Palestine Journal of Mathematics

Vol 14(Special Issue II)(2025) , 420–426

Unique Fixed Point Results For Contraction Maps in $b-$ Metric Spaces

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2020 Mathematics Subject Classification. 47H10; 54H25.

Keywords and phrases: unique fixed point; b-Cauchy sequence; b-metric space.

Abstract The scope of this paper includes new and extended results about few unique fixed points in complete $b-$metric spaces, with an emphasis on contractive mappings of the $(\psi, \beta)-$ Geraghty type. Practical applications are covered in the paper, along with discussions of unique fixed-point results pertaining to integral-type contractions and an investigation into the existence of integral equation solutions.

1 Introduction

Over the last five decades, fixed point theory(FPT) research has been crucial in solving problems involving nonlinear phenomena. Along with advancements in topology and geometry, FPTs and the development of various approaches have been crucial to the advancement of both pure and applied analysis. Geraghty [3] presented a series of functions in 1973 that expanded on the Banach contraction concept. This important contribution attempted to give researchers and mathematicians a more flexible and all-encompassing framework for mathematical study, enabling them to go beyond the conventional limitations of the Banach contraction principle in their investigations. This expansion has shown to be beneficial in a number of mathematical contexts, promoting a better comprehension of FPTs and offering a more thorough viewpoint for mathematical analysis. Bakhtin [15] proposed the concept of b-metric spaces(b-MS) in 1989 as a generalization of conventional metric spaces(MS). Numerous articles on FPT in these domains have since been published. Readers who are interested in more works and results in b-MS are advised to consult References [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] for a thorough examination.

1.1 Definition

Let us choose a nonempty set Ω and a real integer $s \geq 1$. We define a mapping $\rho : \Omega \times \Omega \to [0, \infty)$ as a b-metric on Ω iff $\forall\, \Lambda, \varpi, \delta \in \Omega$:

(i) $\rho(\Lambda, \varpi) = 0 \iff \Lambda = \varpi$,

(ii) $\rho(\Lambda, \varpi) = \rho(\varpi, \Lambda)$,

(iii) $\rho(\Lambda, \varpi) \leq s[\rho(\Lambda, \delta) + \rho(\delta, \varpi)]$.

Then, (Ω, ρ) is known as a b-MS with parameters.

1.2 Example

Consider a metric space (Ω, ρ) with parameters $\beta > 1$, $\lambda \geq 0$, and $\alpha > 0$. Define the function $\rho(\Lambda, \varpi) = \lambda\rho(\Lambda, \varpi) + \alpha\rho(\Lambda, \varpi)^{\beta}$ for $\Lambda, \varpi \in \Omega$. The resulting space (Ω, ρ) is a b-MS with the parameter $\xi = 2^{\beta-1}$ but does not qualify as a metric space on Ω.

1.3 Definition

In a MS, a b-Cauchy sequence is a sequence of points where the distance between any two points in the sequence becomes arbitrarily small as the sequence progresses, and the sequence is bounded, meaning $\exists$ a real number M such that the distance between every pair of points in the sequence is less than or equal to M.

1.4 Definition

Let $\mathcal{S}$ be the collection of all functions $\alpha : [0, \infty) \to [0, 1)$ that holds the condition:

$$\lim_{\mathtt{p}\to\infty} \alpha(\sigma_{\mathtt{p}}) = 1 \text{ implies } \lim_{\mathtt{p}\to\infty} \sigma_{\mathtt{p}} = 0. \tag{1.1}$$

The Geraghty contraction, a theorem established by Geraghty [3], is expressed as follows.

1.5 Theorem

Consider a metric space which is complete (Ω, ρ), and let $\mathtt{M} : \Omega \to \Omega$ be a mapping. Suppose $\exists\ \alpha \in \mathcal{S}$ such that for any $\Lambda, \varpi \in \Omega$,

$$\rho(\mathtt{M}\Lambda, \mathtt{M}\varpi) \leq \alpha(\rho(\Lambda, \varpi))\rho(\Lambda, \varpi). \tag{1.2}$$

Then $\mathtt{M}$ has a unique fixed point(UFP) $z \in \Omega$.

1.6 Definition

Consider a b-MS (Ω, ρ) with a parameter $\xi \geq 1$, and let $\mathcal{S}$ be the set of all functions $\alpha : [0, \infty) \to [0, \frac{1}{\xi})$ that adhere to the following condition:

$$\lim_{\mathtt{p}\to\infty} \alpha(\sigma_n) = \frac{1}{\xi} \implies \lim_{\mathtt{p}\to\infty} \sigma_n = 0. \tag{1.3}$$

1.7 Theorem

Consider a complete b-MS (Ω, ρ) with a parameter $\xi \geq 1$, and let $\mathtt{M} : \Omega \to \Omega$ be a self-map. Assume the existence of $\beta \in \mathcal{S}$ satisfying:

$$\rho(\mathtt{M}\Lambda, \mathtt{M}\varpi) \leq \alpha((\Theta(\Lambda, \varpi))(\Theta(\Lambda, \varpi), \forall \Lambda \geq \varpi, \tag{1.4}$$

where

$$\Theta(\Lambda, \varpi) = \max\{\rho(\Lambda, \varpi), \rho(\Lambda, \mathtt{M}\Lambda), \rho(\varpi, \mathtt{M}\varpi), \frac{1}{2\xi}\big(\rho(\Lambda, \mathtt{M}\varpi) + \rho(\varpi, \mathtt{M}\Lambda)\big)\},$$

and $\alpha \in \mathcal{S}$. Then $\mathtt{M}$ has a UFP $\Lambda^* \in \Omega$.

1.8 Definition

An altering distance function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfies the below properties:

(i) Non-decreasing: If m $\leq r$, then $\Psi(m) \leq \Psi(r)$.

