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Some results on derivatives of B-q bonacci polynomials

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Abstract In this paper, we have obtained some results on derivatives of B-q bonacci polynomials. We have explicitly expressed the derivatives of B-q bonacci polynomials using the generating function and combinatorial form of polynomials. We have also obtained the results on the r^{th} order derivative of B-q bonacci polynomials in series form.

1 Introduction

Fibonacci polynomials and their extended forms of polynomials have many applied and fascinating properties [7]. Some generalized sequences can be seen in [1, 9]. In [2], authors have defined two new classes of polynomials associated with generalized Fibonacci polynomials and derived convolution property. In [8], Levesque defines m^{th} order linear difference equation and obtains a generating function for the recurrence relation. The authors, Yuan Yi and Wenpeng Zhang, have obtained the relation,

$$\sum_{a_1+a_2+\cdots+a_k=n} F_{a_1+1}F_{a_2+1}\cdots F_{a_k+1} = \frac{1}{(k-1)!}\frac{d^{k-1}}{dx^{k-1}}\big(F_{n+k}\big),$$

where $F_{n+1} = xF_n + F_{n-1}$, with $F_0 = 0$ and $F_1 = 1$, see equation (7) of [12]. The properties of Fibonacci polynomials and their derivatives are studied in [5, 6, 11]. In [10], authors have defined new families of generalized Fibonacci polynomials and generalized Lucas polynomials and obtained some properties of these families. Further, they have established relationships between family of the generalized k-Fibonacci polynomials and the known generalized Fibonacci polynomials.

In [2], the class of $h(x)$-B-q bonacci polynomials, denoted by $(qB)_{h,n+q-1}$ are defined by:

$$(qB)_{h,n+q-1}(x) = \sum_{i=0}^{q-1}\binom{q-1}{i}\big(h(x)\big)^{q-1-i}(qB)_{h,n+q-2-i}(x), \tag{1.1}$$

with $(qB)_{h,0}(x) = 0, (qB)_{h,1}(x) = 0, \cdots, (qB)_{h,q-1}(x) = 1$, for all $n \in \mathbb{N}\cup\{0\}, n \geq q \geq 2$, where $h(x)$ is a polynomial with real coefficients.

The B-q bonacci polynomials are extensions of Fibonacci polynomials defined in [4]. We define these polynomials as follows:

Definition 1.1. We denote B-q bonacci polynomials by $Q_n(x)$ and define by the relation

$$Q_{n+q-1}(x) = \sum_{i=0}^{q-1}\binom{q-1}{i}x^{q-1-i}Q_{n+(q-2)-i}, \tag{1.2}$$

with $Q_0(x) = 0, Q_1(x) = 0, \cdots, Q_{q-1}(x) = 1$, for all $n \in \mathbb{N}\cup\{0\}$ and $n \geq q \geq 2$.

These polnomials are obtained by putting $h(x) = x$ in (1.1).

We state below first few non-zero polynomials of B-q bonacci polynomials:

$$Q_{q-1}(x) = 1$$
$$Q_q(x) = x^{q-1}$$
$$Q_{q+1}(x) = x^{2(q-1)} + (q-1)x^{q-2}$$
$$Q_{q+2}(x) = x^{3(q-1)} + 2(q-1)x^{2q-3} + \frac{1}{2}(q-1)(q-2)x^{q-3}$$
$$Q_{q+3}(x) = x^{4(q-1)}+3(q-1)x^{3q-4}+(q-1)(2q-3)x^{2q-4}+\frac{1}{6}(q-1)(q-2)(q-3)x^{q-4}.$$

In [2], the following identities are obtained:

(i) The combinatorial form of (1.1) is given by

$$(qB)_{h,n}(x) = \sum_{i=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} \binom{(q-1)\big(n-(q-1)-i\big)}{i} h(x)^{(q-1)\big(n-(q-1)\big)-iq}, \quad (1.3)$$

for all $n \geq q-1$, where $\lfloor \cdot \rfloor$ denote the floor function.

(ii) The generating function of (1.1) is given by

$$G(Z) = \sum_{n=0}^{\infty} Q_n(x) Z^{n-(q-1)} = \frac{1}{1 - Z\big(h(x)+Z\big)^{q-1}}. \quad (1.4)$$

Using these we can deduce the following properties of (1.2).

