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A revisit to totally bounded functions and totally positive functions

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Abstract *The concept of totally positive functions and totally bounded functions arise in the study of Lagrange interpolation. In this paper, we prove that the function* $f(x) = e^x$ *is totally positive as well as totally bounded on* $[0, 1]$. *We also derive a necessary and sufficient condition for the total positvity of the function* $f(x) = \frac{1}{b-x}$ *on* $[0, 1]$.

1 Introduction and preliminaries

Alan L.Horwitz and Lee A.Rubel in [2] introduced totally positive functions and totally bounded functions on $[-1, 1]$ *and they have studied some properties of these functions in [2]. In this paper, we present a review of some results associated with totally positive functions and totally bounded functions and give examples of these functions in detail.*

Throughout this paper, the closed bounded interval $[a, b]$ *of* $(-\infty, \infty)$ *is denoted by* I. *The space of all polynomials is denoted by* $\mathcal{P}$ *and it is generated by* $P = \{1, x, x^2, \ldots, \}$. *The space of all polynomials of degree atmost* $n-1$ *is denoted by* $\mathcal{P}_n$ *and it is generated by* $P_n = \{1, x, x^2, \ldots, x^{n-1}\}$. *The primary reference for definitions and results included in this section is [1].*

Definition 1.1. Let f be a real valued function defined on I. A polynomial p of degree $n-1$ is said to be a Lagrange interpolant of f if there are n distinct points $\{x_1, x_2, \ldots, x_n\}$ in I such that

$$p(x_i) = f(x_i), \qquad i = 1, 2, \ldots, n.$$

The Lagrange interpolation formula for polynomial interpolation is as follows:

Theorem 1.2. *Let* f *be a real valued function defined on* I. *Let* x_1, x_2, ..., x_n *be* n *distinct points in* I. *Let* p *be the unique polynomial of degree less than* n *interpolating to* f *at* x_1, x_2, ..., x_n. *For* $x \in I$, $x \neq x_1, x_2, \ldots, x_n$, $p(x)$ *has the following expression,*

$$p(x) = \left\{ \sum_{i=1}^{n} \frac{f(x_i)}{(x - x_i)W'(x_i)} \right\} W(x),$$

where $W(x) = (x - x_1)(x - x_2) \ldots (x - x_n)$ *and* $W'(x)$ *denotes the derivative of* $W(x)$.

Definition 1.3. Let f be a function defined on I. Let $a \leq x_1 < x_2 < \ldots < x_n \leq b$. Then the divided difference of f with respect to x_1, x_2, ..., x_n, denoted by $[x_1, x_2, \ldots, x_n]f$ is defined as

$$[x_1, x_2, \ldots, x_n]f = \sum_{i=1}^{n} \frac{f(x_i)}{W'(x_i)},$$

where $W(x) = (x - x_1)(x - x_2) \ldots (x - x_n)$.

Using divided differences, the Lagrange interpolation formula can be converted to the Newton's general interpolation formula which is as follows:

Theorem 1.4. *Let f be a real valued function defined on I. Let $x_1 < x_2 < \ldots < x_n$ be distinct points on I. An explicit expression for the unique polynomial p_n in $\mathcal{P}_n$ interpolating to f at x_1, x_2, ..., x_n is given by*

$$p_n(x) = f(x_1) + \sum_{k=2}^{n} (x - x_1)(x - x_2) \ldots (x - x_{k-1}) \, [x_1, \ldots, x_k] \, f.$$

Moreover the error of interpolation is given by

$$f(x) - p_n(x) = (x - x_1)(x - x_2) \ldots (x - x_n) \, [x_1, \ldots, x_n, x] \, .$$

The forthcoming property of divided differences is needed to prove the main result.

Theorem 1.5. *If f is in $C^{n-1}[a, b]$ and if $a \leq x_1 < \ldots < x_n \leq b$ are n points, then*

$$[x_1, \ldots, x_n] \, f = \frac{f^{(n-1)}(\xi)}{(n-1)!}$$

for some ξ, $x_1 < \xi < x_n$.

Definition 1.6. A polynomial $p(x)$ is said to be positive on $[a, b]$ if $p(x) > 0$ for all $x \in [a, b]$.

The following definitions of totally positive functions and totally bounded functions are given in [2].

Definition 1.7. A function f defined on I is said to be totally positive if all Lagrange interpolants to f on I are positive.

Definition 1.8. A function f defined on I is said to be totally bounded on I if there exists a constant $M > 0$ such that whenever p is a Lagrange interpolant to f, $|p(x)| \leq M$ for all x in I.

The following properties of totally bounded functions and totally positive functions have been proved in [2] for the interval $[-1, 1]$. Those results clearly holds in any interval $I = [a, b]$.

