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An Exponential Diophantine equation $x^2 + 3^a 89^b = y^n$

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Abstract The paper aims to find all positive integer solutions for the Diophantine equation $x^2 + 3^a 89^b = y^n$, where $x, y \geq 1$, with non-negative exponents a and b, and an integer $n \geq 3$, under the constraint that $\gcd(x, y) = 1$.

1 Introduction

Consider the Diophantine equation of the form

$$x^2 + C = y^n, \quad x, y \geq 1, \quad n \geq 3, \tag{1.1}$$

where C is a fixed positive integer. The initial discovery of positive integer solutions for the given equation dates back almost 17 decades ago [24]. It has been proven that the equation invariably has only a finite number of solutions that are positive integers [22]. Early investigations focused on Eq. (1.1) when $C = c_0$ is a constant integer [21, 31, 32]. In reference [17], Cohn solved equation (1.1) under the assumption that $\gcd(x, y) = 1$, examining the parameter C in the interval $1 \leq C \leq 100$, with some specific values of C excluded. Subsequent research, as outlined in [29], further investigated additional C values within the same range, while the remaining values were addressed in [11]. Over time, researchers have examined not just situations where $C = p^k$ with a particular prime p [1, 2, 3, 4, 6, 16, 28] but also the case involving a prime number p in general [7, 9, 23, 33, 39].

Consider a prime set $S = \{p_1, p_2, \ldots, p_k\}$. Recent studies have concentrated on Eq. (1.1), particularly when C is expressed as the product of prime powers p^k, where $p \in S$ and k is a non-negative integer [5, 10, 13, 14, 18, 19, 20, 25, 26, 27, 30, 34, 35, 36, 37, 38]. Additionally, in [40], the authors looked at a wider range of positive integer solutions to Eq. (1.1), taking into account $C = 2^a p^b$ for any odd prime p.

The main emphasis of this paper lies in the investigation of the Diophantine equation expressed as

$$x^2 + 3^a 89^b = y^n, \text{where } n \geq 3, \tag{1.2}$$

where $\gcd(x, y) = 1$ is the condition under which $a, b \geq 0$ and $x, y \geq 1$. Next, the subsequent result is established.

Theorem 1.1. *The equation (1.2) has the following solutions:*

$$(x, y, n, a, b) = (6, 5, 3, 0, 1), (46, 13, 3, 4, 0), (10, 7, 3, 5, 0), (40, 7, 4, 2, 1)$$

excluding the case where $11 \nmid n$.

2 Preliminaries

Consider algebraic integers denoted as η and its conjugate $\overline{\eta}$. A Lucas pair, denoted as $(\eta, \overline{\eta})$, is characterized by the requirement that the sum $\eta + \overline{\eta}$ and the product $\eta\overline{\eta}$ are non-zero coprime rational integers, and $\frac{\eta}{\overline{\eta}}$ is not a root of unity. The sequences of Lucas numbers are defined in correspondence with any given Lucas pair $(\eta, \overline{\eta})$ as follows:

$$L_n(\eta, \overline{\eta}) = \frac{\eta^n - \overline{\eta}^n}{\eta - \overline{\eta}}, \quad n = 0, 1, 2, \ldots$$

The presence of primitive divisors within $L_n(\eta, \overline{\eta})$ holds significant importance within the domain of Lucas sequences.

A primitive divisor of $L_n(\eta, \overline{\eta})$ is defined as a prime number p that satisfies $p \mid L_n(\eta, \overline{\eta})$ and $p \nmid (\eta - \overline{\eta})^2 \prod_{i=1}^{n-1} L_i(\eta, \overline{\eta})$ for $n > 1$. Additionally, a primitive divisor q of $L_n(\eta, \overline{\eta})$ satisfies the congruence $q \equiv \left(\frac{(\eta-\overline{\eta})^2}{q}\right) \pmod{n}$, where $\left(\frac{*}{q}\right)$ denotes the Legendre symbol [15].

For $n > 4$ and $n \neq 6$, every n-th term within any Lucas sequence $L_n(\eta, \overline{\eta})$ is characterized by the presence of primitive divisors, with the exception of specific finite configurations of parameters η, $\overline{\eta}$, and n [8].