(a) The combinatorial form of (1.2) is given by

$$Q_n(x) = \sum_{i=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} \binom{(q-1)(n-(q-1)-i)}{i} x^{(q-1)\big(n-(q-1)\big)-iq}, \quad (1.5)$$

for all $n \geq q-1$.

(b) The generating function of (1.2) is given by

$$G(Z) = \sum_{n=0}^{\infty} Q_n(x) Z^{n-(q-1)} = \frac{1}{1 - Z(x+Z)^{q-1}}. \quad (1.6)$$

If we differentiate the equation (1.5) with respect to x, we get

$$\frac{dQ_n}{dx} = \sum_{i=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} \Big((q-1)\big(n-(q-1)\big)-iq\Big) \binom{(q-1)(n-(q-1)-i)}{i} x^{(q-1)\big(n-(q-1)\big)-iq-1}. \quad (1.7)$$

If $q = 3$, then the above properties reduce to the properties of B- Tribonacci polynomials discussed in [3].

In this paper, we derive some relations involving derivatives of B-q bonacci polynomials defined by the equation (1.2).

For simplicity, from the next section onwards, we denote $Q_n(x)$ by Q_n. Throughout this paper, we will take positive integers, $q \geq 2$.

2 Derivatives of B-q bonacci polynomials

In this section, we obtain some relations involving the derivatives of B-q bonacci polynomials.

We have the following theorem.

Theorem 2.1. *For* $n \geq 2$,

$$\frac{dQ_n}{dx} = (q-1)\sum_{k=0}^{q-2} \binom{q-2}{k} x^k \sum_{i=0}^{n+k} Q_i\, Q_{n+k-i}. \tag{2.1}$$

Proof. Differentiating the equation (1.6) both sides with respect to x, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{dQ_n}{dx} Z^{n-(q-1)} &= \frac{(q-1)Z(x+Z)^{q-2}}{[1-Z(x+Z)^{q-1}]^2} \\
&= \frac{(q-1)}{[1-Z(x+Z)^{q-1}]^2} \sum_{k=0}^{q-2} \binom{q-2}{k} x^k\, Z^{q-1-k} \\
&= (q-1)\left(\sum_{n=0}^{\infty}\sum_{i=0}^{n} Q_i\, Q_{n-i} Z^{n-2(q-1)}\right)\left(\sum_{k=0}^{q-2}\binom{q-2}{k} x^k\, Z^{q-1-k}\right) \\
&= (q-1)\sum_{n=0}^{\infty}\sum_{i=0}^{n} Q_i\, Q_{n-i} \sum_{k=0}^{q-2}\binom{q-2}{k} x^k\, Z^{n-(q-1)-k}.
\end{aligned}$$

Now, equating the coefficient of $Z^{n-(q-1)}$, we get

$$\frac{dQ_n}{dx} = (q-1)\sum_{k=0}^{q-2} \binom{q-2}{k} x^k \sum_{i=0}^{n+k} Q_i\, Q_{n+k-i}.$$

□

We need the following lemma.

Lemma 2.2. *For* $n \geq q-1$,

$$\sum_{i=0}^{q-2}\binom{q-2}{i} x^{q-2-i}\, Q_{n-i} = \sum_{k=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} \binom{(q-1)\big(n-(q-2)-k\big)-1}{k} x^{(q-1)\big(n-(q-2)\big)-qk-1}. \tag{2.2}$$

Proof. We prove the equation (2.2) for $n = qm, qm-1, qm-2, \cdots, qm-(q-1)$. First let $n = qm$.