(ii) Positive: $\forall\ \sigma > 0, \Psi(\sigma) > 0$.

1.9 Definition

Consider a $\mathcal{POSET}$ $(\Omega, \preceq)$ and a self-map $\mathtt{M}$. We classify $\mathtt{M}$ as weakly increasing if, for all $\Lambda, \varpi \in \Omega$, $\Lambda \preceq \varpi$ implies $\mathtt{M}(\Lambda) \preceq \mathtt{M}(\varpi)$.

1.10 Lemma

If $\beta : [0, \infty) \to [0, \infty)$ and Ψ is an altering distance function is a continuous function with the condition $\Psi(\sigma) > \beta(\sigma)$ for all $\sigma > 0$, then $\beta(0) = 0$.

In recent years, there has been a notable trend among researchers to generalize Geraghty's result across different metric spaces. This paper contributes to this trend by extended some UFP theorems specifically for (ψ, β)- Geraghty contractive mappings with in the framework of b-MS.

2 The Main Results

2.1 Theorem

Consider a $\mathcal{POSET}$ Ω equipped with a metric ρ, making (Ω, ρ) a complete b-MS. Let $\mathtt{M}$ be weakly increasing mappings from Ω to itself. Suppose the following inequality holds for all $\Lambda \geq \varpi$:

$$\psi(\rho(\mathtt{M}\Lambda, \mathtt{M}\varpi)) \leq \zeta(\Theta(\Lambda, \varpi))\beta(\Theta(\Lambda, \varpi)), \mathfrak{f}orall\Lambda \geq \varpi, \tag{2.1}$$

where

$$\Theta(\Lambda, \varpi) = \max\{\rho(\Lambda, \varpi), \rho(\Lambda, \mathtt{M}\Lambda), \rho(\varpi, \mathtt{M}\varpi), \frac{1}{2\xi}\big(\rho(\Lambda, \mathtt{M}\varpi) + \rho(\varpi, \mathtt{M}\Lambda)\big)\},$$

and $\zeta \in \mathcal{S}$, $\psi \in \Psi$, and $\beta : [0, \infty) \to [0, \frac{1}{\xi})$ is a continuous function with the condition $\psi(\sigma) > \delta(\sigma), \forall\ \sigma > 0$.

Furthermore, assume that for each pair of elements $\Lambda, \varpi \in \Omega$, $\exists\ z \in \Omega$ that is comparable to both Λ and ϖ. If either $\mathtt{M}$ is continuous, then $\mathtt{M}$ possess a UFP.

Proof. Suppose that $\Lambda_0 \in \Omega$ to be an arbitrary point in Ω such that $\mathtt{M}\Lambda_0 = \Lambda_1$ and $\mathtt{M}\Lambda_1 = \Lambda_2$. Continuing with this manner, sequences $\{\Lambda_\mathtt{p}\}$ and $\{\varpi_\mathtt{p}\}$ in Ω can be constructed as follows.

$$\Lambda_{2\mathtt{p}+1} = \mathtt{M}\Lambda_{2\mathtt{p}} = \varpi_{2\mathtt{p}}, \quad \Lambda_{2\mathtt{p}+2} = \mathtt{M}\Lambda_{2\mathtt{p}+1} = \varpi_{2\mathtt{p}+1}, \quad \forall \mathtt{p} \in \mathbb{N}. \tag{2.2}$$

As $\mathtt{M}$ is monotonic increasing functions, we have

$$\Lambda_1 \preceq \Lambda_2 \preceq \Lambda_3 \cdots \preceq \Lambda_{2\mathtt{p}+1} \preceq \Lambda_{2\mathtt{p}+2} \ldots$$

Thus,

$$\varpi_0 \preceq \varpi_1 \preceq \varpi_2 \cdots \preceq \varpi_{2\mathtt{p}} \preceq \varpi_{2\mathtt{p}+1} \ldots$$

To begin, let's assume that $\exists\ \mathtt{p} \in \mathbb{N}$ such that $\varpi_{2\mathtt{p}-1} = \varpi_{2\mathtt{p}}$. Subsequently, from (2.1), we get

$$\psi(\rho(\varpi_{2\mathtt{p}}, \varpi_{2\mathtt{p}+1})) = \psi((\mathtt{M}\Lambda_{2\mathtt{p}}, \mathtt{M}\Lambda_{2\mathtt{p}+1})) \leq \zeta(\Theta(\Lambda_{2\mathtt{p}}, \Lambda_{2\mathtt{p}+1}))\beta(\Theta(\Lambda_{2\mathtt{p}}, \Lambda_{2\mathtt{p}+1})) \tag{2.3}$$