Theorem 1.9. *(i) If f is totally positive on I, then f is infinitely differentiable on I.*

(ii) If f is totally bounded on I, then for some $M > 0$, $f + M$ is totally positive on I.

Consequently, if f is totally bounded on I, then f is infinitely differentiable on I.

Extended Complete Tchebycheff Spaces(ECT- spaces) are natural generalizations of polynomial spaces and the following property of interpolation associated with ECT-spaces is given in [3]. The particular case for polynomials can be stated as follows:

Lemma 1.10. *Let f be a sufficiently differentiable function defined on I. If p is the unique polynomial interpolating to f at n distinct points, then p' is the unique polynomial interpolating to f' at $(n-1)$ distinct points.*

2 Main Results

Alan.L.Horwitz and Lee.A.Rubel in [2] have mentioned some exampes of totally positive functions and totally bounded functions. In this article, a detailed proof of these facts have been given. In this section, we mainly consider functions on the particular closed bounded interval $J = [0, 1]$.

Proposition 2.1. *The function $f(x) = e^x$ is totally positive on $J = [0, 1]$.*

Proof. We will use the method of induction to prove that the function $f(x) = e^x$ is totally positive on $J = [0,1]$. Let p_1 be the Lagrange interpolant to f from $\mathcal{P}_1$ interpolating at the point $x_1 \in J$. Then

$$p_1(x) = e^{x_1} > 0.$$

Since x_1 is an arbitrary point of J, this shows that all interpolants to f from $\mathcal{P}_1$ is positive.

Now, let x_1, x_2 be in J with $x_1 < x_2$. Let p_2 be the unique Lagrange interpolant to f from $\mathcal{P}_2$ interpolating to f at the points x_1 and x_2. Then

$$p_2(x) = e^{x_1} + \frac{e^{x_2} - e^{x_1}}{x_2 - x_1} \cdot (x - x_1)$$

Since p_2 is the Lagrange interpolant to f from $\mathcal{P}_2$, by Lemma 1.10, p_2' is the Lagrange interpolant to f' from $\mathcal{P}_1$. But, here, $f'(x) = e^x$. Since all interpolants to e^x from $\mathcal{P}_1$ are positive, $p_2' > 0$. So p_2 is an increasing function. Now

$$\begin{aligned} p_2(0) &= e^{x_1} + \frac{e^{x_2} - e^{x_1}}{x_2 - x_1}.(0 - x_1) \\ &= \frac{e^{x_1}.(x_2 - x_1) - x_1(e^{x_2} - e^{x_1})}{x_2 - x_1} \\ &= \left(\frac{e^{x_1}}{x_1} - \frac{e^{x_2}}{x_2}\right) \cdot \frac{x_1 x_2}{x_2 - x_1} \end{aligned} \tag{2.1}$$

Now

$$\left(\frac{e^x}{x}\right)' = \frac{x.e^x - e^x}{x^2} = \frac{-e^x(1 - x)}{x^2} < 0, \quad \text{for} \quad 0 < x < 1.$$

Thus $\frac{e^x}{x}$ is a decreasing function on $J = [0,1]$. So $\frac{e^{x_1}}{x_1} - \frac{e^{x_2}}{x_2} > 0$. Therefore, from (2.1), $p_2(0) > 0$. Since p_2 is an increasing function, $p_2(x) > 0$ for all $x \in J = [0,1]$. Thus all Lagrange interpolants to $f(x) = e^x$ from $\mathcal{P}_2$ are positive. Fix $m > 2$. Suppose that all Lagrange interpolants to f from $\mathcal{P}_n$ are positive for all $n < m$. Let $x_1 < x_2 < \ldots < x_m$ be distinct points in J. Let p_m be the unique generalized polynomial from $\mathcal{P}_m$ interpolating to f at $x_1 < x_2 < \ldots < x_m$. By Lemma 1.10, p_m' is the Lagrange interpolant to f' from $\mathcal{P}_{m-1}$. Since $f'(x) = e^x$, p_m' is the Lagrange interpolant to e^x from $\mathcal{P}_{m-1}$. But all Lagrange interpolants to e^x from $\mathcal{P}_{m-1}$ are strictly positive. Therefore, $p_m' > 0$. Hence p_m is an increasing function. So if we prove that $p_m(0) > 0$, then $p_m(x) > 0$ for all $x \in J$. At $x = 0$, from Theorem 1.4,

$$f(0) - p_m(0) = [x_1, x_2, \ldots, x_m, 0]\, f \cdot (0 - x_1)(0 - x_2)\ldots(0 - x_m).$$

