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Exploring Diophantine Equations Involving Twin Primes: A Note on $3^x + p^y = z^2$

K. Manikandan and R. Venkatraman

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Corresponding Author: R. Venkatraman

Abstract This study delves into the exploration of solutions for the Diophantine equation $3^x + p^y = z^2$, focusing on a twin prime pair $(227, 229)$. For $p = 227$, the equation has a unique non-negative integer solution, denoted as (1, 0, 2). When $p = 229$, the equation presents two non-negative integer solutions: (1, 0, 2) and (3, 1, 16). The research sheds light on the connection between the equation and twin prime pairs, providing insights into the properties of solutions in the realm of Diophantine equations.

1 Introduction

Diophantine equations, which involve finding integer solutions, have captivated mathematicians for centuries, leading to a rich history of exploration and discovery. Researchers have employed diverse methods to investigate equations like $a^x + b^y = c^z$, unraveling solutions and patterns within these intricate mathematical structures.

This research provides an overview of the progress made in solving various Diophantine equations involving sums of powers and squares. In 2011, Suvarnamani [1] presented comprehensive solutions to equations of the form $2^x + p^y = z^2$, where p includes primes such as 2, 3, and $2^{k+1} + 1$. However, Sroysang's [2] work in 2013 refined the understanding of these solutions by addressing an initial oversight regarding the integral nature of certain terms. Hadano [3] in 1976 revealed solutions for equations like $3^y + 11^z = 2^x$ and $3^y + 13^z = 2^x$. Similarly, Rabago [4] in 2013 explored equations such as $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$, uncovering multiple $n \geq 0$ solutions. Sroysang [5, 6, 7] further enriched the understanding of equations like $3^x + 5^y = z^2$ and $3^x + 85^y = z^2$ in 2012 and 2014, respectively, providing additional insights into these Diophantine structures. Numerous other scholars [8, 9, 10, 11, 12] have contributed to the exploration of related Diophantine equations using various methods. In 2017, Asthana and Singh [14] addressed the Diophantine equation $3^x + 13^y = z^2$, revealing four $n \geq 0$ solutions. K. Manikandan and R. Venkatraman [17, 18] investigates a Diophantine equation of the form $5(x^2 + y^2) - 6xy + 4(x + y + 1) = 6100z^3$, as well as a novel equation in the form of $945(x^2+y^2)-1889xy+10(x+y)+100 = pz^q$. More recently, Asthana and Singh[15] worked on the Diophantine equation $3^x + 117^y = z^2$, unveiling four solutions in 2020. Viriyapong and Viriyapong[16] in 2023 considered the equation $255^x + 323^y = z^2$, proving two solutions. In 2024, Malavika and Venkatraman [19] explored the equation $3^x + 121^y = z^2$, demonstrating two solutions. K. Manikandan and R. Venkatraman [20] presents a comprehensive analysis of the Diophantine equation $8^x + 161^y = z^2$, proving that it possesses precisely two solutions: (1, 0, 3) and (1, 1, 13) . In this paper, we delve into a specific type of Diophantine equation, namely $3^x + p^y = z^2$, here p is a twin prime. Our focus lies on the cases where p takes values of 227 and 229. The study of Diophantine equations involving powers of primes has a rich history, with numerous researchers contributing insights and solutions over the years. Our investigation builds upon this foundation, aiming to deepen our understanding of the solution space for equations of this form.

Recent decades have seen the development of various techniques and methodologies to tackle Diophantine equations, ranging from algebraic manipulations to more advanced number theoretic approaches. By judiciously applying these methods, we aim to uncover patterns and properties specific to the equations under consideration.

Our research reveals intriguing results regarding the existence and uniqueness of solutions for the Diophantine equations $3^x + 227^y = z^2$ and $3^x + 229^y = z^2$. Through rigorous mathematical analysis, we demonstrate the unique $n \geq 0$ solution $(1, 0, 2)$. Similarly, for the latter equation, we find two $n \geq 0$ solutions: $(1, 0, 2)$ and $(3, 1, 16)$.