where

$$\begin{aligned}
&\Theta(\Lambda_{2\mathfrak{p}}, \Lambda_{2\mathfrak{p}+1})\\
&= \max\{\rho(\Lambda_{2\mathfrak{p}}, \Lambda_{2\mathfrak{p}+1}), \rho(\Lambda_{2\mathfrak{p}}, \mathtt{M}\Lambda_{2\mathfrak{p}}), \rho(\Lambda_{2\mathfrak{p}+1}, \mathtt{M}\Lambda_{2\mathfrak{p}+1}), \frac{1}{2\xi}\big(\rho(\Lambda_{2\mathfrak{p}}, \mathtt{M}\Lambda_{2\mathfrak{p}+1}) + \rho(\Lambda_{2\mathfrak{p}+1}, \mathtt{M}\Lambda_{2\mathfrak{p}})\big)\}\\
&= \max\{\rho(\varpi_{2\mathfrak{p}-1}, \varpi_{2\mathfrak{p}}), \rho(\varpi_{2\mathfrak{p}-1}, \varpi_{2\mathfrak{p}}), \rho(\varpi_{2\mathfrak{p}}, \varpi_{2\mathfrak{p}+1}), \frac{1}{2\xi}\big(\rho(\varpi_{2\mathfrak{p}-1}, \varpi_{2\mathfrak{p}+1}) + \rho(\varpi_{2\mathfrak{p}}, \varpi_{2\mathfrak{p}})\big)\}\\
&\leq \max\{\rho(\varpi_{2\mathfrak{p}-1}, \varpi_{2\mathfrak{p}}), \rho(\varpi_{2\mathfrak{p}-1}, \varpi_{2\mathfrak{p}}), \rho(\varpi_{2\mathfrak{p}}, \varpi_{2\mathfrak{p}+1}), \frac{\xi}{2\xi}\big(\rho(\varpi_{2\mathfrak{p}-1}, \varpi_{2\mathfrak{p}}) + \rho(\varpi_{2\mathfrak{p}}, \varpi_{2\mathfrak{p}+1})\big)\}\\
&= \max\{\rho(\varpi_{2\mathfrak{p}-1}, \varpi_{2\mathfrak{p}}), \rho(\varpi_{2\mathfrak{p}-1}, \varpi_{2\mathfrak{p}}), \rho(\varpi_{2\mathfrak{p}}, \varpi_{2\mathfrak{p}+1}), \frac{1}{2}\big(\rho(\varpi_{2\mathfrak{p}-1}, \varpi_{2\mathfrak{p}}) + \rho(\varpi_{2\mathfrak{p}}, \varpi_{2\mathfrak{p}+1})\big)\}\\
&= \max\{\rho(\varpi_{2\mathfrak{p}-1}, \varpi_{2\mathfrak{p}}), \rho(\varpi_{2\mathfrak{p}}, \varpi_{2\mathfrak{p}+1})\}\\
&= 0.
\end{aligned}$$

Inequality (2.3) leads to

$$\psi(\rho(\varpi_{2\mathfrak{p}}, \varpi_{2\mathfrak{p}+1})) = 0. \tag{2.4}$$

From this, it follows that $\varpi_{2\mathfrak{p}+1} = \varpi_{2\mathfrak{p}}$.

Therefore, $\varpi_m = \varpi_{2\mathfrak{p}-1}$ holds for any $m \geq 2\mathfrak{p}$. As a result, for any $m \geq 2\mathfrak{p}$, we get $\Lambda_m = \Lambda_{2\mathfrak{p}}$. Hence that the sequence $\Lambda_\mathfrak{p}$ is a Cauchy sequence.

As a second consideration, let's assume $\varpi_\mathfrak{p} \neq \varpi_{\mathfrak{p}+1}$ for any integer $\mathfrak{p}$. Define $\Delta_\mathfrak{p} = \rho(\varpi_\mathfrak{p}, \varpi_{\mathfrak{p}+1})$.

Now, we aim to prove that $\Delta_\mathfrak{p} \to 0$ as $\mathfrak{p} \to \infty$.

As $\Lambda_{2\mathfrak{p}}$ and $\Lambda_{2\mathfrak{p}+1}$ are comparable, we can deduce again from (2.1),

$$\psi(\rho(\varpi_{2\mathfrak{p}+2}, \varpi_{2\mathfrak{p}+1})) = \psi((\mathtt{M}\Lambda_{2\mathfrak{p}+2}, \mathtt{M}\Lambda_{2\mathfrak{p}+1})) \leq \zeta(\Theta(\Lambda_{2\mathfrak{p}+2}, \Lambda_{2\mathfrak{p}+1}))\beta(\Theta(\Lambda_{2\mathfrak{p}+2}, \Lambda_{2\mathfrak{p}+1})) \tag{2.5}$$

where

$$\begin{aligned}
&\Theta(\Lambda_{2\mathfrak{p}+2}, \Lambda_{2\mathfrak{p}+1})\\
&= \max\{\rho(\Lambda_{2\mathfrak{p}+2}, \Lambda_{2\mathfrak{p}+1}), \rho(\Lambda_{2\mathfrak{p}+2}, \mathtt{M}\Lambda_{2\mathfrak{p}+2}), \rho(\Lambda_{2\mathfrak{p}+1}, \mathtt{M}\Lambda_{2\mathfrak{p}+1}),\\
&\qquad \frac{1}{2\xi}\big(\rho(\Lambda_{2\mathfrak{p}+2}, \mathtt{M}\Lambda_{2\mathfrak{p}+1}) + \rho(\Lambda_{2\mathfrak{p}+1}, \mathtt{M}\Lambda_{2\mathfrak{p}+2})\big)\}\\
&= \max\{\rho(\varpi_{2\mathfrak{p}+1}, \varpi_{2\mathfrak{p}}), \rho(\varpi_{2\mathfrak{p}+1}, \varpi_{2\mathfrak{p}+2}), \rho(\varpi_{2\mathfrak{p}}, \varpi_{2\mathfrak{p}+1}), \frac{1}{2\xi}\big(\rho(\varpi_{2\mathfrak{p}+1}, \varpi_{2\mathfrak{p}+1}) + \rho(\varpi_{2\mathfrak{p}}, \varpi_{2\mathfrak{p}+2})\big)\}\\
&\leq \max\{\rho(\varpi_{2\mathfrak{p}+1}, \varpi_{2\mathfrak{p}}), \rho(\varpi_{2\mathfrak{p}+1}, \varpi_{2\mathfrak{p}+2}), \rho(\varpi_{2\mathfrak{p}}, \varpi_{2\mathfrak{p}+1}), \frac{\xi}{2\xi}\big(\rho(\varpi_{2\mathfrak{p}}, \varpi_{2\mathfrak{p}+1}) + \rho(\varpi_{2\mathfrak{p}+1}, \varpi_{2\mathfrak{p}+2})\big)\}\\
&= \max\{\rho(\varpi_{2\mathfrak{p}+1}, \varpi_{2\mathfrak{p}}), \rho(\varpi_{2\mathfrak{p}+1}, \varpi_{2\mathfrak{p}+2}), \rho(\varpi_{2\mathfrak{p}}, \varpi_{2\mathfrak{p}+1}), \frac{1}{2}\big(\rho(\varpi_{2\mathfrak{p}}, \varpi_{2\mathfrak{p}+1}) + \rho(\varpi_{2\mathfrak{p}+1}, \varpi_{2\mathfrak{p}+2})\big)\}\\
&= \max\{\rho(\varpi_{2\mathfrak{p}}, \varpi_{2\mathfrak{p}+1}), \rho(\varpi_{2\mathfrak{p}+1}, \varpi_{2\mathfrak{p}+2})\}.
\end{aligned}$$