3 Proof of Theorem 1.1

When $n = 3$, $n = 4$, and $n \geq 5$, Equation (1.2) will be investigated independently as outlined below:

Proposition 3.1. *If $n = 3$, the Eq. (1.2) has the following solutions:*

$$(x, y, a, b) = (6, 5, 0, 1), (46, 13, 4, 0), (10, 7, 5, 0)$$

Proof. When $n = 3$, represent $a = 6a_1 + i$ and $b = 6b_1 + j$, where $i, j \in \{0, 1, \ldots, 5\}$. Subsequently, Eq. (1.2) takes the form of an elliptic curve

$$L^2 = M^3 - 3^i 89^j$$

where $L = \frac{x}{3^{3a_1} 89^{3b_1}}$ and $M = \frac{y}{3^{2a_1} 89^{2b_1}}$.

Therefore, the task of discovering positive integer solutions for Eq. (1.2) is transformed into identifying all $\{3, 89\}$-integer points on the corresponding 36 elliptic curves for each i and j. It's essential to highlight that for any finite prime number set S, an S-integer is defined as a rational number $\frac{r}{s}$, where r and $s > 0$ are coprime integers, and any prime factor of s belongs to the set S.

The MAGMA function SIntegralPoints is utilised to find all S-integral points on the provided curves. For the set $S = \{3, 89\}$ [12], the recognised points are shown below:

$$(M, L, i, j) = (1, 0, 0, 0), (5, 6, 0, 1), (89, 0, 0, 3), \left(\frac{7387}{9}, \frac{633680}{27}, 1, 3\right), (3, 0, 3, 0),$$
$$(87, 810, 3, 1), (267, 0, 3, 3), (13, 46, 4, 0), (7, 10, 5, 0)$$

Considering that x and y are positive integers with no common factors, it's important to highlight that only three among these points yield a solution for Eq. (1.2). With this, the proof comes to an end. □

Proposition 3.2. *If $n = 4$, the Eq. (1.2) has only one solution $(x, y, a, b) = (40, 7, 2, 1)$*

Proof. Let $n = 4$. Initially, express $a = 4\alpha_1 + i$ and $b = 4\beta_1 + j$, where $i, j \in \{0, 1, 2, 3\}$. Consequently, Eq. (1.2) takes the form

$$A^2 = B^4 - 3^i 89^j$$

where $A = \frac{x}{3^{2\alpha_1} 89^{2\beta_1}}$ and $B = \frac{y}{3^{\alpha_1} 89^{\beta_1}}$.

Determining each of the 16 quartic curves' corresponding $S = \{3, 89\}$-integral points equates to finding each integer solution of Eq. (1.2).

By employing the SIntegralLjunggrenPoints, we successfully determined all S-Integral Points on these curves, resulting in

$$(A, B, i, j) = (\mp 1, 0, 0, 0), (\mp 7, 40, 2, 1)$$

Given the condition on the values of x and y, it is evident that Eq. (1.2) have only one solution $(x, y, a, b) = (40, 7, 2, 1)$. □

Proposition 3.3. *If $n \geq 5$, the Eq. (1.2) does not possess any positive integer solutions.*

Proof. Assume that n is greater than or equal to 5. If a solution for Equation (1.2) exists with $n = 2^k$ and $k \geq 3$, it may be derived from solutions with $n = 4$ using the relationship $y^{2^k} = \left(y^{2^{k-2}}\right)^4$. Hence, there are no solutions for equation (1.2) when n is equal to 2^k and k is greater than or equal to 3.

Consequently, Eq. (1.2) also lacks a solution when n equals 3^k and k is equal to 2. Therefore, it may be assumed that n is an odd prime without any loss of generality.
Let's initiate the analysis of the factorization of Eq. (1.2) in the field $K = Q(\sqrt{-d})$ as follows

$$(x + e\sqrt{-d})(x - e\sqrt{-d}) = y^n$$

where $e = 3^\alpha 89^\beta$ for some integers $\alpha, \beta \geq 0$ and $d \in \{1, 3, 89, 267\}$.