Applying Theorem 1.5, we have

$$1 - p_m(0) = \frac{f^{(m)}(\xi)}{m!}.(-x_1)(-x_2)\ldots(-x_m)$$

for some ξ, $0 < \xi < 1$. Thus

$$1 - p_m(0) = \frac{e^\xi}{m!}.(-x_1)(-x_2)\ldots(-x_m)$$

So

$$\begin{aligned} |1 - p_m(0)| &= \frac{e^\xi}{m!}.|x_1.x_2.\ldots.x_m| \\ &\le \frac{e^\xi}{m!} \\ &\le \frac{e}{6} < 1 \end{aligned}$$

since $m \ge 3$. Thus

$$|1 - p_m(0)| < 1.$$

So $p_m(0) > 0$. Thus $p_m(x) > 0$ for all $x \in J$. Hence, by induction, all Lagrange interpolants to f are positive on $J = [0,1]$. So the function $f(x) = e^x$ is totally positive on J. □

Remark 2.2. The function $f(x) = e^x$ is not totally positive on any interval $[0, b]$, $b > 1$. For, if x_1, x_2 are two points in $[0, b]$ with $1 < x_1 < x_2$ and if p_2 is the Lagrange interpolant to f in $\mathcal{P}_2$ interpolating to f at x_1 and x_2, then as in Proposition 2.1

$$\begin{aligned} p_2(0) &= -\left(\frac{e^x}{x}\right)'_{x=\xi} .x_1x_2, \quad \text{for} \quad 1 < x_1 < \xi < x_2 \\ &= \frac{e^\xi}{\xi^2}(1-\xi)x_1x_2 < 0, \end{aligned}$$

since $\xi > x_1 > 1$. Thus $f(x) = e^x$ has a Lagrange interpolant which is not positive on $[0, b]$. Hence e^x is not totally positive on $[0, b]$.

Proposition 2.3. *The function* $f(x) = e^x$ *is totally bounded on* $I = [a, b]$, *where* $[a, b]$ *is any subinterval of* $(-\infty, \infty)$.

Proof. Let $x_1, x_2, \ldots, x_m$ be any m distinct points in $[a, b]$ and let p_m be the Lagrange interpolant to f from $\mathcal{P}_{m-1}$ interpolating to f at $x_1, x_2, \ldots, x_m$. Then, by Theorem 1.4, for all x in $[a, b]$

$$p_m(x) = f(x_1) + \sum_{k=2}^{m}(x-x_1)(x-x_2)\ldots(x-x_{k-1})\,[x_1,\ldots,x_k]\,f \tag{2.2}$$

Now, by Theorem 1.5, for $k = 2, 3, \ldots, m$,

$$[x_1,\ldots,x_k]\,f = \frac{f^{(k-1)}(\xi_k)}{(k-1)!}, \tag{2.3}$$

for some point ξ_k in (a, b). Applying Equation 2.3 in (2.2), we have,

$$p_m(x) = f(x_1) + \sum_{k=2}^{m}\frac{f^{(k-1)}(\xi_k)}{(k-1)!}.\,(x-x_1)\ldots.(x-x_{k-1}) \tag{2.4}$$

Therefore,

$$|p_m(x)| \le |f(x_1)| + \sum_{k=2}^{m}\frac{|f^{(k-1)}(\xi_k)|}{(k-1)!}.\,\left(|x-x_1|\ldots|x-x_{k-1}|\right)$$

Also, since $f(x) = e^x$, f', f'' , . . ., all coincide with f on $[a, b]$. Therefore, for all x in $[a, b]$,

$$\begin{aligned} |p_m(x)| &\le e^{x_1} + \sum_{k=2}^{m}\frac{e^{\xi_k}}{(k-1)!}.\,\left(|x-x_1|\ldots.|x-x_{k-1}|\right) \\ &\le e^{x_1} + \sum_{k=2}^{m}\frac{e^{\xi_k}}{(k-1)!}.\,(b-a)^{k-1} \\ &\le e^b + e^b\sum_{k=2}^{m}\frac{(b-a)^{k-1}}{(k-1)!} \\ &= e^b\left\{1 + \sum_{k=2}^{m}\frac{(b-a)^{k-1}}{(k-1)!}\right\} < e^b.e^{b-a} = e^{2b-a} \end{aligned}$$

That is, for all x in $[a, b]$,

$$|p_m(x)| \le e^{2b-a}$$

Since p_m is an arbitrary Lagrange interpolant to $f(x) = e^x$, this shows that e^x is totally bounded on $[a, b]$. □

An analogous result of the following Theorem is given in [2] for the interval $[-1, 1]$. *We have observed a slight modification in the result for the interval* $[0, 1]$.

Theorem 2.4. *The function* $f(x) = \frac{1}{b-x}$ *is totally positive on* $J = [0, 1]$ *if and only if* $b \ge 2$.