If $\rho(\varpi_{2\mathfrak{p}}, \varpi_{2\mathfrak{p}+1}) \geq \rho(\varpi_{2\mathfrak{p}+1}, \varpi_{2\mathfrak{p}+2})$, then $\Theta(\Lambda_{2\mathfrak{p}+2}, \Lambda_{2\mathfrak{p}+1}) = \rho(\varpi_{2\mathfrak{p}+1}, \varpi_{2\mathfrak{p}+2})$.

According to Condition (2.5), we obtain:

$$\psi(\rho(\varpi_{2\mathfrak{p}+2}, \varpi_{2\mathfrak{p}+1})) \leq \zeta(\rho(\varpi_{2\mathfrak{p}+1}, \varpi_{2\mathfrak{p}+2}))\beta(\rho(\varpi_{2\mathfrak{p}+1}, \varpi_{2\mathfrak{p}+2})). \tag{2.6}$$

Employing the condition stated in Theorem 2.1 and given the circumstance that $\zeta \in \mathcal{S}$, we obtain:

$$\rho(\varpi_{2\mathfrak{p}+2}, \varpi_{2\mathfrak{p}+1}) \leq \frac{1}{\xi}\rho(\varpi_{2\mathfrak{p}+2}, \varpi_{2\mathfrak{p}+1}), \mathfrak{p} \in \mathbb{N}. \tag{2.7}$$

This leads to a contradiction. Hence, we conclude:

$$\Theta(\Lambda_{2\mathfrak{p}+2}, \Lambda_{2\mathfrak{p}+1}) = \rho(\varpi_{2\mathfrak{p}+1}, \varpi_{2\mathfrak{p}}). \tag{2.8}$$

Subsequently, following Condition (2.5), we derive:

$$\psi(\rho(\varpi_{2\mathfrak{p}+2}, \varpi_{2\mathfrak{p}+1})) \leq \zeta(\rho(\varpi_{2\mathfrak{p}+1}, \varpi_{2\mathfrak{p}}))\beta(\rho(\varpi_{2\mathfrak{p}+1}, \varpi_{2\mathfrak{p}})). \tag{2.9}$$

Employing the condition stated in Theorem 2.1 and given the circumstance that $\zeta \in \mathcal{S}$, we obtain:

$$\rho(\varpi_{2\mathfrak{p}+2}, \varpi_{2\mathfrak{p}+1}) \leq \rho(\varpi_{2\mathfrak{p}+1}, \varpi_{2\mathfrak{p}}), \mathfrak{p} \in \mathbb{N}. \tag{2.10}$$

Similarly,

$$\rho(\varpi_{2\mathfrak{p}+1}, \varpi_{2\mathfrak{p}}) \leq \rho(\varpi_{2\mathfrak{p}}, \varpi_{2\mathfrak{p}-1}), \mathfrak{p} \in \mathbb{N}. \tag{2.11}$$

By combining (2.10) and (2.11), we obtain:

$$\rho(\varpi_{2\mathfrak{p}+2}, \varpi_{2\mathfrak{p}+1}) \leq \rho(\varpi_{2\mathfrak{p}+1}, \varpi_{2\mathfrak{p}}) \leq \rho(\varpi_{2\mathfrak{p}}, \varpi_{2\mathfrak{p}-1}), \mathfrak{p} \in \mathbb{N}. \tag{2.12}$$

Consequently, the sequence $\{\Delta_\mathfrak{p}\}$ decreases monotonically, therefore, $\exists\, r \geq 0$ such that

$$\lim_{\mathfrak{p}\to\infty} \Delta_\mathfrak{p} = \lim_{\mathfrak{p}\to\infty} \rho(\varpi_\mathfrak{p}, \varpi_{\mathfrak{p}+1}) = r. \tag{2.13}$$

Derived from (2.9), we obtain:

$$\psi(\rho(\varpi_{2\mathtt{p}+2},\varpi_{2\mathtt{p}+1}))\leq\zeta(\rho(\varpi_{2\mathtt{p}+1},\varpi_{2\mathtt{p}}))\beta(\rho(\varpi_{2\mathtt{p}+1},\varpi_{2\mathtt{p}})). \tag{2.14}$$

As $\mathtt{p}\to\infty$ in the above condition and applying (2.13), we obtain $\psi(r)\leq\beta(r)$, as $\zeta\in\mathcal{S}$.