Assuming that y is even leads to a contradiction, as x must be odd according to (1.2), resulting in $1 + 3^a \equiv 0 \pmod 8$.

The ideals generated by $x + e\sqrt{-d}$ and $x - e\sqrt{-d}$ are relatively prime in the field K because y is an odd integer as a consequence.

The class number $h(K)$ takes on one of three values: 1, 2, or 12 for the specific choice of d.

Thus, we can deduce that the greatest common divisor of n and $h(K)$ is 1.

Given an algebraic integer $\xi \in K$ and units u_1 and u_2 in the ring of algebraic integers of K, this can be expressed as:

$$x + e\sqrt{-d} = \xi^n u_1$$
$$x - e\sqrt{-d} = \overline{\xi}^n u_2$$

Taking into account that the orders of the multiplicative group of units in the ring of algebraic integers of K are 2, 4, or 6, depending on the value of d, and observing that these orders are relatively prime to n, the units u_1 and u_2 can be eliminated from the equations. Consequently, the units u_1 and u_2 can be integrated into the factors ξ^n and $\overline{\xi}^n$.

Let us examine the two cases separately, where d belongs to $\{1, 89\}$ or $\{3, 267\}$. These cases correspond to distinct integral bases for $\mathcal{O}_K$, specifically $\{1, \sqrt{-d}\}$ for $d \in \{1, 89\}$ and $\left\{1, \frac{1+\sqrt{-d}}{2}\right\}$ for $d \in \{3, 267\}$. To begin with, the case $d \in \{1, 89\}$. Consequently, we have the following equations:

$$x + e\sqrt{-d} = \xi^n = (s + t\sqrt{-d})^n$$
$$x - e\sqrt{-d} = \overline{\xi}^n = (s - t\sqrt{-d})^n$$

where $y = s^2 + dt^2$ for certain rational integers s and t. By examining these equations, we can deduce that

$$e = L_n t,$$

where $L_n = \frac{\xi^n - \overline{\xi}^n}{\xi - \overline{\xi}}$. It is important to note that the sequence L_n forms a Lucas sequence.

The Lucas sequences lacking primitive divisors are explicitly enumerated in [8], where it is verified that L_n does not correspond to any of these sequences. Consequently, we explore the potential existence of a primitive divisor for L_n. Let q be any such primitive divisor of L_n. In this context, q must be either 3 or 89. Given that any primitive divisor is congruent to ± 1 modulo n, and since $n \geq 5$, we can exclude $q = 3$. Therefore, we proceed under the assumption that $q = 89$. According to the definition of a primitive divisor, $q \nmid (\xi - \overline{\xi})^2 = -4dt^2$, which implies that $d = 1$. Furthermore, considering that $\left(\frac{-4t^2 d}{q}\right) = \left(\frac{-1}{89}\right) = 1$, we conclude that $89 \equiv 1 \pmod n$, which indicates $n = 11$. This leads to a contradiction, as it implies $11 \nmid n$, contrary to our initial assumption.

Now, consider the case $d \in \{3, 267\}$. In these cases, the integral basis for $\mathcal{O}_K$ is given by $\left\{1, \frac{1+\sqrt{-d}}{2}\right\}$. Consequently, we can represent

$$x + e\sqrt{-d} = \xi^n = \left(\frac{u + v\sqrt{-d}}{2}\right)^n$$

$$x - e\sqrt{-d} = \overline{\xi}^n = \left(\frac{u - v\sqrt{-d}}{2}\right)^n$$

with $u \equiv v \pmod 2$. Then

$$2e = L_n v,$$

where $L_n = \dfrac{\xi^n - \overline{\xi}^n}{\xi - \overline{\xi}}$.

Upon observing that L_n forms a Lucas sequence, the absence of a primitive divisor for L_n suggests its correspondence with one of the Lucas sequences listed in the table referenced [8]. However, this inference does not apply to $d \in \{3, 267\}$.

Hence, L_n possesses a primitive divisor, denoted as q, where q can either be 3 or 89.