Proof. Let $f(x) = \frac{1}{b-x}$. Since f must be positive on J, we must have $b > 1$. Let $x_1 < x_2 < \ldots < x_n$ be n distinct points in $J = [0, 1]$. Let p be the unique Lagrange polynomial from $\mathcal{P}_n$ interpolating to f at the points $x_1, x_2, \ldots, x_n$. Since p interpolates to f at $x_1, x_2, \ldots, x_n$, we have

$$p(x_i) = \frac{1}{b - x_i} \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Then

$$p(x_i)\,(b - x_i) - 1 = 0 \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Thus $x_1, x_2, \ldots, x_n$ are the zeros of $p(x)(b - x) - 1$. So, there exists a K such that

$$p(x)(b - x) - 1 = K(x - x_1)(x - x_2)\ldots(x - x_n).$$

When $x = b$, from the above equation, we have,

$$-1 = K(b - x_1)(b - x_2)\ldots(b - x_n).$$

Thus

$$K = \frac{-1}{(b - x_1)(b - x_2)\ldots(b - x_n)}.$$

So

$$p(x)(b - x) - 1 = \frac{-1}{(b - x_1)(b - x_2)\ldots(b - x_n)} \cdot (x - x_1)(x - x_2)\ldots(x - x_n)$$

Thus

$$p(x) = \frac{1}{b - x} - \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(b - x)(b - x_1)(b - x_2)\ldots(b - x_n)}$$

So the error of interpolation $E(x)$ at the $x \in J$ is given by

$$E(x) = f(x) - p(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(b - x)(b - x_1)(b - x_2)\ldots(b - x_n)}$$

Now

$$\begin{aligned}
p > 0 \quad \text{on } J &\Leftrightarrow E(x) < f(x) \;\; \text{for} \quad \text{all} \quad x \quad \text{on } J \\
&\Leftrightarrow \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(b - x)(b - x_1)(b - x_2)\ldots(b - x_n)} < \frac{1}{b - x} \\
&\Leftrightarrow (x - x_1)(x - x_2)\ldots(x - x_n) < (b - x_1)(b - x_2)\ldots(b - x_n) \quad \text{on} \quad J.
\end{aligned}$$

Thus

$$f \quad \text{is totally positive on } J \Leftrightarrow (x - x_1)(x - x_2)\ldots(x - x_n) < (b - x_1)(b - x_2)\ldots(b - x_n)$$

whenever $x_1, x_2, \ldots, x_n$ are any n distinct points in $[0, 1]$ and x is any point in $[0, 1]$.

Case(i) : Suppose $1 < b < 2$

Consider $n = 2$. Take $x_1 = \frac{b}{2}$, $x_2 = 1$, $x = 0$. Then

$$(0 - x_1)(0 - x_2) = (-x_1)(-x_2) = \frac{b}{2}$$

and

$$(b - x_1)(b - x_2) = (b - b/2)(b - 1) = \frac{b}{2}(b - 1) < \frac{b}{2}.$$

Therefore,

$$(0 - x_1)(0 - x_2) > (b - x_1)(b - x_2).$$

This implies that f is not totally positive in this case.

Case(ii) :Suppose $b \geq 2$

Let n be any positive integer and x_1, x_2, ..., x_n any n distinct points in $[0, 1]$ with $0 < x_1 < x_2 < \ldots < x_n \leq 1$. If $n \geq 2$, then $b - x_i > 1$ for $i = 1, \ldots, n-1$ and $b - x_n \geq 1$. Therefore, for all x in $[0, 1]$,

$$(b - x_1)(b - x_2) \ldots (b - x_n) > 1 \geq (x - x_1)(x - x_2) \ldots (x - x_n).$$

If $n = 1$, then $x_n = x_1$. If $b > 2$ and if $0 \leq x_1 < 1$, then $b - x_1 > 1 \geq x - x_1$. If $b = 2$ and $x_1 = 1$, then $b - x_1 = 1$ and $x - x_1 = x - 1 \leq 0$ for all x in $[0, 1]$. Therefore, in this case also

$$b - x_1 > x - x_1.$$

Hence, $f(x) = \frac{1}{b-x}$ is totally positive on $J = [0, 1]$ if and only if $b \geq 2$.

□

References

[1] Davis, P.J., Interpolation and Approximation, Dover Publications, New York, 1975.

[2] Horwitz, A.L., and Rubel, L.A., Totally positive functions and totally bounded functions on $[-1, 1]$., J. Approx. Theory **52** *(1988), 204–216.*

[3] Thalmi. B and Jayasri. C, A connection between generalized totally positive functions and generalized totally bounded functions, Bulletin of Calcutta Mathematical society, 109(4), 2017, 305-314.

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