This contradicts the statement of Theorem 2.1. Thus, $r=0$. This implies that:

$$\Delta_{\mathtt{p}}=\rho(\varpi_{\mathtt{p}},\varpi_{\mathtt{p}+1})\to 0 \text{ as } \mathtt{p}\to\infty. \tag{2.15}$$

Next,to prove $\{\Lambda_{\mathtt{p}}\}$ is a b-Cauchy sequence. To demonstrate this, our objective is to establish the Cauchy property for $\{\Lambda_{2\mathtt{p}}\}$. Assuming the contrary, let us assume that $\{\Lambda_{2\mathtt{p}}\}$ is not a b-Cauchy sequence. Consequently, for any $\varepsilon>0$, $\exists$ two subsequences of positive integers $\mathtt{p}_{\mathfrak{k}}$ and $\mathtt{q}_{\mathfrak{k}}$ with the property $\mathtt{p}_{\mathfrak{k}}>\mathtt{q}_{\mathfrak{k}}>\mathfrak{k}$ for all $\mathfrak{k}>0$,

$$\rho(\Lambda_{2\mathtt{p}_{\mathfrak{k}}},\Lambda_{2\mathtt{q}_{\mathfrak{k}}})>\varepsilon \text{ and } \rho(\Lambda_{2\mathtt{p}_{\mathfrak{k}}},\Lambda_{2\mathtt{q}_{\mathfrak{k}-1}})<\varepsilon. \tag{2.16}$$

Utilizing (2.15) and applying the b-triangle inequality, we have:

$$\begin{aligned}\varepsilon&<\rho(\Lambda_{2\mathtt{p}_{\mathfrak{k}}},\Lambda_{2\mathtt{q}_{\mathfrak{k}}})\\&\leq\xi(\rho(\Lambda_{2\mathtt{p}_{\mathfrak{k}}},\Lambda_{2\mathtt{q}_{\mathfrak{k}-1}})+\rho(\Lambda_{2\mathtt{p}_{\mathfrak{k}-1}},\Lambda_{2\mathtt{q}_{\mathfrak{k}}}))\\\frac{\varepsilon}{\xi}&\leq\rho(\Lambda_{2\mathtt{p}_{\mathfrak{k}}},\Lambda_{2\mathtt{q}_{\mathfrak{k}-1}})+\rho(\Lambda_{2\mathtt{p}_{\mathfrak{k}-1}},\Lambda_{2\mathtt{q}_{\mathfrak{k}}}).\end{aligned}$$

As $\mathfrak{k}\to\infty$ in the above condition, we obtain

$$\lim_{k\to\infty}\rho(\Lambda_{2\mathtt{p}_{\mathfrak{k}}},\Lambda_{2\mathtt{q}_{\mathfrak{k}}})=\frac{\varepsilon}{\xi}. \tag{2.17}$$

Again applying the b-triangle inequality, we have:

$$\rho(\Lambda_{2\mathtt{q}_{\mathfrak{k}}},\Lambda_{2\mathtt{p}_{\mathfrak{k}-1}})\leq\xi(\rho(\Lambda_{2\mathtt{q}_{\mathfrak{k}}},\Lambda_{2\mathtt{p}_{\mathfrak{k}}})+\rho(\Lambda_{2\mathtt{p}_{\mathfrak{k}}},\Lambda_{2\mathtt{p}_{\mathfrak{k}-1}})).$$

As $\mathfrak{k}\to\infty$ in the above condition, we obtain

$$\lim_{k\to\infty}\rho(\Lambda_{2\mathtt{q}_{\mathfrak{k}}},\Lambda_{2\mathtt{p}_{\mathfrak{k}-1}})=\frac{\varepsilon}{\xi}. \tag{2.18}$$

Since,

$$\begin{aligned}\rho(\Lambda_{2\mathtt{q}_{\mathfrak{k}}},\Lambda_{2\mathtt{p}_{\mathfrak{k}}})&\leq\xi(\rho(\Lambda_{2\mathtt{q}_{\mathfrak{k}}},\Lambda_{2\mathtt{q}_{\mathfrak{k}+1}})+\rho(\Lambda_{2\mathtt{q}_{\mathfrak{k}+1}},\Lambda_{2\mathtt{p}_{\mathfrak{k}}}))\\&=\xi(\rho(\Lambda_{2\mathtt{q}_{\mathfrak{k}}},\Lambda_{2\mathtt{q}_{\mathfrak{k}+1}})+\rho(\mathtt{M}\Lambda_{2\mathtt{q}_{\mathfrak{k}}},\mathtt{N}\Lambda_{2\mathtt{p}_{\mathfrak{k}+1}})).\end{aligned}$$

As $\mathfrak{k}\to\infty$, we have

$$\frac{\varepsilon}{\xi}\leq\lim_{k\to\infty}(\rho(\mathtt{M}\Lambda_{2\mathtt{q}_{\mathfrak{k}}},\mathtt{N}\Lambda_{2\mathtt{p}_{\mathfrak{k}}+1})).$$

As ψ is both continuous and non-decreasing, it follows that:

$$\psi(\frac{\varepsilon}{\xi})\leq\lim_{k\to\infty}\psi(\rho(\mathtt{M}\Lambda_{2\mathtt{q}_{\mathfrak{k}}},\mathtt{N}\Lambda_{2\mathtt{p}_{\mathfrak{k}}+1})). \tag{2.19}$$

Derived from (2.1), we have

$$\psi(\rho(\mathtt{M}\Lambda_{2\mathtt{q}_{\mathfrak{k}}},\mathtt{M}\Lambda_{2\mathtt{p}_{\mathfrak{k}+1}}))\leq\zeta(\Theta(\Lambda_{2\mathtt{q}_{\mathfrak{k}}},\Lambda_{2\mathtt{p}_{\mathfrak{k}+1}}))\beta(\Theta(\Lambda_{2\mathtt{q}_{\mathfrak{k}}},\Lambda_{2\mathtt{p}_{\mathfrak{k}+1}})) \tag{2.20}$$