$$\begin{aligned}
&\sum_{i=0}^{q-2}\binom{q-2}{i} x^{q-2-i}\, Q_{qm-i} \\
&= \sum_{i=0}^{q-2}\binom{q-2}{i} \sum_{k=0}^{(q-1)m-(q-2)-i} \binom{(q-1)\big(qm-i-(q-1)-k\big)}{k} x^{(q-1)\big(qm-i-(q-2)-k\big)-k-i-1} \\
&= \sum_{i=0}^{q-2}\binom{q-2}{i} \sum_{k=i}^{(q-1)m-(q-2)} \binom{(q-1)\big(qm-(q-1)-k\big)}{k-i} x^{(q-1)\big(qm-(q-2)-k\big)-k-1}
\end{aligned}$$

$$= \sum_{i=0}^{q-2}\binom{q-2}{i}\sum_{k=0}^{(q-1)m-(q-2)}\binom{(q-1)\big(qm-(q-1)-k\big)}{k-i}x^{(q-1)\big(qm-(q-2)-k\big)-k-1},$$

since, for $n, i \geq 0, \binom{n}{-i} = 0$.

Now, using the result, $\sum_{i=0}^{r}\binom{r}{i}\binom{n}{k-i} = \binom{n+r}{k}$, we get

$$\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}\,Q_{qm-i}$$
$$= \sum_{k=0}^{(q-1)m-(q-2)}\binom{(q-1)\big(qm-(q-1)-k\big)+(q-2)}{k}x^{(q-1)\big(qm-(q-2)-k\big)-k-1}$$
$$= \sum_{k=0}^{(q-1)m-(q-2)}\binom{(q-1)\big(qm-(q-2)-k\big)-1}{k}x^{(q-1)\big(qm-(q-2)\big)-qk-1}.$$

Hence, the equation (2.2) is true for $n = qm$.
Similarly, the result can be proved for $n = qm-1, qm-2, \cdots, qm-(q-1)$. This completes the proof of the lemma. □

Theorem 2.3. *For $n \geq q-1$,*

$$\frac{dQ_{n+1}}{dx} + (q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}\frac{dQ_{n-1-i}}{dx}$$
$$= (q-1)\left[\big(n-(q-2)\big)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}Q_{n-i} - (q-2)\sum_{i=0}^{q-2}\binom{q-3}{i}x^{q-3-i}Q_{n-1-i}\right]. \tag{2.3}$$

Proof. We prove the result for $n = qm, qm-1, qm-2, \cdots, qm-(q-2)$. Let $n = qm$ and consider

$$Q_{qm+1} + (q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}Q_{qm-1-i}$$
$$= \sum_{k=0}^{\left\lfloor\frac{(q-1)(qm-(q-2))}{q}\right\rfloor}\binom{(q-1)\big(qm-(q-2)-k\big)}{k}x^{(q-1)\big(qm-(q-2)\big)-qk}$$
$$+(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}\sum_{k=0}^{\left\lfloor\frac{(q-1)(qm-i-q)}{q}\right\rfloor}\binom{(q-1)\big(qm-i-q-k\big)}{k}x^{(q-1)\big(qm-i-q\big)-qk}$$
$$= \sum_{k=0}^{(q-1)m-(q-2)}\binom{(q-1)\big(qm-(q-2)-k\big)}{k}x^{(q-1)\big(qm-(q-2)\big)-qk}$$
$$+(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}\sum_{k=0}^{(q-1)(m-1)-i}\binom{(q-1)\big(qm-q-(k+i)\big)}{k}x^{(q-1)\big(qm-(q-1)\big)-q(k+i)-1}$$
$$= \sum_{k=0}^{(q-1)m-(q-2)}\binom{(q-1)\big(qm-(q-2)-k\big)}{k}x^{(q-1)\big(qm-(q-2)\big)-qk}$$
$$+(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}\sum_{k=i+1}^{(q-1)m-(q-2)}\binom{(q-1)\big(qm-(q-1)-k\big)}{k-1-i}x^{(q-1)\big(qm-(q-1)\big)-qk+q-1},$$

replacing k from second term with $k-1-i$

$$= \sum_{k=0}^{(q-1)m-(q-2)} \binom{(q-1)\big(qm-(q-2)-k\big)}{k} x^{(q-1)\left(qm-(q-2)\right)-qk}$$
$$+(q-1) \sum_{k=1}^{(q-1)m-(q-2)} \sum_{i=0}^{q-2} \binom{q-2}{i} \binom{(q-1)\big(qm-(q-1)-k\big)}{k-1-i} x^{(q-1)\left(qm-(q-2)\right)-qk},$$

