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ON THE EXPONENTIAL DIOPHANTINE EQUATION $5^x + 11^y = z^2$

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Abstract This research examines the exponential Diophantine equation $5^x + 11^y = z^2$ within the set of numbers in $\mathbb{Z}_{\geq 0}$. Through rigorous analysis, it is established that this equation admits exactly three integer solutions (x, y, z): $(1, 1, 4)$, $(2, 1, 6)$, and $(5, 1, 56)$. Notably, the bases of the exponents, 5 and 11, are consecutive Sophie Germain primes. This research contributes to the understanding of Diophantine equations involving prime numbers, particularly those with special properties such as Sophie Germain primes.

1 Introduction

Diophantine equations are polynomial equations with integer coefficients and unknowns that are assumed to be rational integers. This formulation is often used for any kind of equations involving integers where the unknowns are also integers. Fermat's equation,

$$x^n + y^n = z^n, \tag{1.1}$$

is a fundamental example of a Diophantine equation, where x, y, z are positive integers and $n > 2$. Fermat's Last Theorem states that this equation has no nontrivial integer solutions for $n > 2$, a result proven by Andrew Wiles in 1994. When one or more exponents are unknown, they are often called exponential Diophantine equations. We investigated the Diophantine problem $3^x + 5^y = z^2$ proposed by B. Sroysang[3] discovered in 2012 that there exists a single solution $(x = 1, y = 0, z = 2)$ within the set of numbers in $\mathbb{Z}_{\geq 0}$. Later, J.F.T. Rabago[4] in 2013 showed the complete solution to two Polynomial equation with integer constraints: $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$, where x, y, and z are positive numbers. Rabago found two solutions to each equation: (1,0,2), (4,1,10) and (1,0,2), (2,1,10). In 2020, Shivangi Asthana and Madan Mohan Singh [1] explored the integer equation $3^x + 117^y = z^2$ and identified four solutions within the set of non-negative integral values: $(1, 0, 2)$, $(3, 1, 12)$, $(7, 1, 48)$, and $(7, 2, 126)$. Similarly, in 2023, Nonglauk Viriyapong and Chokchai Viriyapong [5] examined the problem $255^x + 323^y = z^2$ and determined two valid solutions: $(1, 0, 16)$ and $(1, 0, 18)$, where x, y, and z are non-negative integral values. Further, in 2024, Malavika N and R. Venkatraman [6] investigated the exponential integer equation $3^x + 121^y = z^2$ and established that it has precisely two solutions: $(1, 0, 2)$ and $(5, 2, 122)$. Many of these solutions are based on Catalan's conjectures, which were solved by Mihailescu[2] in 2004.

2 Prerequisites

Definition 2.1. A prime number p is referred to as a Sophie Germain prime if both p and $2p + 1$ are prime numbers.

Example 2.2. The initial Sophie Germain primes include the numbers 2, 3, 5, 11, 23, 29, 41, 53, 83, 89, 113, and 131.

Theorem 2.3 (Mihăilescu's Theorem). *The only solution in the natural numbers of $x^a - y^b = 1$ for $a, b > 1$, $x, y > 0$ is $x = 3, a = 2, y = 2, b = 3$.*

Lemma 2.4. *The Exponential Diophantine equation $5^x + 1 = z^2$ has no solutions in non-negative integers.*

Proof. Examining the provided equation, it's evident that for $x = 0$, we conclude $z^2 = 2$, leading to a contradiction. Consequently, we deduce $x \geq 1$. Leveraging Catalan's Conjecture, we infer $y = 1$. Subsequently, the equation simplifies to $z^2 = 6$, which proves to be untenable. Hence, we conclude the impossibility of such a scenario. Hence, the proof.

□

Lemma 2.5. *The Exponential Diophantine equation $1 + 11^y = z^2$ has no solutions in $\mathbb{Z}_{\geq 0}$.*

Proof. Consider the given equation. If x equals zero, then it leads to z^2 being 2, contradicting our assumptions. Thus, we infer that x must be greater than or equal to 1. Utilizing Catalan's Conjecture, we deduce that y equals 1. Consequently, this results in z^2 equating to 6, which is untenable within our framework. Hence, the demonstration is concluded.

□

3 Main Result

Theorem 3.1. *The Exponential Diophantine equation $5^x + 11^y = z^2$ precisely only has three non-negative integer solutions $(1, 1, 4)$, $(2, 1, 6)$ and $(5, 1, 56)$ respectively for x, y and z.*

Proof. **CASE I:** Given that x equals zero, according to Lemma 2.5, there are no solutions.
CASE II: If x is assumed to be even, it can be expressed as $x = 2m$. If the equation $5^x + 11^y = z^2$ has a solution for some value of z, then $11^y = z^2 - 5^x = z^2 - 5^{2m} = (z - 5^m)(z + 5^m)$. Let $11^u = z - 5^m$ and $11^{y-u} = z + 5^m$, where $y > 2u$. Consequently, we derive $11^u[11^{y-2u} - 1] = 2{\cdot}5^m$. For $m = 1$, we have $11^u[11^{y-2u} - 1] = 2{\cdot}5 = 10$. Thus $u = 0$, yielding $11^0[11^y - 1] = 10$. This implies $11^y = 11^1$, hence $y = 1$, $x = 2$, and $z = 6$.
CASE III: Assuming x to be an odd number, expressed as $x = 2m + 1$, we can derive the equation
$5^{2m+1} + 11^y = z^2$, leading to $5 \cdot 5^{2m} + 11^y = z^2$. This further simplifies to $(9 - 4)^{2m} + 11^y = z^2$, resulting in $11^y - 4 \cdot 5^{2m} = z^2 - 9 \cdot 5^{2m}$. Through subsequent computations, it is inferred that for $m = 0$, the solution is $x = 1, y = 1, z = 4$, and for m equal to 2, the solution obtained is $(x, y, z) = (5, 1, 56)$. Thus, the proof is completed. □

Corollary 3.2. *The solutions to the Diophantine Equation $5^x + 11^y = 4l^2$, where x, y, and l arenon-negative integral values, are precisely* $(1, 1, 2)$ *and* $(5, 1, 28)$.

Proof. Let x, y, and l represent non-negative integers that satisfy the equation

$$5^x + 11^y = 4l^2.$$

Define $z = 2l$, which leads to the equation

$$5^x + 11^y = z^2.$$

Based on Theorem 3.1, the solutions to this equation are $(1, 1, 4)$, $(2, 1, 6)$, and $(5, 1, 56)$. As a result, the possible values for l are 2 and 28. Therefore, the complete set of non-negative integer solutions for the Diophantine equation

$$5^x + 11^y = 4l^2$$

is given by $(1, 1, 2)$ and $(5, 1, 28)$.

□

4 Conclusion remarks

In this study, it has been demonstrated that the Exponential Diophantine equation $5^x + 11^y = z^2$ possesses precisely three solutions in non-negative integers. These solutions are given by $(1, 1, 4), (2, 1, 6)$ and $(5, 1, 56)$, respectively.

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