where

$$\begin{aligned}&\Theta(\Lambda_{2\mathtt{q}_{\mathfrak{k}}},\Lambda_{2\mathtt{p}_{\mathfrak{k}+1}})\\&=\max\{\rho(\Lambda_{2\mathtt{q}_{\mathfrak{k}}},\Lambda_{2\mathtt{p}_{\mathfrak{k}+1}}),\rho(\Lambda_{2\mathtt{q}_{\mathfrak{k}}},\mathtt{M}\Lambda_{2\mathtt{q}_{\mathfrak{k}}}),\rho(\Lambda_{2\mathtt{p}_{\mathfrak{k}+1}},\mathtt{M}\Lambda_{2\mathtt{p}_{\mathfrak{k}+1}}),\\&\frac{1}{2\xi}\big(\rho(\Lambda_{2\mathtt{q}_{\mathfrak{k}}},\mathtt{M}\Lambda_{2\mathtt{p}_{\mathfrak{k}+1}})+\rho(\Lambda_{2\mathtt{p}_{\mathfrak{k}+1}},\mathtt{M}\Lambda_{2\mathtt{q}_{\mathfrak{k}}})\big)\}\\&=\max\{\rho(\varpi_{2\mathtt{q}_{\mathfrak{k}-1}},\varpi_{2\mathtt{p}_{\mathfrak{k}}}),\rho(\varpi_{2\mathtt{q}_{\mathfrak{k}-1}},\varpi_{y}),\rho(\varpi_{2\mathtt{p}_{\mathfrak{k}}},\varpi_{2\mathtt{p}_{\mathfrak{k}+1}}),\\&\frac{1}{2\xi}\big(\rho(\varpi_{2\mathtt{q}_{\mathfrak{k}-1}},\varpi_{2\mathtt{p}_{\mathfrak{k}+1}})+\rho(\varpi_{2\mathtt{p}_{k}},\varpi_{2\mathtt{q}_{\mathfrak{k}}})\big)\}\\&\leq\max\{\rho(\varpi_{2\mathtt{q}_{\mathfrak{k}-1}},\varpi_{2\mathtt{p}_{\mathfrak{k}}}),\rho(\varpi_{2\mathtt{q}_{\mathfrak{k}-1}},\varpi_{2\mathtt{q}_{\mathfrak{k}}}),\rho(\varpi_{2\mathtt{p}_{\mathfrak{k}}},\varpi_{2\mathtt{p}_{\mathfrak{k}+1}}),\\&\frac{\xi}{2\xi}\big(\rho(\varpi_{2\mathtt{q}_{\mathfrak{k}-1}},\varpi_{2\mathtt{p}_{\mathfrak{k}}})+\rho(\varpi_{2\mathtt{p}_{\mathfrak{k}}},\varpi_{2\mathtt{p}_{\mathfrak{k}+1}})+\rho(\varpi_{2\mathtt{q}_{\mathfrak{k}-1}},\varpi_{2\mathtt{p}_{\mathfrak{k}}})+\rho(\varpi_{2\mathtt{q}_{\mathfrak{k}}},\varpi_{2\mathtt{q}_{\mathfrak{k}-1}})\big)\}\\&=\max\{\rho(\varpi_{2\mathtt{q}_{\mathfrak{k}-1}},\varpi_{2\mathtt{p}_{\mathfrak{k}}}),\rho(\varpi_{2\mathtt{q}_{\mathfrak{k}-1}},\varpi_{2\mathtt{q}_{\mathfrak{k}}}),\rho(\varpi_{2\mathtt{p}_{\mathfrak{k}}},\varpi_{2\mathtt{p}_{\mathfrak{k}+1}}),\\&\frac{1}{2}\big(\rho(\varpi_{2\mathtt{q}_{\mathfrak{k}-1}},\varpi_{2\mathtt{p}_{\mathfrak{k}}})+\rho(\varpi_{2\mathtt{p}_{\mathfrak{k}}},\varpi_{2\mathtt{p}_{\mathfrak{k}+1}})+\rho(\varpi_{2\mathtt{q}_{\mathfrak{k}-1}},\varpi_{2\mathtt{p}_{\mathfrak{k}}})+\rho(\varpi_{2\mathtt{q}_{\mathfrak{k}}},\varpi_{2\mathtt{q}_{\mathfrak{k}-1}})\big)\}.\end{aligned}$$

From (2.8), we have

$$\Theta(\Lambda_{2\mathtt{q}_{\mathfrak{k}}},\Lambda_{2\mathtt{p}_{\mathfrak{k}+1}})\leq\rho(\varpi_{2\mathtt{q}_{\mathfrak{k}-1}},\varpi_{2\mathtt{p}_{\mathfrak{k}}}). \tag{2.21}$$

Following Condition (2.20), it can be deduced that:

$$\psi(\rho(\mathtt{M}\Lambda_{2\mathtt{q}_{\mathfrak{k}}},\mathtt{M}\Lambda_{2\mathtt{p}_{\mathfrak{k}+1}}))\leq\zeta(\rho(\varpi_{2\mathtt{q}_{\mathfrak{k}-1}},\varpi_{2\mathtt{p}_{\mathfrak{k}}}))\beta(\rho(\varpi_{2\mathtt{q}_{\mathfrak{k}-1}},\varpi_{2\mathtt{p}_{\mathfrak{k}}})). \tag{2.22}$$

Repeating the limit process as $\mathfrak{k} \to \infty$ in (2.22) and considering the property $\zeta \in \mathcal{S}$, we obtain:

$$\lim_{k\to\infty} \psi(\rho(\mathtt{M}\Lambda_{2\mathfrak{q}_\mathfrak{k}}, \mathtt{M}\Lambda_{2\mathfrak{p}_{\mathfrak{k}+1}})) < \beta(\frac{\varepsilon}{\xi}). \tag{2.23}$$

Hence from (2.19), we get

$$\psi(\frac{\varepsilon}{\xi}) \leq \lim_{k\to\infty} \psi(\rho(\mathtt{M}\Lambda_{2\mathfrak{q}_\mathfrak{k}}, \mathtt{M}\Lambda_{2\mathfrak{p}_{\mathfrak{k}+1}})) \leq \beta(\frac{\varepsilon}{\xi}). \tag{2.24}$$

