interaction between two groups may cause spreading of disease. Dilation (whether it is for vertices or edges) focuses on determining the effect of neighbouring vertices or edges on risk measurement. To attain this, we exclude edge's (or vertex's) own membership and non-membership function at the time of calculating dilation of edge (or vertex). Dilated membership function of vertices and edges $(\mu_1 \vee \mu_2)$ gives at what extent the disease likely to spread and non-membership function of vertices and edges $(\gamma_1 \vee \gamma_2)$ projects at what extent the disease not likely to spread. Non-membership function measures the uncertainty on risk transmission.

$\mu_1 =$ Supremum of membership functions of neighbouring vertices of a vertex
$\mu_2 =$ Supremum of membership functions of neighbouring edges of an edge
$\gamma_1 =$ Infimum of non-membership functions of neighbouring vertices of a vertex
$\gamma_2 =$ Infimum of non-membership functions of neighbouring edges of an edge

The measure of risk is defined as $\beta_i = \mu_i^2 - \gamma_i^2$; $i = 1, 2$ for each vertices and edges.
Dilated membership and non-membership functions of vertices and edges are defined as follows:

| Vertex | Neighbouring Vertices | $\mu_1$ | $\gamma_1$ | Dilated vertex | $\beta_1 = \mu_1^2 - \gamma_1^2$ |
|---|---|---|---|---|---|
| A(0.8,0.1) | C,D,F | 0.8 | 0.2 | (0.8,0.2) | 0.6 |
| B(0.3,0.6) | C,D,E | 0.8 | 0.2 | (0.8,0.2) | 0.6 |
| C(0.5,0.5) | A,B,D,E,F | 0.8 | 0.1 | (0.8,0.1) | 0.63 |
| D(0.8,0.2) | A,B,C,F | 0.8 | 0.1 | (0.8,0.1) | 0.63 |
| E(0.4,0.5) | B,C,F | 0.5 | 0.5 | (0.5,0.5) | 0 |
| F(0.4,0.5) | A,D,C,E | 0.8 | 0.1 | (0.8,0.1) | 0.63 |

| Vertex | Neighbouring Edges | $\mu_2$ | $\gamma_2$ | Dilated vertex | $\beta_2 = \mu_2^2 - \gamma_2^2$ |
|---|---|---|---|---|---|
| AC(0.5,0.6) | AF, AD, CF, CE, BC, CD | 0.8 | 0.3 | (0.8,0.3) | 0.55 |
| AD(0.8,0.3) | AF, AC, DF, DC, BD | 0.5 | 0.5 | (0.5,0.5) | 0 |
| AF(0.2,0.7) | AD, AC, DF, CF, EF | 0.8 | 0.3 | (0.8,0.3) | 0.55 |
| BC(0.2,0.6) | CF, CE, AC, CD, BE, BD | 0.5 | 0.5 | (0.5,0.5) | 0 |
| BD(0.2,0.7) | DF, AD, CD, BC, BE | 0.8 | 0.3 | (0.8,0.3) | 0.55 |
| BE(0.3,0.7) | EF, CE, BC, BD | 0.4 | 0.6 | (0.4,0.6) | -0.2 |
| CD(0.3,0.5) | BC, CE, CF, AC, BD, AD, DF | 0.8 7 | 0.3 | (0.8,0.3) | 0.55 |
| CE(0.4,0.7) | EF, BE, BC, CF, AC, BC, CD | 0.5 | 0.5 | (0.5,0.5) | 0 |
| CF(0.2,0.7) | BC, AC, CD, CE, EF, DF, AF | 0.5 | 0.5 | (0.5,0.5) | 0 |
| DF(0.3,0.9) | EF, CF, AF, AD, CD, BD | 0.8 | 0.3 | (0.8,0.3) | 0.55 |
| EF(0.2,0.6) | BE, CE, CF, AF, DF | 0.4 | 0.7 | (0.4,0.7) | -0.33 |

Dilated PFG (Figure 13) of a Pythagorean Fuzzy Graph (Figure 14) is given below:

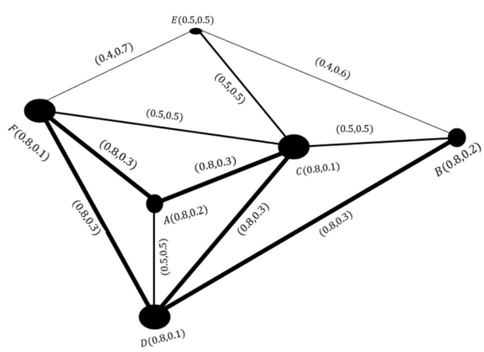


**Figure 14. Dilated PFG.**

Both $\mu_i$ and $\gamma_i$ are considered for making decision using above PFG model for disease spread. High values of $\mu_i$ and $\gamma_i$ predicts high transmission risk and high uncertainty on risk assessment. That is, high value of $\mu_i$ and $\beta_i$ is small suggests high transmission risk and high uncertainty on risk assessment.

**Set a threshold.**
Set a threshold value for identifying the risk category vertices and edges. In particular, setting a threshold can separate the transmission risk levels as high, medium and low. So this categorization allocate each vertex and edge to a particular transmission category.
Set threshold value of level of risk $(\mu_i)$ is given below:

| Level of risk | Membership function $(\mu_i)$ |
|---|---|
| High | $\mu_i \geq 0.7$ |
| Medium | $0.4 \leq \mu_{1i} < 0.7$ |
| Low | $\mu_{1i} < 0.4$ |

**Disease Spread Prediction & Strategy Adaptation**

| Risk Level | Vertices | Edges |
|---|---|---|
| High | A, B, C, D, F | AC, AF, BD, CD, DF |
| Medium | E | AD, BC, BE, CE, CF, EF |
| Low | — | — |

The dilated membership value is described as an indicator of the vertex's transmission risk. A higher dilated membership value implies a higher likelihood of transmitting the disease due to the influence of nearby individuals or areas. For each vertex and edge, analyse the dilated membership values to assess its likelihood of being infected. Measure of risk indicates the reliability of the result. High risk vertices and edges are represented using large dots and possible high-risk transmission edges are marked as large width lines in Figure 14.

Different containment strategies can be implemented to different categorized risk level. Strict isolations measures can be applied in high risk areas and proper monitoring is possible in medium and low risk levels. Awareness programs regarding hygiene should also be conducted.

**Continuous Monitoring and Adaptation:**

Continuous monitoring and evaluation are necessary for vertices with risk factor according to changing situation and upcoming risks.

## 5 Conclusion

In this paper, we introduced vertex dilation, vertex erosion, edge dilation and edge erosion on PFG using $\beta_n$-adjacency. Combining vertex dilation and edge dilation resulted dilated PFG and combination of vertex erosion and edge erosion gave eroded PFG. An algorithm for determining dilated PFG and Eroded PFG was illustrated with numerical example. We proposed the definition of morphological dilation induced strong products of these dilated PFGs. We also derived the some properties of dilated PFG and eroded PFG. An algorithm for predicting and controlling the spreading of disease is explained as an application of vertex dilation on PFG in decision making.

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