From the properties $q \nmid (\xi - \overline{\xi})^2 = -dv^2$ and $q \equiv \left(\dfrac{-v^2 d}{q}\right) \pmod n$, it follows that $q = 89$, $d = 3$, and $n = 5$. So we get that

$$x + 3^\alpha 89^\beta \sqrt{-3} = \left(\frac{u + v\sqrt{-3}}{2}\right)^5$$

where $\alpha = \frac{a-1}{2}$ and $\beta = \frac{b}{2}$. Equating the imaginary parts in the above equation we find that

$$v(5u^4 - 30u^2v^2 - 9v^4) = 2^5 3^\alpha 89^\beta \tag{3.1}$$

since $89 \nmid v$ and $\gcd(u, v) = 1$ we have the following possibilities $v = \pm 1, v = \pm 2^5, v = \pm 3^\alpha, v = \pm 2^5 3^\alpha$.

If $v = \pm 1$, then Eq. (3.1) transforms into the following equations:

$$5u^4 - 30u^2 - 9 = \pm 2^5 3^\alpha 89^\beta$$

By letting $\alpha = 2a_1 + i$ and $\beta = 2b_1 + j$ for $i, j \in \{0, 1\}$, we obtain:

$$5u^4 - 30u^2 - 9 = \delta Y^2$$

$\delta = \pm 2 \cdot 3^i 89^j$ and $Y = 2^2 3^{a_1} 89^{b_1}$. By multiplying δ both sides of the above equation, we obtain:

$$5\delta u^4 - 30\delta u^2 - 9\delta = (\delta Y)^2$$

We employ the IntegralQuarticPoints function in MAGMA to calculate all integral points on the quartic curves described above. However, the computations reveal that no solutions exist for this equation. Hence, equation (1.2) lacks a solution. If $v = \pm 2^5$, then Eq. (3.1) transforms into

$$5u^4 - 30 \cdot 2^{10} u^2 - 9 \cdot 2^{20} = \pm 3^\alpha 89^\beta$$

Write $\alpha = 2a_1 + i$ and $\beta = 2b_1 + j$ for $i, j \in \{0, 1\}$, we get

$$5u^4 - 30 \cdot 2^{10} u^2 - 9 \cdot 2^{20} = \delta Y^2$$

$\delta = \pm 3^i 89^j$ and $Y = 3^{a_1} 89^{b_1}$. Therefore, it is necessary to determine all $\{2\}$-integral points on the aforementioned curves. Employing the SIntegralLjunggrenPoints function in MAGMA facilitated the identification of these points, yielding $(u, Y) = (0, 3072)$ and $(\pm 96, 12288)$ for $\delta = 1$, and $(u, Y) = (\pm 32, 4096)$ for $\delta = -1$. However, none of these points satisfy equation (1.2).

For $v = \pm 3^\alpha$, by dividing both sides of Eq. (3.1) by v^4 and introducing $\beta = 2b_1 + j$ for $j \in \{0, 1\}$, we obtain:

$$5U^4 - 30U^2 - 9 = \delta Y^2$$

where $\delta=\pm 2^5 89^j$, $Y=\dfrac{89^{b_1}}{v^2}$, $U=\frac{u}{v}$. To determine all $\{3\}$-integral points on the elliptic curve described above for every value of δ, we seek solutions to the equation. However, no solutions are found for this equation. Hence, equation (1.2) has no solution.

For $v=\pm 2^5 3^\alpha$, we proceed analogously to the preceding case, yielding:

$$5U^4-30U^2-9=\delta Y^2$$

where $\delta=\pm 89^j$, $Y=\dfrac{89^{b_1}}{v^2}$, $U=\frac{u}{v}$. To determine all $\{2,3\}$-integral points on the aforementioned elliptic curves for every value of δ, we identify the subsequent points:

$$(U,Y)=(0,3),(\pm 3,12) \text{ for } \delta=1,$$
$$(U,Y)=\left(\pm\frac{5}{9},\frac{4}{81}\right) \text{ for } \delta=89,$$
$$(U,Y)=(\pm 1,4) \text{ for } \delta=-1,$$

It is easy to verify that none of them leads to a solution of Eq. (1.2). This completes the proof. □

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