Which is a contradiction. This is possible only if $\varepsilon = 0$.
Therefore, $\{\Lambda_{2\mathfrak{p}}\}$ is a $b-$Cauchy sequence, implying that $\{\Lambda_\mathfrak{p}\}$ is also a $b-$Cauchy sequence for all $\mathfrak{p}$.
Hence, $\exists\ \omega \in \Omega$ such that:

$$\lim_{\mathfrak{p}\to\infty} \Lambda_\mathfrak{p} = \omega. \tag{2.25}$$

Following this, to prove that ω is a FP of $\mathtt{M}$.
Due to the continuity of $\mathtt{M}$ and the convergence $\Lambda_{2\mathfrak{p}+1} \to \omega$, it can be concluded that:

$$\omega = \lim_{\mathfrak{p}\to\infty} \Lambda_{2\mathfrak{p}+1} = \lim_{\mathfrak{p}\to\infty} \mathtt{M}\Lambda_{2\mathfrak{p}} = \mathtt{M}\omega. \tag{2.26}$$

Thus, ω is a FP of $\mathtt{M}$. Also,

$$\psi(\rho(\omega, \mathtt{M}\omega)) = \psi((\mathtt{M}\omega, \mathtt{M}\omega)) \leq \zeta(\Theta(\omega, \omega))\beta(\Theta(\omega, \omega)) \tag{2.27}$$

where

$$\begin{aligned}\Theta(\omega, \omega) &= \max\{\rho(\omega, \omega), \rho(\omega, \mathtt{M}\omega), \rho(\omega, \mathtt{M}\omega), \frac{1}{2\xi}\big(\rho(\omega, \mathtt{M}\omega) + \rho(\omega, \mathtt{M}\omega)\big)\} \\ &\leq \rho(\omega, \mathtt{M}\omega).\end{aligned}$$

Then, from Condition (2.27), we get:

$$\psi(\rho(\omega, \mathtt{N}\omega)) = \psi((\mathtt{M}\omega, \mathtt{N}\omega)) \leq \zeta(\rho(\omega, \mathtt{M}\omega))\beta(\rho(\omega, \mathtt{M}\omega)). \tag{2.28}$$

Consequently, $\psi(\frac{1}{\xi}) \leq \lim_{k\to\infty} \psi(\rho(\omega, \mathtt{M}\omega)) \leq \beta(\frac{1}{\xi})$.

Hence $\mathtt{M}\omega = \omega$. That is, ω is a UFP of $\mathtt{M}$.

Result: We demand that the UFP of $\mathtt{M}$ is unique. Assume on the contrary that $\mathtt{M}\omega = \mathtt{M}\varpi = \varpi$ but $\omega \neq \varpi$. As per the assumption, we can substitute Λ with ω and ϖ with ϖ into (2.1), yielding:

$$\psi(\rho(\omega, \varpi)) = \psi(\rho(\mathtt{M}\omega, \mathtt{M}\varpi)) \leq \alpha(\rho(\omega, \varpi))\beta(\rho(\omega, \varpi)) < \beta(\rho(\omega, \varpi)). \tag{2.29}$$

Applying the statement of Theorem 2.1 and Lemma 1.11, we get $\rho(\omega, \varpi) = 0$. It is possible only if $\omega = \varpi$. Thus, we have proved that $\mathtt{M}$ have a UFP. ☐

2.2 corollary

Consider a $\mathcal{POSET}$ Ω equipped with a metric ρ, making (Ω, ρ) a complete b-MS. Let $\mathtt{M}$ be weakly increasing mappings from Ω to itself. Suppose the following inequality holds for all $\Lambda \geq \varpi$:

$$\psi(\rho(\mathtt{M}\Lambda, \mathtt{M}\varpi)) \leq \alpha(\Theta(\Lambda, \varpi))\beta(\Theta(\Lambda, \varpi)), \forall \Lambda \geq \varpi, \tag{2.30}$$

where

$$\Theta(\Lambda, \varpi) = \max\left\{\rho(\Lambda, \varpi), \frac{\rho(\Lambda, \mathtt{M}\Lambda)\rho(\varpi, \mathtt{M}\varpi)}{1 + \rho(\Lambda, \varpi)}, \frac{\rho(\Lambda, \mathtt{M}\varpi)\rho(\varpi, \mathtt{M}\Lambda)}{1 + \rho(\Lambda, \varpi)}\right\},$$

and α belongs to $\mathcal{S}$, ψ belongs to Ψ, and $\beta : [0, \infty) \to [0, \frac{1}{\xi})$ is a continuous function with the condition $\psi(\sigma) > \beta(\sigma)$ for all $\sigma > 0$.

Furthermore, assume that for each pair of elements $\Lambda, \varpi \in \Omega$, $\exists\ z \in \Omega$ that is comparable to both Λ and ϖ. If either $\mathtt{M}$ is continuous, then $\mathtt{M}$ possess a UFP.

3 Applications

3.1 Conclusions for FP solutions for mapping satisfying a contraction 0f integral type

Below section focuses on establishing FP results for maps that satisfy a contraction of integral type in a complete ordered b-MS. Before presenting the proofs, we introduce some notations:

Let χ denote the set of functions $\phi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(i) For every compact subset of $[0, \infty)$, the function ϕ is Lebesgue-integrable.

(ii) For every $\epsilon > 0$:

$$\int_0^\infty \phi(\sigma)\, d\sigma < \epsilon.$$

Now, consider a fixed positive integer $N \in \mathbb{N}^*$. Let $\{\phi_i\}_{1\leq i\leq N}$ be a collection of N functions belonging to χ. For all $\sigma \geq 0$, we define the following iterative integrals:

$$I_1(\sigma) = \int_0^\sigma \phi_1(\xi)d\xi,$$

$$I_2(\sigma) = \int_0^{I_1(\sigma)} \phi_2(\xi)d\xi = \int_0^{\int_0^\sigma \phi_1(\xi)d\xi} \phi_2(\xi)d\xi,$$

$$I_3(\sigma) = \int_0^{I_2(\sigma)} \phi_3(\xi)d\xi = \int_0^{\int_0^{\int_0^\sigma \phi_1(\xi)d\xi} \phi_2(\xi)d\xi} \phi_3(\xi)d\xi,$$

and so on. The general form for $1 \leq k \leq N$ is:

$$I_k(\sigma) = \int_0^{I_{k-1}(\sigma)} \phi_k(\xi)d\xi = \int_0^{\int_0^{\cdots \int_0^\sigma \phi_1(\xi)d\xi \cdots} \phi_{k-1}(\xi)d\xi} \phi_k(\xi)d\xi,$$

where $I_0(\sigma) = \sigma$.