These findings contribute to the broader body of knowledge surrounding Diophantine equations, shedding light on the behavior of such equations in relation to twin prime pairs. Additionally, our study showcases the efficacy of mathematical techniques in exploring and understanding complex mathematical structures, offering valuable insights into the realm of number theory and integer solutions.

2 Preliminaries

This section discusses foundational concepts crucial for understanding the advanced sections of this paper.

The variable $n \geq 0$ represents a non-negative integer.

Definition 2.1. A twin prime pair consists of two prime numbers that have a difference of two.

Example 2.2. Notable examples of twin primes include pairs such as $(3, 5), (11, 13), and (17, 19)$.

Proposition 2.3. *The only known solution to the equation $a^x - b^y = 1$, for integers greater than one, involves $a = 3, b = 2, x = 2, y = 3$ [13].*

Lemma 2.4. *According to [5] , the equation $3^x + 1 = z^2$, where x and $z \geq 0$, has a unique solution of* $(1, 2)$.

Proof. Considering the cases where x or z equals zero leads to impossibilities. If $x = 0$, then $z^2 = 2$, which is not feasible. Similarly, if $z = 0$, then $3^x + 1 = 0$, implying $3^x = -1$, which is not possible. Therefore, we focus on the scenario where both x and $z \geq 0$.

Rewriting the equation as $3^x = z^2 - 1$, we obtain $3^x = (z - 1)(z + 1)$. Let $3^u = (z - 1)$ and $3^{x-u} = (z + 1)$, where u represents a non-negative integer and $x > 2u$. This yields $2 = (z + 1) - (z - 1) = 3^{x-u} - 3^u$, or $2 = 3^u(3^{x-2u} - 1)$. From this, we deduce that $3^u = 3^0$ and $3^{x-2u} - 1 = 2$. Consequently, u equals zero, leading to $x = 1$. Thus, the only solution (x, z) for $3^x + 1 = z^2$ is $(1, 2)$. □

3 Main Results

Theorem 3.1. *The Diophantine equation $3^x + 227^y = z^2$ has a unique $n \geq 0$ solution, where $x \leq 100$, $y \leq 100$, and z are the variables. The solution is* $(1, 0, 2)$.

Proof. Let us take $0 \leq x \leq 100$, $0 \leq y \leq 100$, and $z \geq 0$ satisfies $3^x + 227^y = z^2$. Initially, we analyze the case where $y = 0$, as proven in Lemma (2.4), resulting in the solution $(x, y, z) = (1, 0, 2)$.

Subsequently, we delve into the scenario where all variables, $0 \leq x \leq 100, 0 \leq y \leq 100$, and $z \geq 0$, leading to two distinct situations:

Condition (1): When $y = 2k$ with $k > 0$. By expressing the equation as $3^x + 227^{2k} = z^2$, we derive $3^x = (z - 227^k)(z + 227^k)$. This implies $z - 227^k = 3^u$ for a non-negative integer u, and $z + 227^k = 3^{x-u}$. Consequently, $2 \cdot 227^k = 3^{x-u} - 3^u$, with $x > 2u$. Setting $u = 0$, we get $3^x - 1 = 2 \cdot 227^k$. However, since $k \geq 1$, $3^x \geq 455$, rendering this equation unsolvable. Hence, no solution exists in this case.

Condition (2): When $y = 2l + 1$ with $l \geq 0$. Expressing $3^x + 227 \cdot 227^{2l} = z^2$ as $3^x - 12769 \cdot 227^{2l} = z^2 - 12996 \cdot 227^{2l}$, we have $3^x - 12769 \cdot 227^{2l} = (z - 114 \cdot 227^l)(z + 114 \cdot 227^l)$. This equation leads to two potential scenarios:

- In the first scenario, $z - 114 \cdot 227^l = 1$ and $z + 114 \cdot 227^l = 3^x - 12769 \cdot 227^{2l}$.
- In the second scenario, $z + 114 \cdot 227^l = 1$ and $z - 114 \cdot 227^l = 3^x - 12769 \cdot 227^{2l}$.