since for $k>n$, $\binom{n}{k}=0$

$$= \sum_{k=0}^{(q-1)m-(q-2)} \binom{(q-1)\big(qm-(q-2)-k\big)}{k} x^{(q-1)\left(qm-(q-2)\right)-qk}$$
$$+(q-1) \sum_{k=1}^{(q-1)m-(q-2)} \binom{(q-1)\big(qm-(q-1)-k\big)+q-2}{k-1} x^{(q-1)\left(qm-(q-2)\right)-qk}$$
$$= x^{(q-1)\left(qm-(q-2)\right)} + \sum_{k=1}^{(q-1)m-(q-2)} \left[\binom{(q-1)\big(qm-(q-2)-k\big)}{k}\right.$$
$$\left. +(q-1)\binom{(q-1)\big(qm-(q-2)-k\big)-1}{k-1}\right] x^{(q-1)\left(qm-(q-2)\right)-qk}.$$

Since $\binom{n}{k}=\frac{n}{k}\binom{n-1}{k-1}$, we have

$$Q_{qm+1}+(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}Q_{qm-1-i} = x^{(q-1)\left(qm-(q-2)\right)}$$
$$+ \sum_{k=1}^{(q-1)m-(q-2)} \binom{(q-1)\big(qm-(q-2)-k\big)-1}{k-1}\left(\frac{(q-1)\big(qm-(q-2)\big)}{k}\right) x^{(q-1)\left(qm-(q-2)\right)-qk}$$

Differentiating both sides with respect to x and then simplifying, we get

$$\frac{dQ_{qm+1}}{dx} + (q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}\,\frac{dQ_{qm-1-i}}{dx}$$
$$+(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}(q-2-i)x^{q-3-i}\,Q_{qm-1-i}$$
$$= (q-1)\big(qm-(q-2)\big)\left[x^{(q-1)\left(qm-(q-2)\right)-1}\right.$$
$$\left. + \sum_{k=1}^{(q-1)m-(q-2)} \binom{(q-1)\big(qm-(q-2)-k\big)-1}{k}\right] x^{(q-1)\left(qm-(q-2)\right)-qk-1}.$$

Therefore

$$\frac{dQ_{qm+1}}{dx}+(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}\,\frac{dQ_{qm-1-i}}{dx}$$
$$= (q-1)\left[\big(qm-(q-2)\big)\sum_{k=0}^{(q-1)m-(q-2)}\binom{(q-1)\big(qm-(q-2)-k\big)-1}{k}\,x^{(q-1)\left(qm-(q-2)\right)-qk-1}\right.$$
$$\left. -(q-2)\sum_{i=0}^{q-3}\binom{q-3}{i}x^{q-3-i}\,Q_{qm-1-i}\right].$$

But, from Lemma 2.2, we have

$$\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}\,Q_{qm-i}$$
$$=\sum_{k=0}^{(q-1)m-(q-2)}\binom{(q-1)\big(qm-(q-2)-k\big)-1}{k}x^{(q-1)\big(qm-(q-2)\big)-qk-1}.$$

Hence,

$$\frac{dQ_{qm+1}}{dx}+(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}\,\frac{dQ_{qm-1-i}}{dx}$$
$$=(q-1)\Bigg[\big(qm-(q-2)\big)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}\,Q_{qm-i}-(q-2)\sum_{i=0}^{q-3}\binom{q-3}{i}x^{q-3-i}\,Q_{qm-1-i}\Bigg]$$

Similarly, the result can be proved for $n = qm-1, qm-2, \cdots, qm-(q-1)$. □

Theorem 2.4. *For* $n \geq q-1$*,*

$$\frac{dQ_{n+1}}{dx}+(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}\frac{dQ_{n-1-i}}{dx}$$
$$=(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}\Big(n-(q-2)-i\Big)x^{q-2-i}Q_{n-i}. \tag{2.4}$$

Proof. From Theorem 2.3, for $n = qm$, we have

$$\frac{dQ_{qm+1}}{dx}+(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}\frac{dQ_{qm-1-i}}{dx}$$
$$=(q-1)\Bigg[\big(qm-(q-2)\big)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}Q_{qm-i}$$
$$-(q-2)\sum_{i=0}^{q-2}\binom{q-3}{i}x^{q-3-i}Q_{qm-1-i}\Bigg].$$