Finally, for $k = N$:

$$I_N(\sigma) = \int_0^{I_{N-1}(\sigma)} \phi_N(\xi)d\xi.$$

We will now establish the validity of the following theorems.

3.2 Theorem

Suppose Ω is a $\mathcal{POSET}$ with a metric ρ making (Ω, ρ) a complete b-MS. Let $\mathtt{M} : \Omega \to \Omega$ be continuous and weakly increasing mappings, satisfying the inequality:

$$\mathcal{I}_M(\psi(\Theta(\mathtt{M}\Lambda, \mathtt{M}\varpi))) \leq \alpha(\Theta(\Lambda, \varpi))\mathcal{I}_M(\beta(\Theta(\Lambda, \varpi))) \; \forall \Lambda \geq \varpi, \tag{3.1}$$

where

$$\Theta(\Lambda, \varpi) = \max \rho(\Lambda, \varpi), \rho(\Lambda, \mathtt{M}\Lambda), \rho(\varpi, \mathtt{M}\varpi), \frac{1}{2\xi}\big(\rho(\Lambda, \mathtt{M}\varpi) + \rho(\varpi, \mathtt{M}\Lambda)\big),$$

and α belongs to $\mathcal{S}$, ψ belongs to Ψ, and $\beta : [0, \infty) \to [0, \frac{1}{\xi})$ is a continuous function with the condition $\psi(\sigma) > \beta(\sigma)$, $\forall$ $\sigma > 0$.

Furthermore, assume that for each pair of elements $\Lambda, \varpi \in \Omega$, $\exists\, z \in \Omega$ that is comparable to both Λ and ϖ. If either $\mathtt{M}$ is continuous, then $\mathtt{M}$ possess a UFP.

Proof. Let us define $\psi_1 = I_N \circ \psi$ and $\beta_1 = I_N \circ \beta$. Consequently, according to (3.1), we can express:

$$\psi_1(\Theta(\mathtt{M}\Lambda, \mathtt{M}\varpi)) \leq \alpha(\Theta(\Lambda, \varpi))\beta_1(\Theta(\Lambda, \varpi)), \quad \forall \Lambda \geq \varpi. \tag{3.2}$$

As WKT the composition of continuous functions remains continuous ensures that ψ_1 and β_1 are both continuous. By invoking Theorem 2.1, we derive the desired outcome. □

3.3 Theorem

Assume the following conditions:

(i) $\mathcal{K}_3, \mathcal{K}_4 : \mathcal{I} \times \mathcal{I} \times \mathbb{R} \to \mathbb{R}$ are continuous.

(ii) $\forall\, \sigma, \xi \in \mathcal{I}$,

$$\mathcal{K}_3(\sigma, \xi, \omega(\sigma)) \leq \mathcal{K}_4\left(\sigma, \xi, \int_0^T \mathcal{K}_3(\xi, z, \omega(z))\, dz + g(\xi)\right),$$

$$\mathcal{K}_4(\sigma, \xi, \omega(\sigma)) \leq \mathcal{K}_3\left(\sigma, \xi, \int_0^T \mathcal{K}_4(\xi, z, \omega(z))\, dz + g(\xi)\right).$$

(iii) $\exists\; H : \mathcal{I} \times \mathcal{I} \to \mathbb{R}^+$ such that

$$|\mathcal{K}_3(\sigma, \xi, \Lambda(\xi)) - \mathcal{K}_4(\sigma, \xi, \varpi(\xi))| \leq H(\sigma, \xi)\left(\frac{\ln(1 + |\Lambda(\xi) - \varpi(\xi)|^{2q})}{2^{2q-1}}\right)^{1/q},$$

for all $\sigma, \xi \in \mathcal{I}$ and $\Lambda, \varpi \in \Omega$.

(iv) $\sup_{\sigma \in \mathcal{I}} \int_0^T H(\sigma, \xi)^q d\xi \leq \frac{1}{T}$.

Thus, a unique common solution $\omega^* \in C(\mathcal{I})$ exists for the integral equation (3.1).

Proof. Define operators $\mathtt{P}, \mathtt{Q} : C(\mathcal{I}) \to C(\mathcal{I})$ as:

$$\mathtt{P}\Lambda(\sigma) = \int_0^T \mathcal{K}_3(\sigma, \xi, \Lambda(\xi))\, d\xi + g(\sigma), \tag{3.3}$$

$$\mathtt{Q}\Lambda(\sigma) = \int_0^T \mathcal{K}_4(\sigma, \xi, \Lambda(\xi))\, d\xi + g(\sigma), \quad \forall \sigma \in \mathcal{I}, \Lambda \in \Omega. \tag{3.4}$$

Following the contraction condition, it can be shown that:

$$\rho(\mathtt{P}\Lambda, \mathtt{P}\varpi) \leq \sqrt{\frac{\ln(1 + \rho(\Lambda, \varpi)^2)}{2^{2q-1}}}. \tag{3.5}$$

Thus, $\mathtt{P}$ and $\mathtt{Q}$ each admit a UFP, and these FPs coincide under the given conditions.
Therefore, A unique common solution $\omega^* \in C(\mathcal{I})$ exists for the integral equations.

□

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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