Solving the first set of equalities results in $227^l(228 + 12769 \cdot 227^l) = 3^x - 1$, implying $227^l = 227^0$ and $3^x - 1 = 228 + 12769 \cdot 227^l$. However, this leads to $l = 0$ and $3^x = 12998$, which is unsolvable. Similarly, we deduce $l = 0$ and $z+114 = 1$ from second scenario, resulting in $z = -113$, contradicting the requirement for $z \in \mathbb{N}$. Therefore, no solution exists in this case.

In conclusion, the exponential Diophantine equation $3^x + 227^y = z^2$ has a unique non-negative integer solution, specifically $(1, 0, 2)$. □

Theorem 3.2. *The Diophantine equation* $3^x + 229^y = z^2$*, where* $0 \le x \le 100$*,* $0 \le y \le 100$*, and* $z \ge 0$*, has two solutions:*

1. $(1, 0, 2)$ *2.* $(3, 1, 16)$

This means that the equation has precisely two non-negative integer solutions for the variables $0 \le x \le 100$*,* $0 \le y \le 100$*, and* $z \ge 0$*.*

Proof. Let $0 \le x \le 100$, $0 \le y \le 100$, and $z \ge 0$ that satisfy the equation $3^x + 229^y = z^2$.

Initially, we explore the case where $y = 0$, as proven by Lemma (2.4), resulting in the solution $(x, y, z) = (1, 0, 2)$.

Next, we investigate the situation where all variables, $0 \le x \le 100$, $0 \le y \le 100$, and $z \ge 0$, leading to two distinct cases:

Condition (1): If $y = 2k$, where $k > 0$.

In this scenario, we have $3^x + 229^{2k} = z^2$, which simplifies to $3^x = (z - 229^k)(z + 229^k)$. This leads to $z - 229^k = 3^u$ and $z + 229^k = 3^{x-u}$. However, the equation $2 \cdot 229^k = 3^u(3^{x-2u} - 1)$ with $x > 2u$, when setting $u = 0$, results in $3^x - 1 = 2 \cdot 229^k$. Since $k \ge 1$, this renders the equation unsolvable. Therefore, no solution exists in this case.

Condition (2): If $y = 2l + 1$, where $l \ge 0$.

Considering $x = 2k + 1 > 0$ and $z = 2m$, where $m > 0$, we have $3^x + 229 \cdot 229^{2l} = (2m)^2$. This simplifies to $3^x + 4 \cdot 229^{2l} = (4m^2) - 225 \cdot 13^{2l}$ or equivalently $3^x + 4 \cdot 229^{2l} = (2m - 15 \cdot 229^l)(2m + 15 \cdot 229^l)$.

This equation leads to two possibilities:

1. First possibility: $2m - 15 \cdot 229^l = 1$ and $2m + 15 \cdot 229^l = 3^x + 4 \cdot 229^{2l}$. 2. Second possibility: $2m + 15 \cdot 229^l = 1$ and $2m - 15 \cdot 229^l = 3^x + 4 \cdot 229^{2l}$.

Solving the first set of equalities results in $229^l(30 - 4 \cdot 229^l) = 3^x - 1$, implying $229^l = 229^0$ and $3^x - 1 = 30 - 4 \cdot 229^l$. This leads to $l = 0$, $x = 3$, and $m = 8$, $(x, y, z) = (3, 1, 16)$ is the solution.

In 2^{nd}, we find $l = 0$, but $2m + 15 = 1 \Rightarrow 2m = -14$, which is not possible, since $m \in \mathbb{N}$.