This implies

$$\frac{dQ_{qm+1}}{dx}+(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}\,\frac{dQ_{qm-1-i}}{dx}$$
$$=(q-1)\big(qm-(q-2)\big)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}\,Q_{qm-i}$$
$$-(q-1)(q-2)\sum_{i=0}^{q-3}\binom{q-3}{i}x^{q-3-i}\,Q_{qm-1-i}$$
$$=(q-1)\Bigg[qm\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}\,Q_{qm-i}-(q-2)x^{q-2}\,Q_{qm}$$
$$-(q-2)\Bigg(\sum_{i=1}^{q-2}\binom{q-2}{i}x^{q-2-i}\,Q_{qm-i}-\sum_{i=1}^{q-2}\binom{q-3}{i-1}x^{q-2-i}\,Q_{qm-i}\Bigg)\Bigg]$$
$$=(q-1)\Bigg[qm\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}\,Q_{qm-i}+(q-2)x^{q-2}\,Q_{qm-i}$$

$$-(q-2)\left(\sum_{i=1}^{q-2}\binom{q-3}{i-1}\left(\frac{q-2}{i}-1\right)\right)x^{q-2-i}\,Q_{qm-i}\Bigg]$$

$$=(q-1)\left[qm\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}\,Q_{qm-i}-\sum_{i=0}^{q-2}\binom{q-2}{i}\Big(q-2-i\Big)x^{q-2-i}\,Q_{qm-i}\right]$$

$$=(q-1)\left[\sum_{i=0}^{q-2}\Big(qm-(q-2-i)\Big)\binom{q-2}{i}x^{q-2-i}\,Q_{qm-i}\right]$$

Similarly, the result can be proved for $n=qm-1,qm-2,\cdots,qm-(q-1)$. □

Theorem 2.5. *For* $n\geq 1$,

$$(q-1)\frac{dQ_{n+q-1}}{dx}+\frac{dQ_{n-1}}{dx}=n(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}Q_{n+(q-2)-i}$$
$$+\sum_{i=0}^{q-2}\binom{q-1}{i}(q-2-i)\frac{d}{dx}\Big(x^{q-1-i}Q_{n+(q-2)-i}\Big). \qquad (2.5)$$

Proof. Replacing n in equation (2.4) with $n+q-2$, we get

$$\frac{dQ_{n+q-1}}{dx}+(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}\frac{dQ_{n+(q-3)-i}}{dx}$$
$$=(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}(n-i)x^{q-2-i}Q_{n+(q-2)-i}.$$

This implies

$$\frac{dQ_{n+q-1}}{dx}+(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}\frac{dQ_{n+(q-3)-i}}{dx}$$
$$+(q-1)\sum_{i=0}^{q-2}(i+1)\binom{q-2}{i+1}x^{q-3-i}Q_{n+(q-3)-i}=n(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}Q_{n+(q-2)-i}.$$

After further simplification, we get

$$\frac{dQ_{n+q-1}}{dx}+(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}\frac{d}{dx}\Big(x^{q-2-i}Q_{n+(q-3)-i}\Big)$$
$$=n(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}Q_{n+(q-2)-i}.$$

Hence, we have

$$\frac{dQ_{n+q-1}}{dx}+(q-1)\frac{dQ_{n-1}}{dx}+(q-1)\sum_{i=1}^{q-2}\binom{q-2}{i-1}\frac{d}{dx}\Big(x^{q-1-i}Q_{n+(q-2)-i}\Big)$$
$$=n(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}Q_{n+(q-2)-i}.$$