In conclusion, $3^x + 229^y = z^2$ has 2 non-negative integer solutions: $(1, 0, 2)$ and $(3, 1, 16)$ for $0 \le x \le 100$ $0 \le y \le 100$, and $z \ge 0$. □

Corollary 3.3. *The positive integer solution* $(3, 1, 4)$ *satisfies the Diophantine equation* $3^x + 229^y = w^4$*, where* $0 \le x \le 100$*,* $0 \le y \le 100$*, and* $w > 0$*.*

Proof. Take $0 \le x \le 100$, $0 \le y \le 100$, and $w > 0$ such that $3^x + 229^y = w^4$. Let $z = w^2$. Then $3^x + 229^y = z^2$. Using $theorem(3.2)$, we have $(x, y, z) = (3, 1, 16)$. Consequently, $w^2 = z = 16$, implying $w = 4$. Thus, $(3, 1, 4)$ satisfies the given equation. Hence, the solution $(3, 1, 4) > 0$ for $3^x + 229^y = w^4$. □

Corollary 3.4. *The Diophantine equation* $3^x + 229^y = v^8$ *has a only solution* $(3, 1, 2) > 0$*.*

Proof. Let positive integers $0 < x \le 100$, $0 < y \le 100$, and v satisfy $3^x + 229^y = v^8$. Let $z = v^4$. Then, $3^x + 229^y = z^2$. Theorem (3.2), we conclude the solution is $(3, 1, 16)$. Hence, $z = v^4 = 16$. Therefore, $v = 2$. Thus, the unique solution to the Diophantine equation $3^x + 229^y = v^8$ in positive integers (x, y, v) is $(3, 1, 2)$. □

Corollary 3.5. *The Diophantine equation* $3^x + 229^y = 4s^2$ *has a solution* $(x, y, s) > 0$*, given by* $(3, 1, 8)$*, where* $x, y \in \mathbb{N}$*.*

Proof. Let positive integers $0 < x \leq 100$, $0 < y \leq 100$, and s satisfy the equation $3^x + 229^y = 4s^2$. Assume $z = 2s$. Consequently, $3^x + 229^y = z^2$. Theorem (3.2), we establish that $(x, y, z) = (3, 1, 16)$. Therefore, $z = 2s = 16$, implying $s = 8$. Thus, the Diophantine equation $3^x + 229^y = 4s^2$ has a sole solution in positive integers (x, y, s), which is $(3, 1, 8)$. □

4 Conclusion remarks

In conclusion, the research article extensively investigated the Diophantine equations $3^x + p^y = z^2$, *with a specific focus on twin prime pairs, notably* $(227, 229)$. *The study rigorously analyzed these equations and established significant results:*

1. For the equation $3^x + 227^y = z^2$, $(1, 0, 2) \geq 0$ *is the solution was identified.*

2. In the case of $3^x + 229^y = z^2$, *2 solutions,* $(1, 0, 2), (3, 1, 16) \geq 0$, *were discovered for the variables* $0 \leq x \leq 100$, $0 \leq y \leq 100$, *and* $z \geq 0$.

The research also derived corollaries to extend the analysis to related equations involving higher powers of za, showcasing unique or singular solutions in positive integers. Overall, this study provides valuable insights into the characteristics and solution space of Diophantine equations, particularly within the realm of twin prime pairs, contributing to a deeper understanding of these mathematical structures.

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Author information

K. Manikandan, Department of Mathematics, Faculty of Engineering and Technology, SRM Institute of Science and Technology, Vadapalani Campus, No.1 Jawaharlal Nehru Salai, Vadapalani, Chennai-600026, Tamilnadu, India.
E-mail: manishkm4439@gmail.com, mk8408@srmist.edu.in

R. Venkatraman, Department of Mathematics, Faculty of Engineering and Technology, SRM Institute of Science and Technology, Vadapalani Campus, No.1 Jawaharlal Nehru Salai, Vadapalani, Chennai-600026, Tamilnadu, India.
E-mail: venkatrr1@srmist.edu.in, venkarmeeraj@gmail.com