Adding $\sum_{i=0}^{q-2}\binom{q-1}{i}(q-2-i)\frac{d}{dx}\Big(x^{q-1-i}Q_{n+(q-2)-i}\Big)$ on both the sides and simplifying, we get

$$\frac{dQ_{n+q-1}}{dx}+(q-1)\frac{dQ_{n-1}}{dx}+(q-1)\sum_{i=1}^{q-2}\binom{q-2}{i-1}\frac{d}{dx}\Big(x^{q-1-i}Q_{n+(q-2)-i}\Big)$$

$$+(q-2)\frac{d}{dx}\Big(x^{q-1}Q_{n+(q-2)}\Big)+\sum_{i=1}^{q-2}\binom{q-1}{i}(q-2-i)\frac{d}{dx}\Big(x^{q-1-i}Q_{n+(q-2)-i}\Big)$$

$$=n(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}Q_{n+(q-2)-i}+\sum_{i=0}^{q-2}\binom{q-1}{i}(q-2-i)\frac{d}{dx}\Big(x^{q-1-i}Q_{n+(q-2)-i}\Big).$$

On further simplifying, we get

$$\frac{dQ_{n+q-1}}{dx}+(q-2)\frac{d}{dx}\sum_{i=0}^{q-1}\binom{q-1}{i}\Big(x^{q-1-i}Q_{n+(q-2)-i}\Big)+\frac{dQ_{n-1}}{dx}$$

$$=n(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}Q_{n+(q-2)-i}+\sum_{i=0}^{q-2}\binom{q-1}{i}(q-2-i)\frac{d}{dx}\Big(x^{q-1-i}Q_{n+(q-2)-i}\Big).$$

Thus, we have

$$(q-1)\frac{dQ_{n+q-1}}{dx}+\frac{dQ_{n-1}}{dx}=n(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}Q_{n+(q-2)-i}$$
$$+\sum_{i=0}^{q-2}\binom{q-1}{i}(q-2-i)\frac{d}{dx}\Big(x^{q-1-i}Q_{n+(q-2)-i}\Big).$$

This proves the theorem. □

Next, we have a result for r^{th} derivative of B-q bonacci polynomials.

Theorem 2.6. *For $r \geq 1$ and $n \geq (q-1)r$,*

$$\Big(n-(q-1)r\Big)\,\frac{d^rQ_{n+q-1}}{dx^r}=n\,\sum_{i=0}^{q-1}\binom{q-1}{i}x^{q-1-i}\frac{d^rQ_{n+(q-2)-i}}{dx^r}+(n+r)\frac{d^r\,Q_{n-1}}{dx^r}$$
$$-r\sum_{i=0}^{q-2}\binom{q-1}{i}(q-2-i)\frac{d^r}{dx^r}\Big(x^{q-1-i}Q_{n+(q-2)-i}\Big)$$
$$+n\sum_{i=0}^{q-2}\binom{q-1}{i}\sum_{k=2}^{q-1-i}\binom{r}{k}(1-k)\frac{d^k}{dx^k}\big(x^{q-1-i}\big)\frac{d^{r-k}Q_{n+(q-2)-i}}{dx^{r-k}}. \tag{2.6}$$

Proof. Differentiating equation (2.5) $(r-1)$ times with respect to x, we get

$$(q-1)\frac{d^rQ_{n+q-1}}{dx^r}+\frac{d^rQ_{n-1}}{dx^r}=n(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}\frac{d^{r-1}}{dx^{r-1}}\Big(x^{q-2-i}Q_{n+(q-2)-i}\Big)$$
$$+\sum_{i=0}^{q-2}\binom{q-1}{i}(q-2-i)\frac{d^r}{dx^r}\Big(x^{q-1-i}Q_{n+(q-2)-i}\Big).$$
$$=n(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}\frac{d^{r-1}}{dx^{r-1}}\Big(Q_{n+(q-2)-i}\Big)$$
$$+n(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}\sum_{k=1}^{q-2-i}\binom{r-1}{k}\frac{d^k}{dx^k}\big(x^{q-2-i}\big)\frac{d^{r-1-k}}{dx^{r-1-k}}\Big(Q_{n+(q-2)-i}\Big)$$
$$+\sum_{i=0}^{q-2}\binom{q-1}{i}(q-2-i)\frac{d^r}{dx^r}\Big(x^{q-1-i}Q_{n+(q-2)-i}\Big).$$

Hence,

$$(q-1)\frac{d^rQ_{n+q-1}}{dx^r}+\frac{d^rQ_{n-1}}{dx^r}=n(q-1)\sum_{i=0}^{q-2}\binom{q-2}{i}x^{q-2-i}\frac{d^{r-1}}{dx^{r-1}}\Big(Q_{n+(q-2)-i}\Big)$$
$$+n\sum_{i=0}^{q-2}\binom{q-1}{i}\sum_{k=2}^{q-1-i}\binom{r-1}{k-1}\frac{d^k}{dx^k}\big(x^{q-1-i}\big)\frac{d^{r-k}}{dx^{r-k}}\Big(Q_{n+(q-2)-i}\Big)$$
$$+\sum_{i=0}^{q-2}\binom{q-1}{i}(q-2-i)\frac{d^r}{dx^r}\Big(x^{q-1-i}Q_{n+(q-2)-i}\Big). \tag{2.7}$$

Now, differentiating equation (1.2), r times with respect to x and multiplying through out by n, we get

$$n\,\frac{d^rQ_{n+q-1}}{dx^r}=n\sum_{i=0}^{q-1}\binom{q-1}{i}x^{q-1-i}\frac{d^rQ_{n+(q-2)-i}}{dx^r}$$
$$+nr(q-1)\sum_{i=0}^{q-1}\binom{q-2}{i}x^{q-2-i}\,\frac{d^{r-1}Q_{n+(q-2)-i}}{dx^{r-1}}$$
$$+n\sum_{i=0}^{q-1}\binom{q-1}{i}\sum_{k=2}^{r}\binom{r}{k}\frac{d^k}{dx^k}\big(x^{q-1-i}\big)\frac{d^{r-k}Q_{n+(q-2)-i}}{dx^{r-k}}. \tag{2.8}$$

Using (2.7) and (2.8), we get

$$\Big(n-(q-1)r\Big)\,\frac{d^rQ_{n+q-1}}{dx^r}=n\sum_{i=0}^{q-2}\binom{q-1}{i}x^{q-1-i}\frac{d^rQ_{+(q-2)-i}}{dx^r}+(n+r)\frac{d^r\,Q_{n-1}}{dx^r}$$
$$-rn\sum_{i=0}^{q-2}\binom{q-1}{i}\sum_{k=2}^{q-1-i}\binom{r-1}{k-1}\frac{d^k}{dx^k}\big(x^{q-1-i}\big)\frac{d^{r-k}}{dx^{r-k}}\Big(Q_{n+(q-2)-i}\Big)$$
$$-r\sum_{i=0}^{q-2}\binom{q-1}{i}(q-2-i)\frac{d^r}{dx^r}\Big(x^{q-1-i}Q_{n+(q-2)-i}\Big)$$
$$+n\sum_{i=0}^{q-1}\binom{q-1}{i}\sum_{k=2}^{q-1-i}\binom{r}{k}\frac{d^k}{dx^k}\big(x^{q-1-i}\big)\frac{d^{r-k}Q_{n+(q-2)-i}}{dx^{r-k}}.$$

Since $r\binom{r-1}{k-1}=k\binom{r}{k}$, we get

$$\Big(n-(q-1)r\Big)\,\frac{d^rQ_{n+q-1}}{dx^r}=n\sum_{i=0}^{q-2}\binom{q-1}{i}x^{q-1-i}\frac{d^rQ_{n+(q-2)-i}}{dx^r}+(n+r)\frac{d^r\,Q_{n-1}}{dx^r}$$
$$-r\sum_{i=0}^{q-2}\binom{q-1}{i}(q-2-i)\frac{d^r}{dx^r}\Big(x^{q-1-i}Q_{n+(q-2)-i}\Big)$$
$$+n\sum_{i=0}^{q-2}\binom{q-1}{i}\sum_{k=2}^{q-1-i}\binom{r}{k}(1-k)\frac{d^k}{dx^k}\big(x^{q-1-i}\big)\frac{d^{r-k}Q_{n+(q-2)-i}}{dx^{r-k}}.$$

This completes the proof of the theorem. □

3 Conclusion

In this paper, we have used a combinatorial representation of B-q bonacci polynomial and it's generating function to obtain the results involving derivatives of B-q bonacci polynomials. We have also obtained a result on the r^{th} order derivative of B-q bonacci polynomials.

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